THOMAS' CALCULUS

MEDIA UPGRADE

Chapter 2

Limits and Continuity

2.1

Rates of Change and Limits

DEFINITION **Average Rate of Change over an Interval**

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

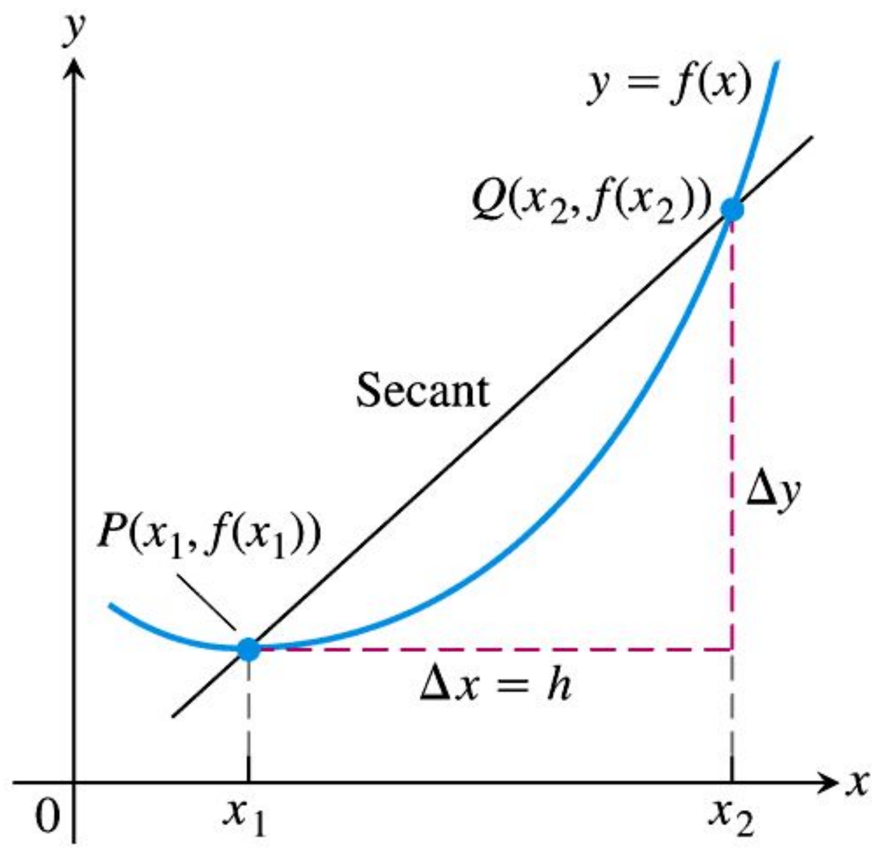


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

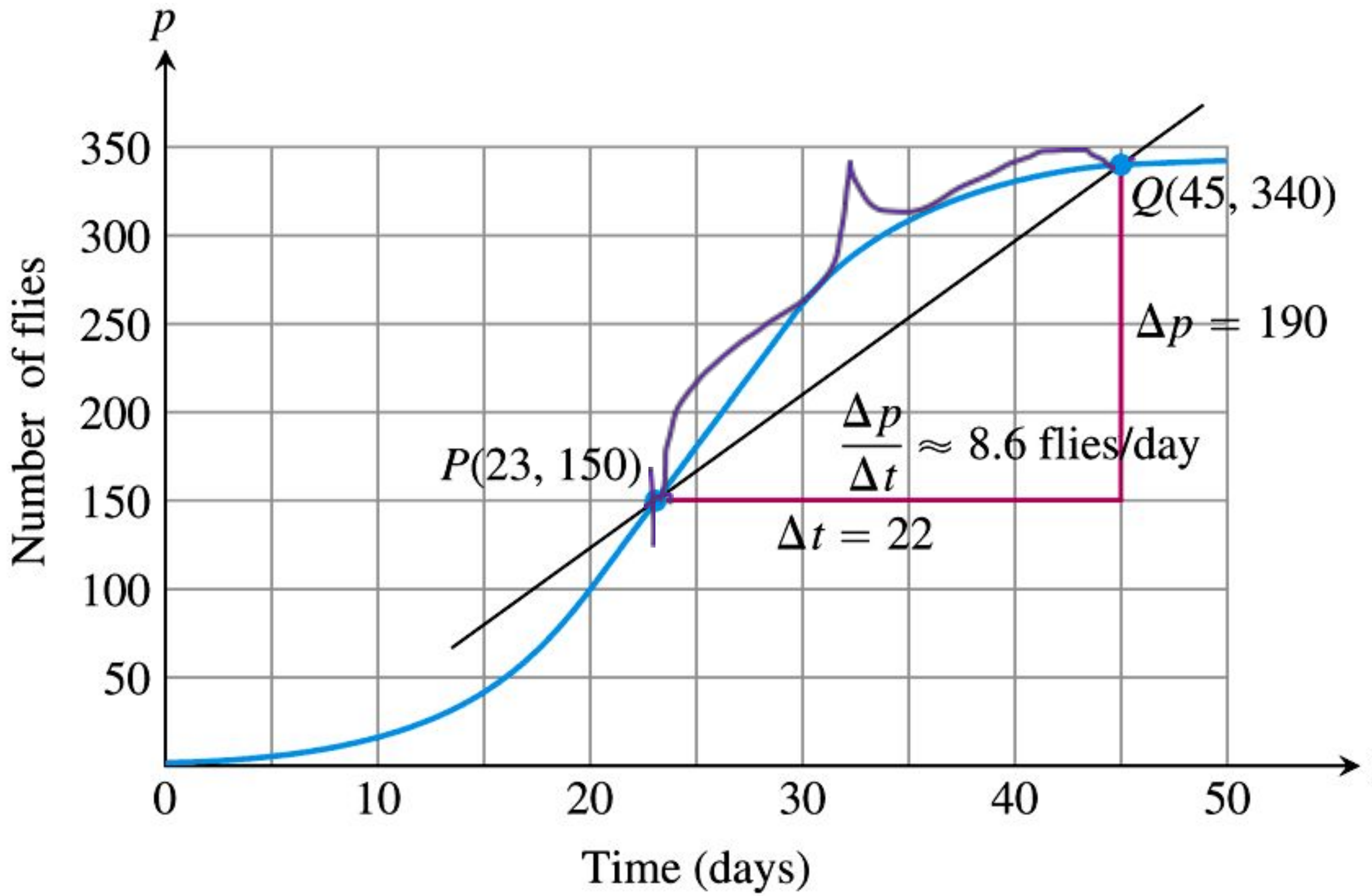


FIGURE 2.2 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line.

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\dfrac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\dfrac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\dfrac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\dfrac{265 - 150}{30 - 23} \approx 16.4$

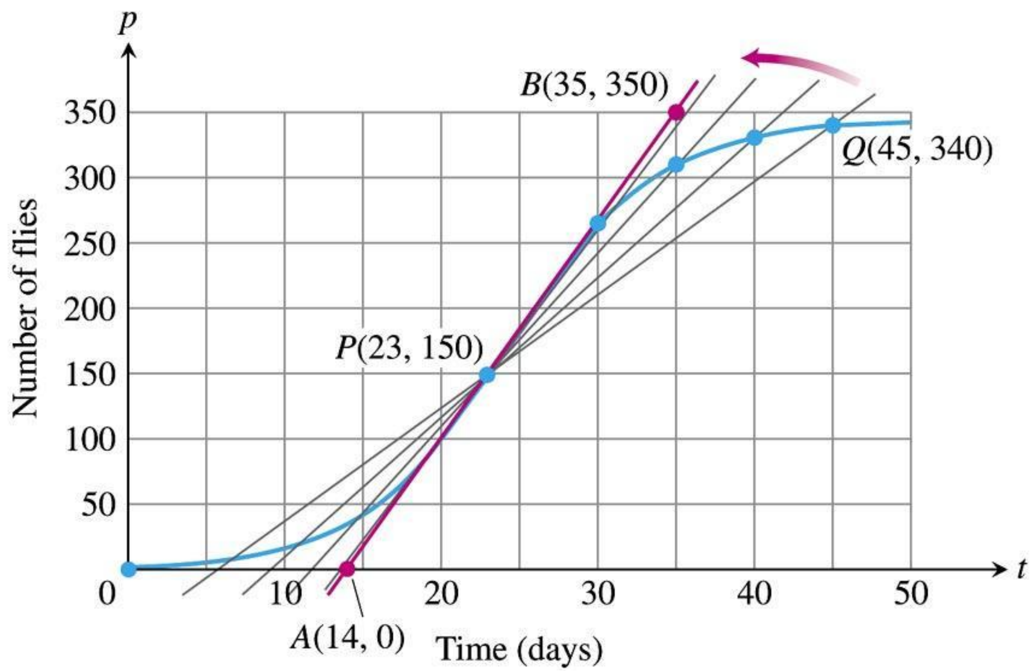


FIGURE 2.3 The positions and slopes of four secants through the point P on the fruit fly graph (Example 4).

TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \dfrac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

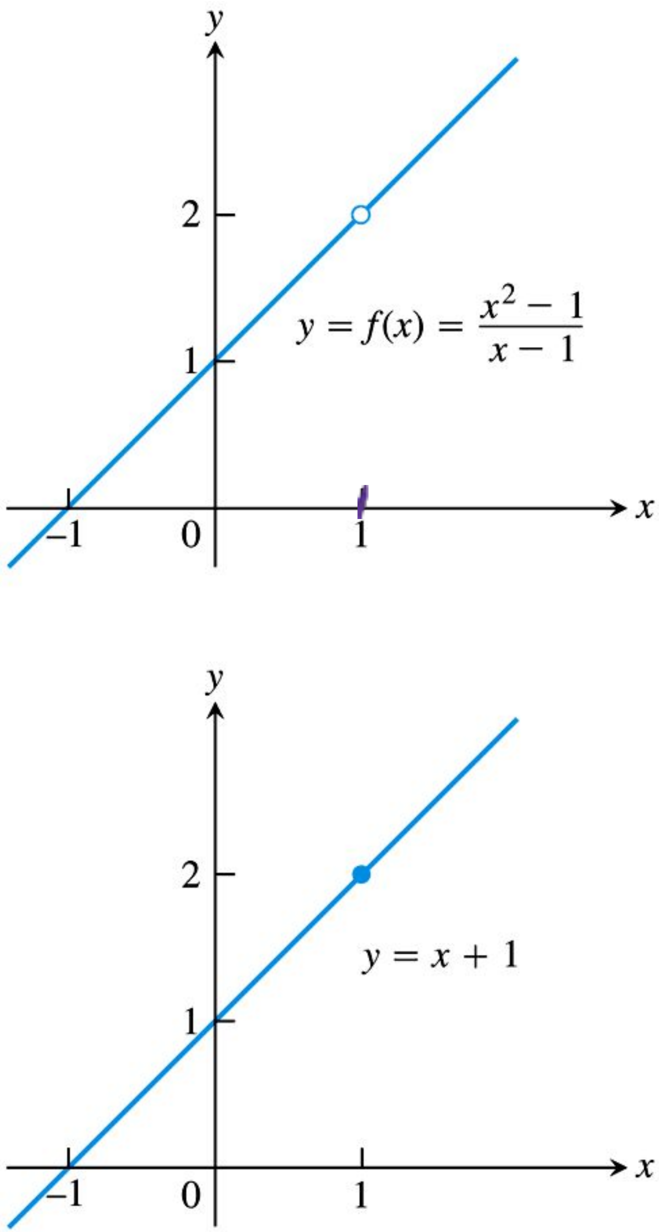


FIGURE 2.4 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 5).

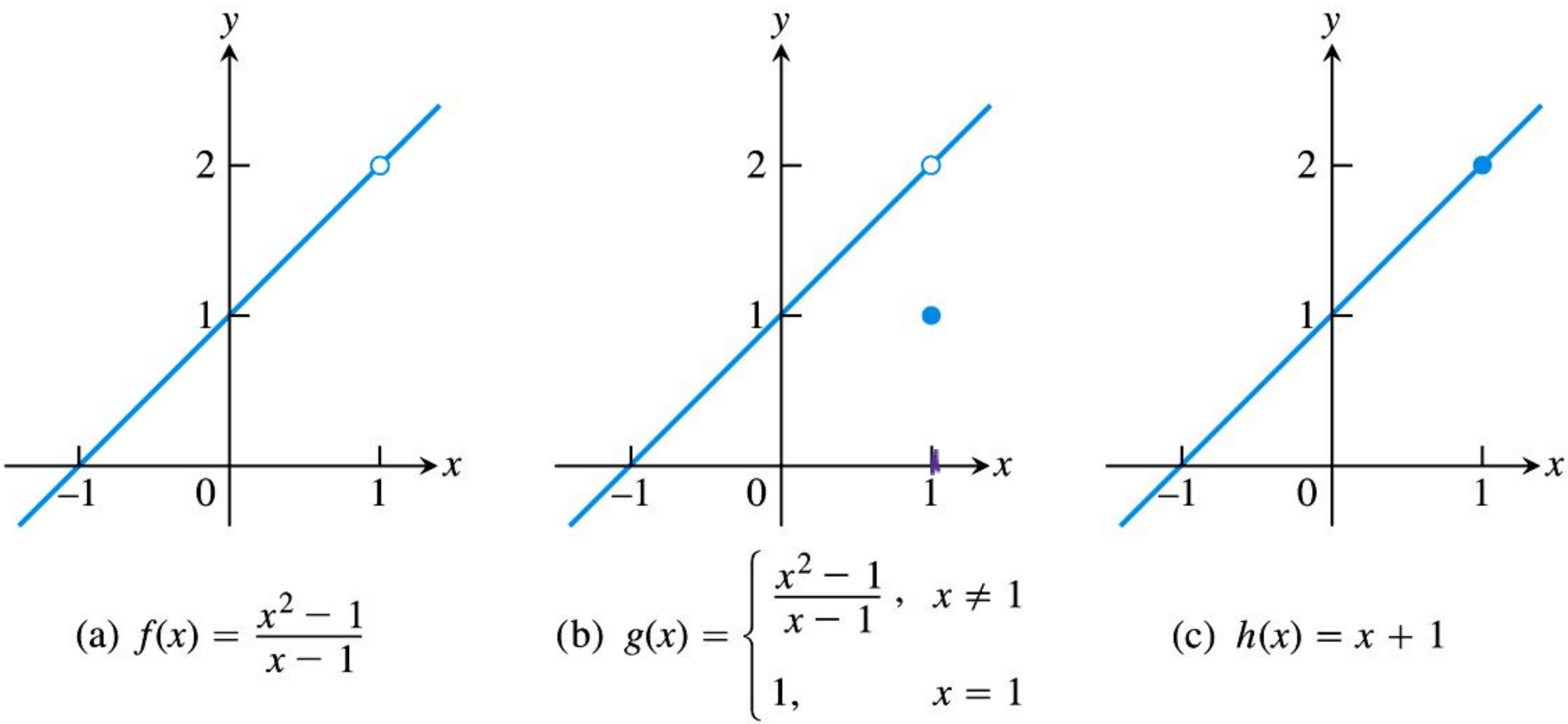
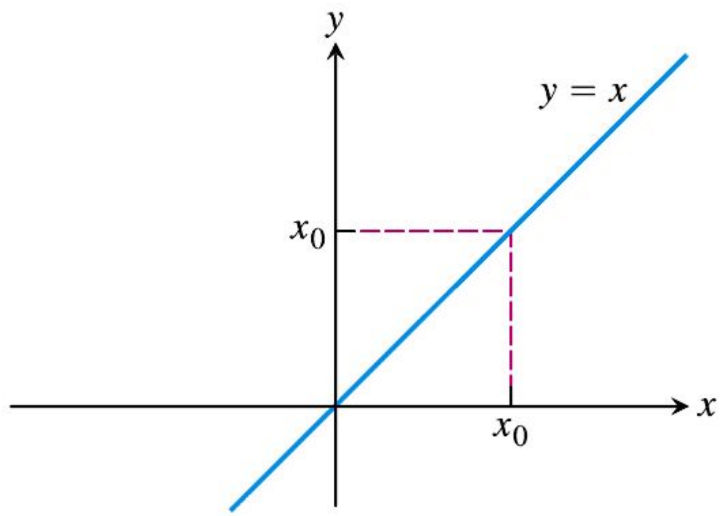
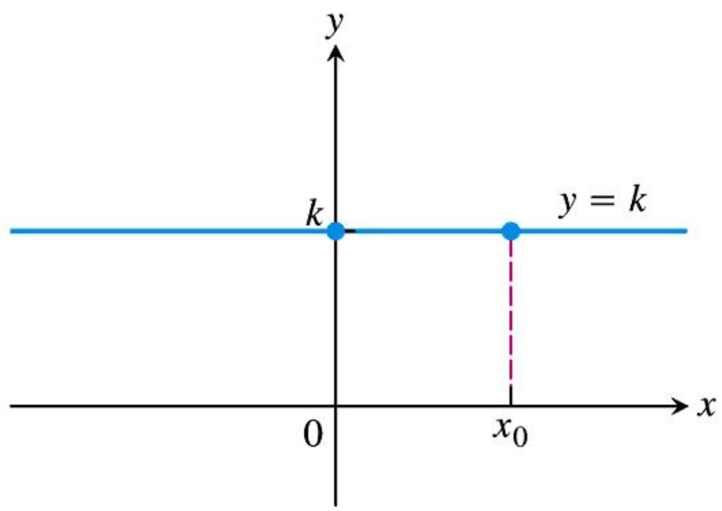


FIGURE 2.5 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 6).



(a) Identity function



(b) Constant function

FIGURE 2.6 The functions in Example 8.

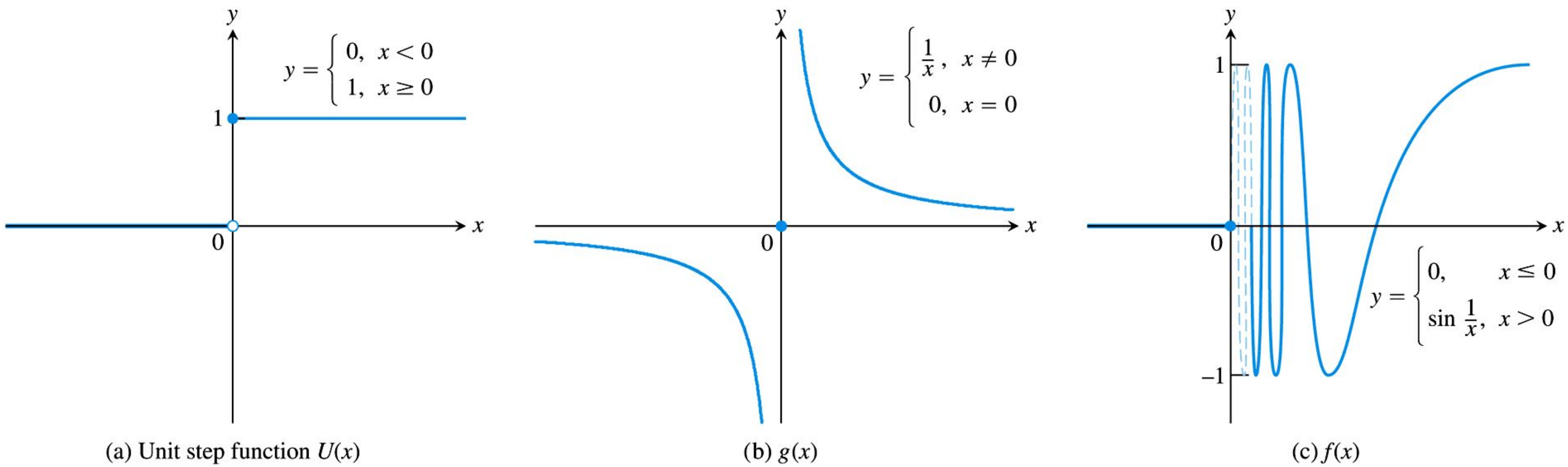


FIGURE 2.7 None of these functions has a limit as x approaches 0 (Example 9).

2.2

Calculating Limits Using the Limits Laws

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $$\lim_{x \to c}(f(x) + g(x)) = L + M$$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $$\lim_{x \to c}(f(x) - g(x)) = L - M$$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $$\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $$\lim_{x \to c}(k \cdot f(x)) = k \cdot L$$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Identifying Common Factors

It can be shown that if $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

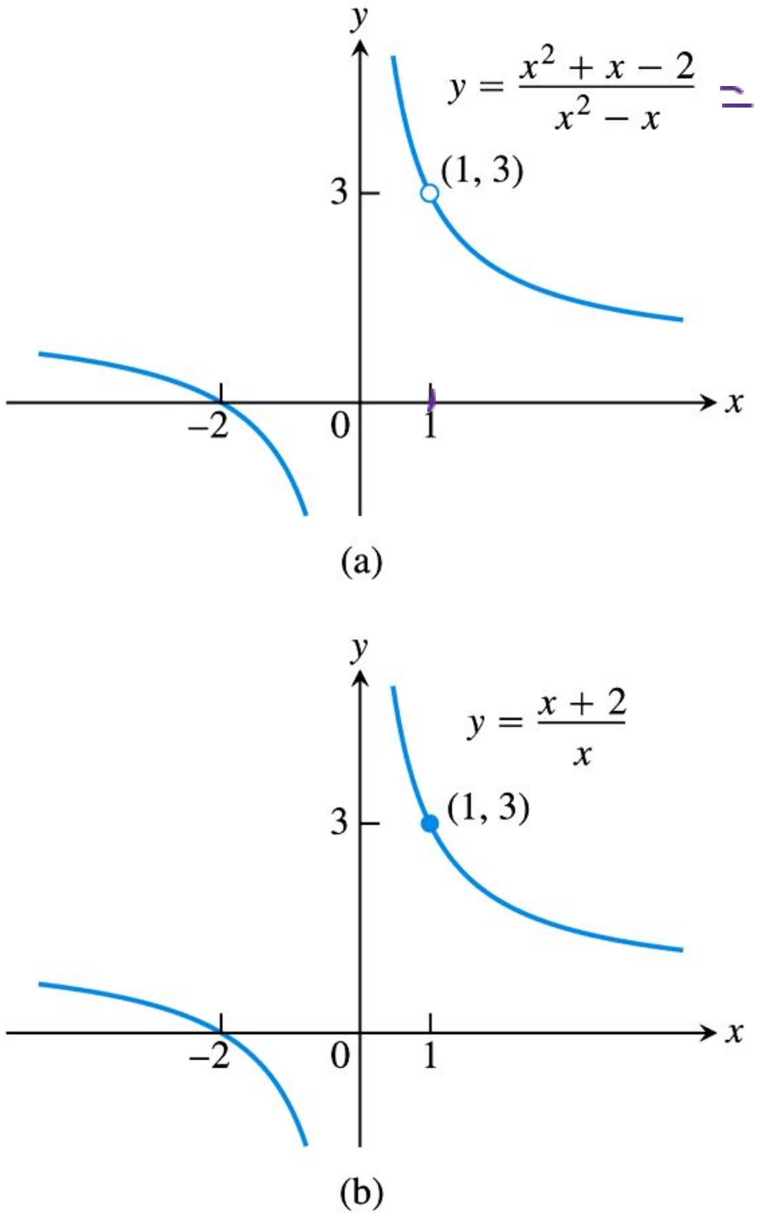


FIGURE 2.8 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 3).

THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

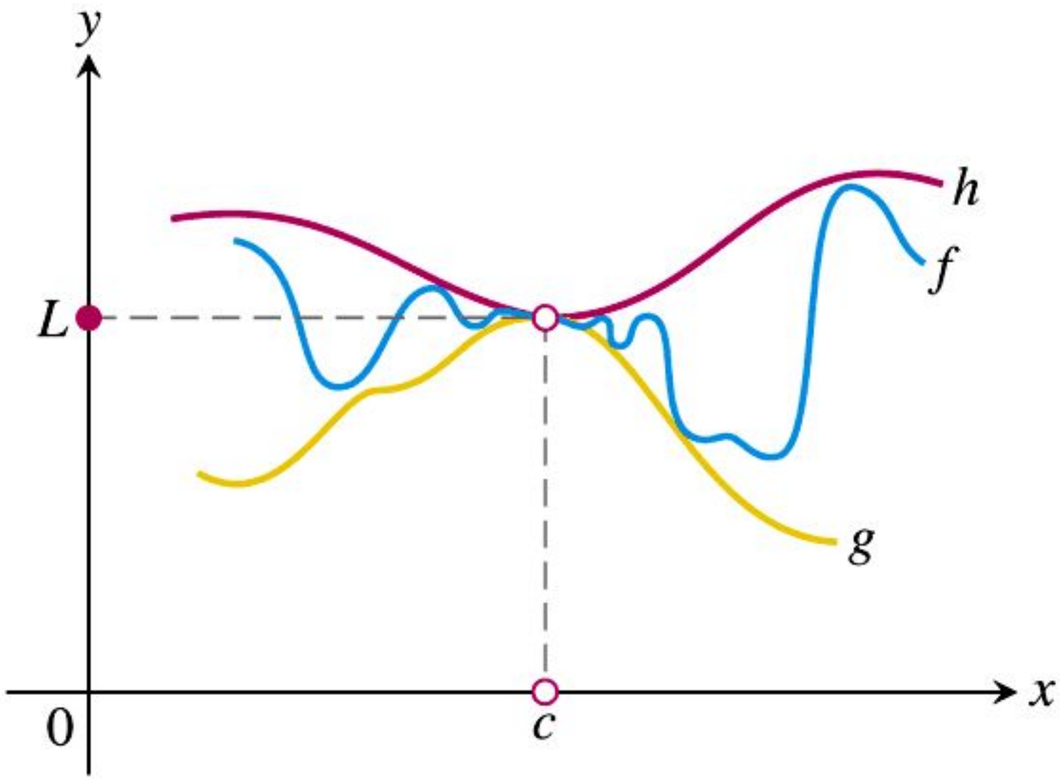


FIGURE 2.9 The graph of f is sandwiched between the graphs of g and h.

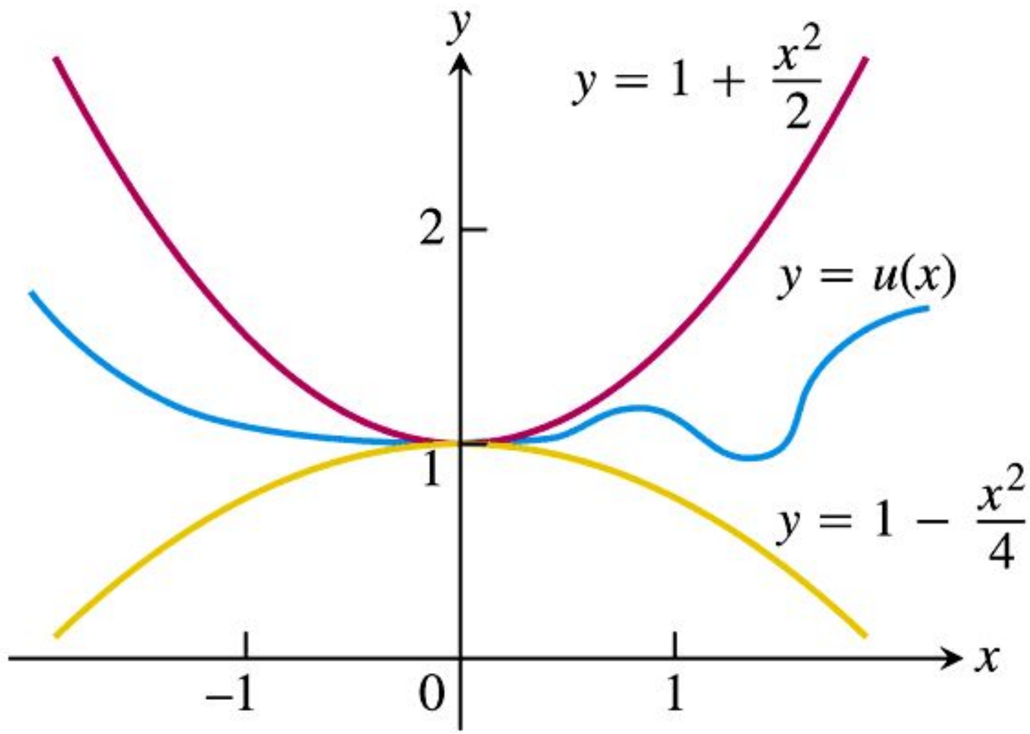


FIGURE 2.10 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 5).

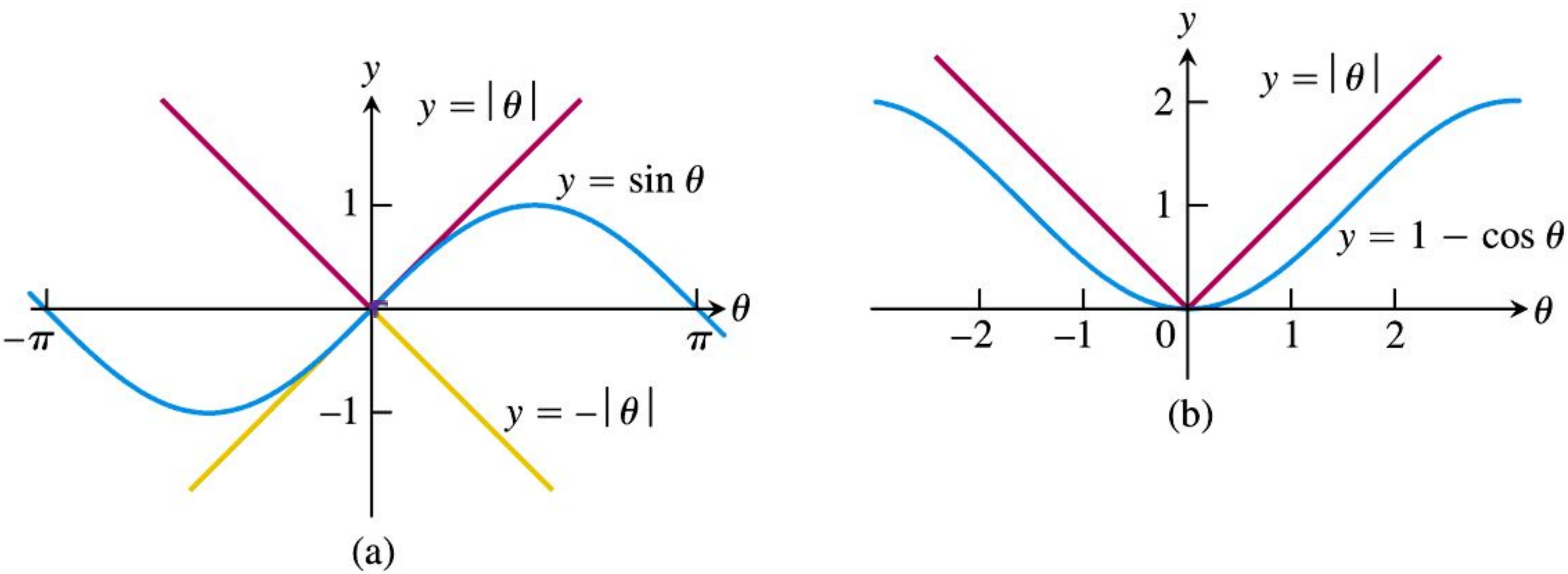


FIGURE 2.11 The Sandwich Theorem confirms that (a) $\lim_{\theta \to 0} \sin \theta = 0$ and (b) $\lim_{\theta \to 0} (1 - \cos \theta) = 0$ (Example 6).

THEOREM 5 If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

2.3

The Precise Definition of a Limit

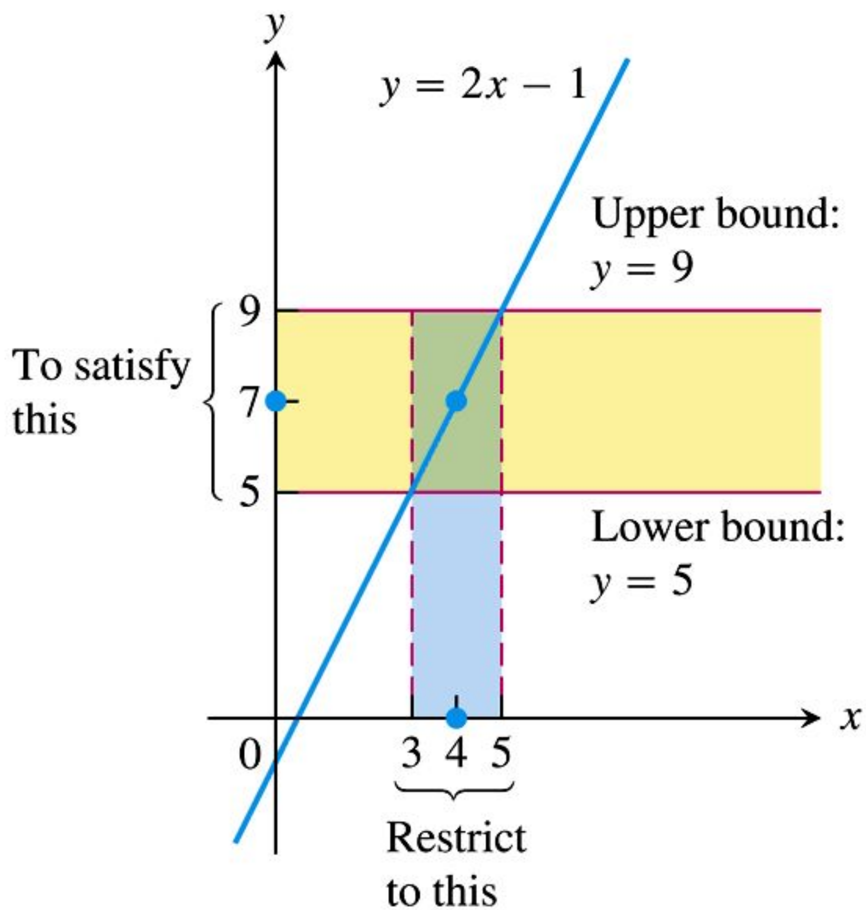


FIGURE 2.12 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Example 1).

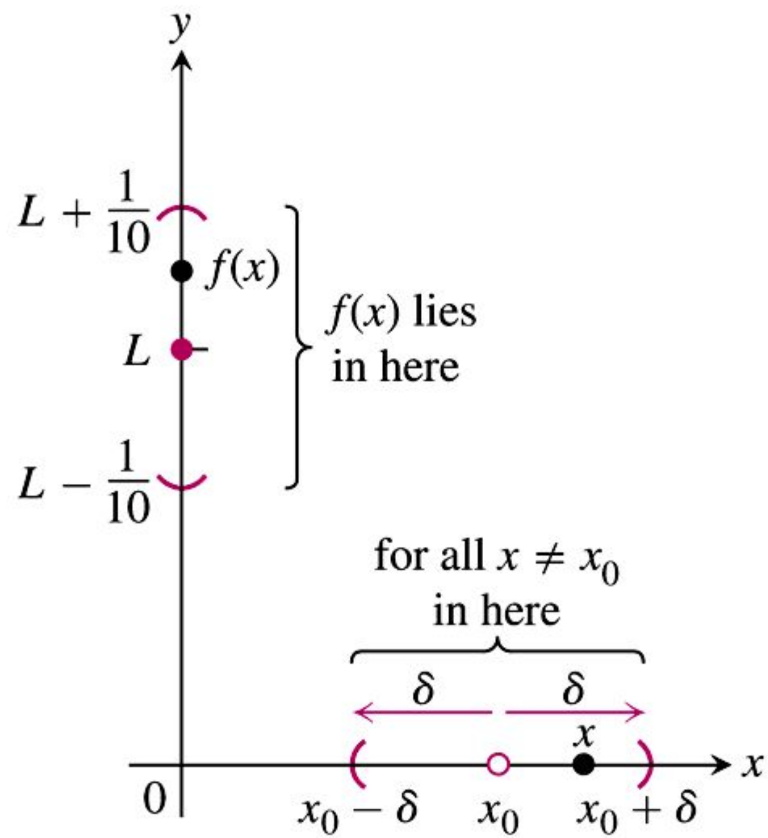


FIGURE 2.13 How should we define $\delta > 0$ so that keeping x within the interval $(x_0 - \delta, x_0 + \delta)$ will keep $f(x)$ within the interval $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$?

DEFINITION **Limit of a Function**

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

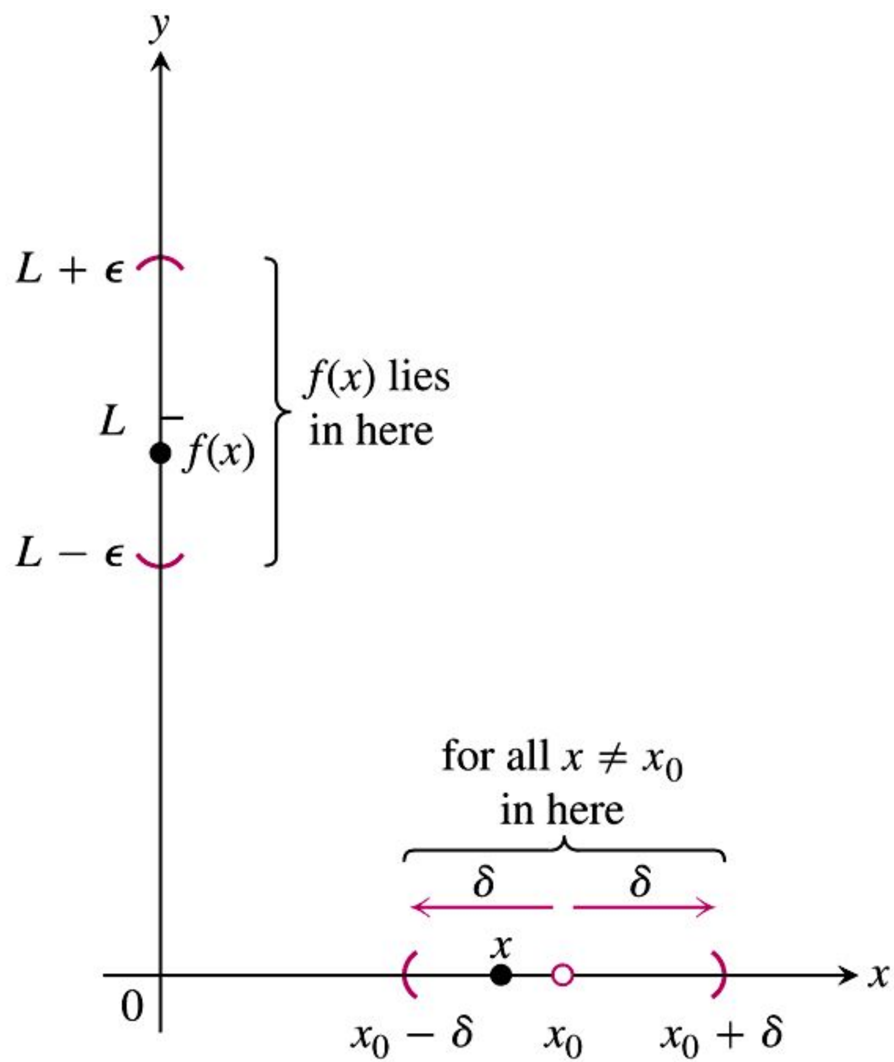


FIGURE 2.14 The relation of δ and ϵ in the definition of limit.

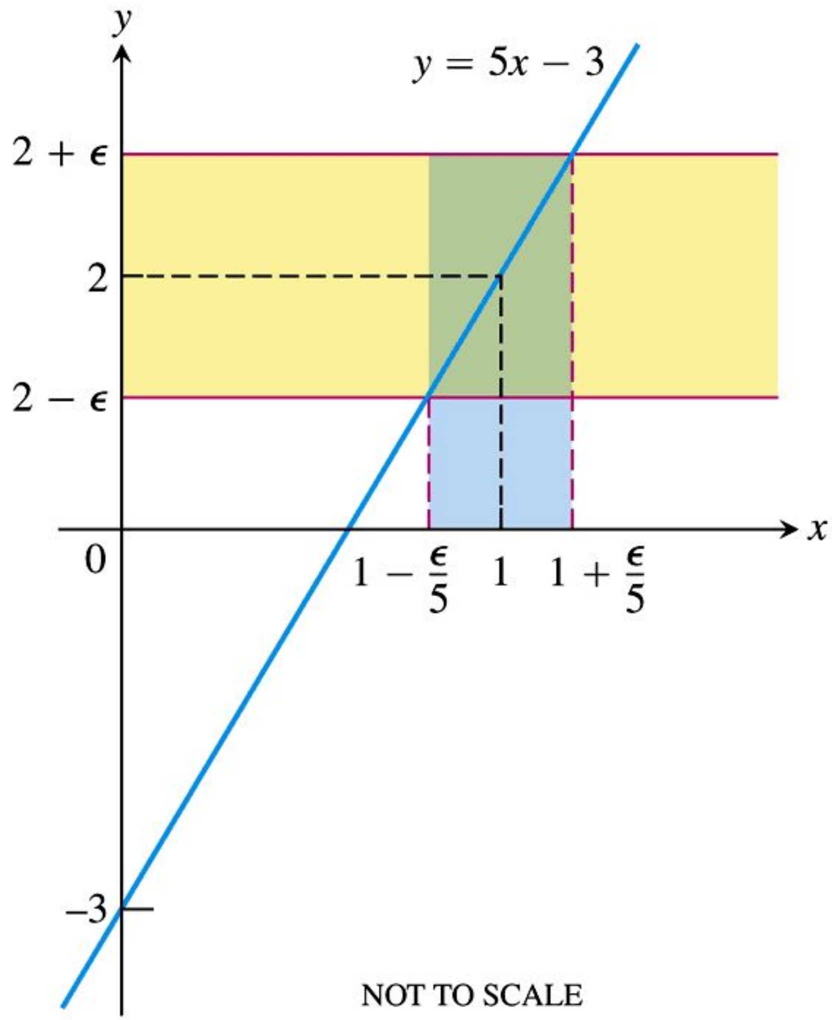


FIGURE 2.15 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

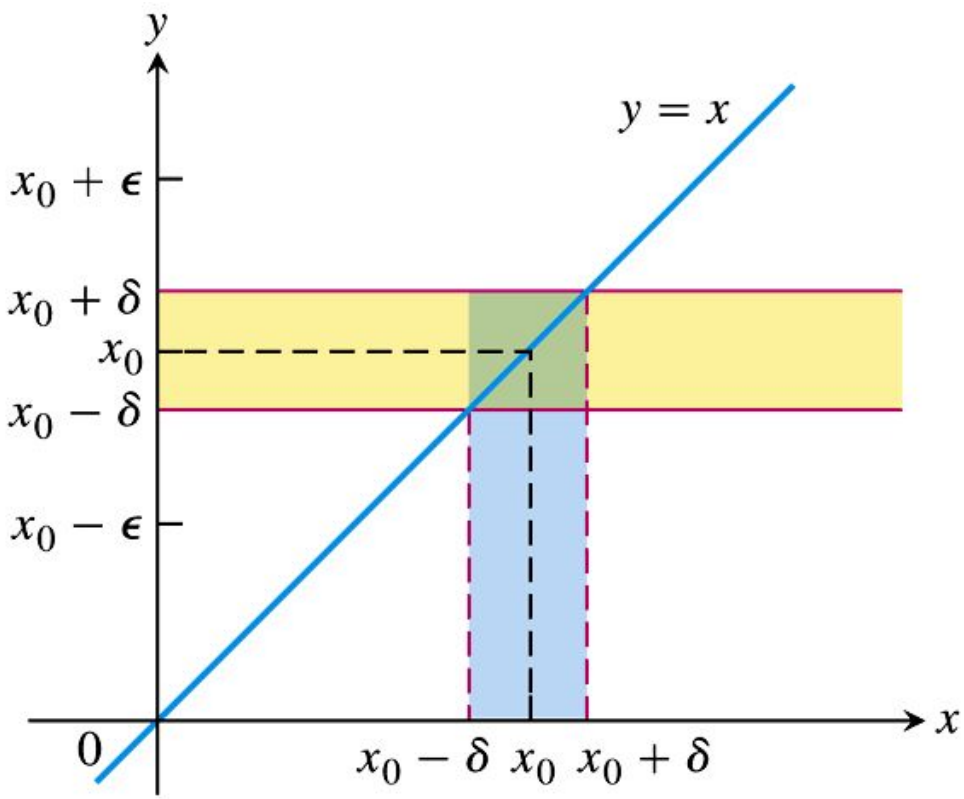


FIGURE 2.16 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \leq \epsilon$ (Example 3a).

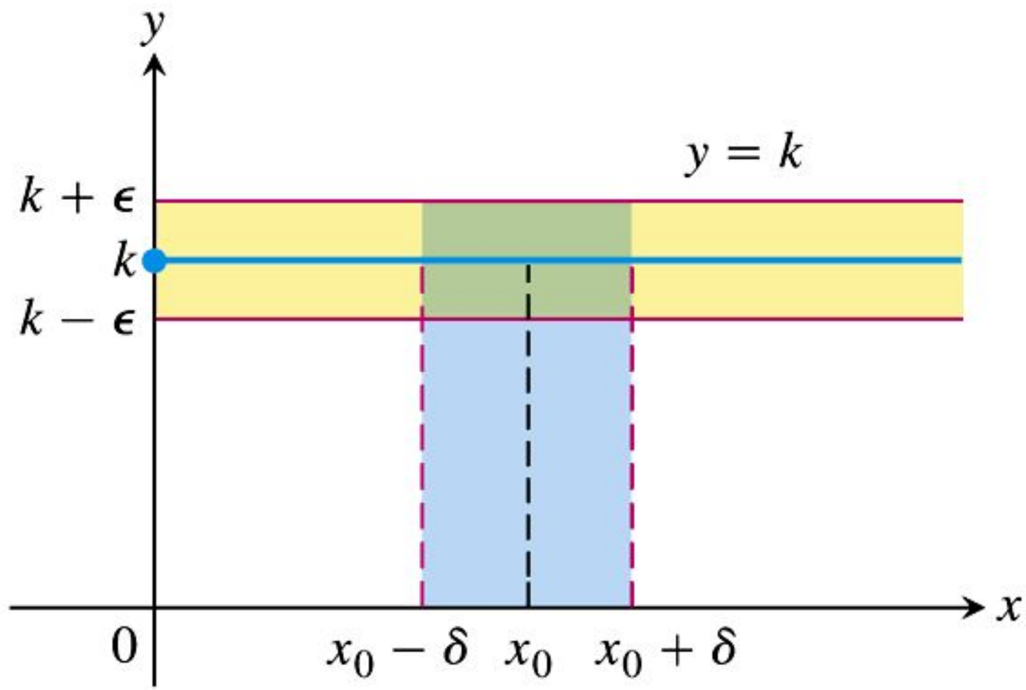


FIGURE 2.17 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).

FIGURE 2.18 An open interval of radius 3 about $x_0 = 5$ will lie inside the open interval (2, 10).

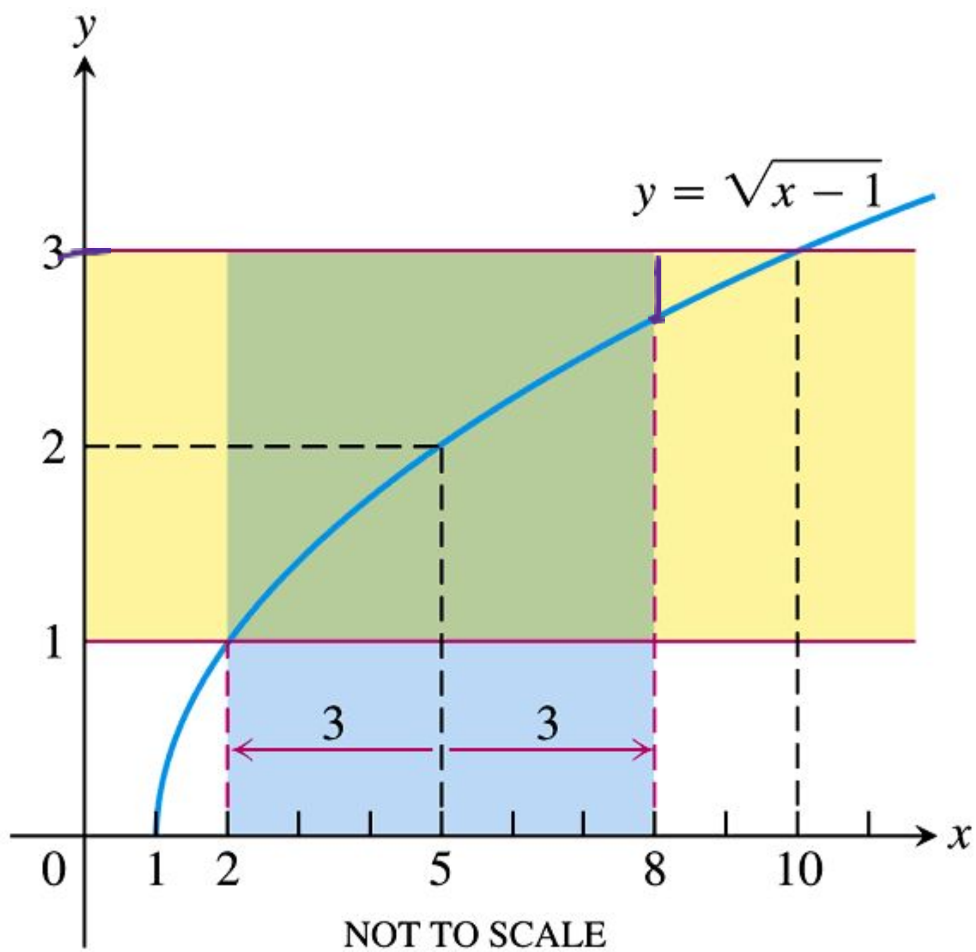


FIGURE 2.19 The function and intervals in Example 4.

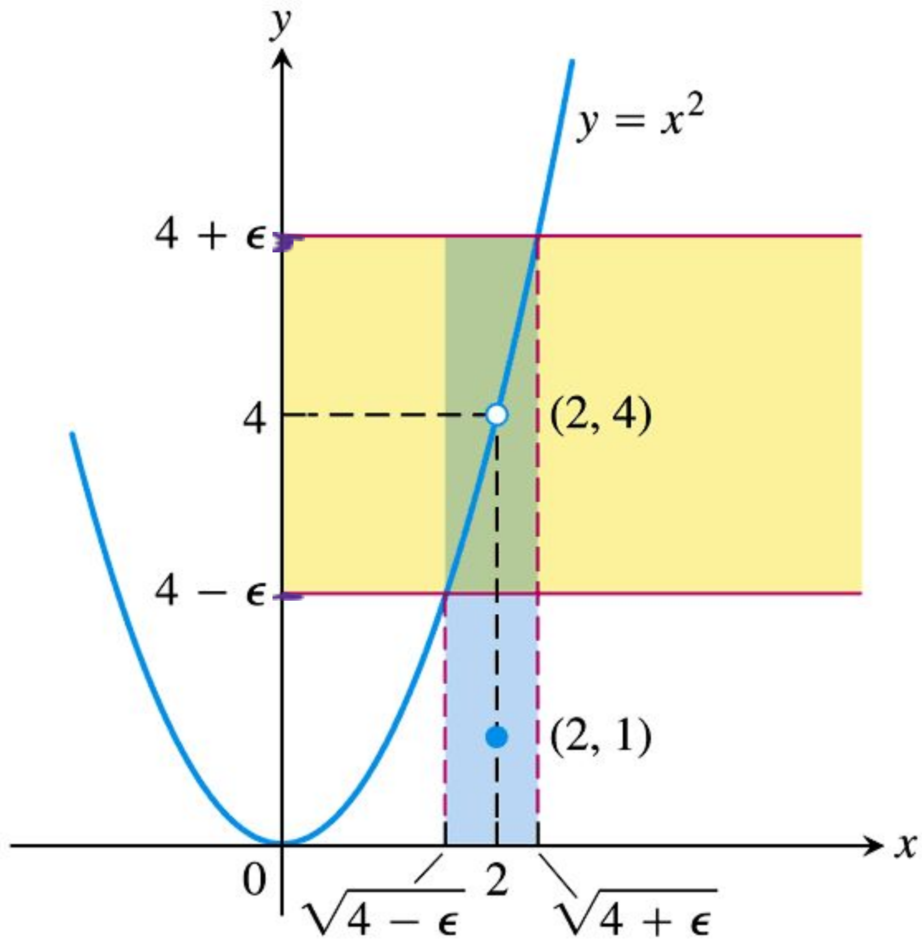


FIGURE 2.20 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \epsilon$.

2.4

One-Sided Limits and Limits at Infinity

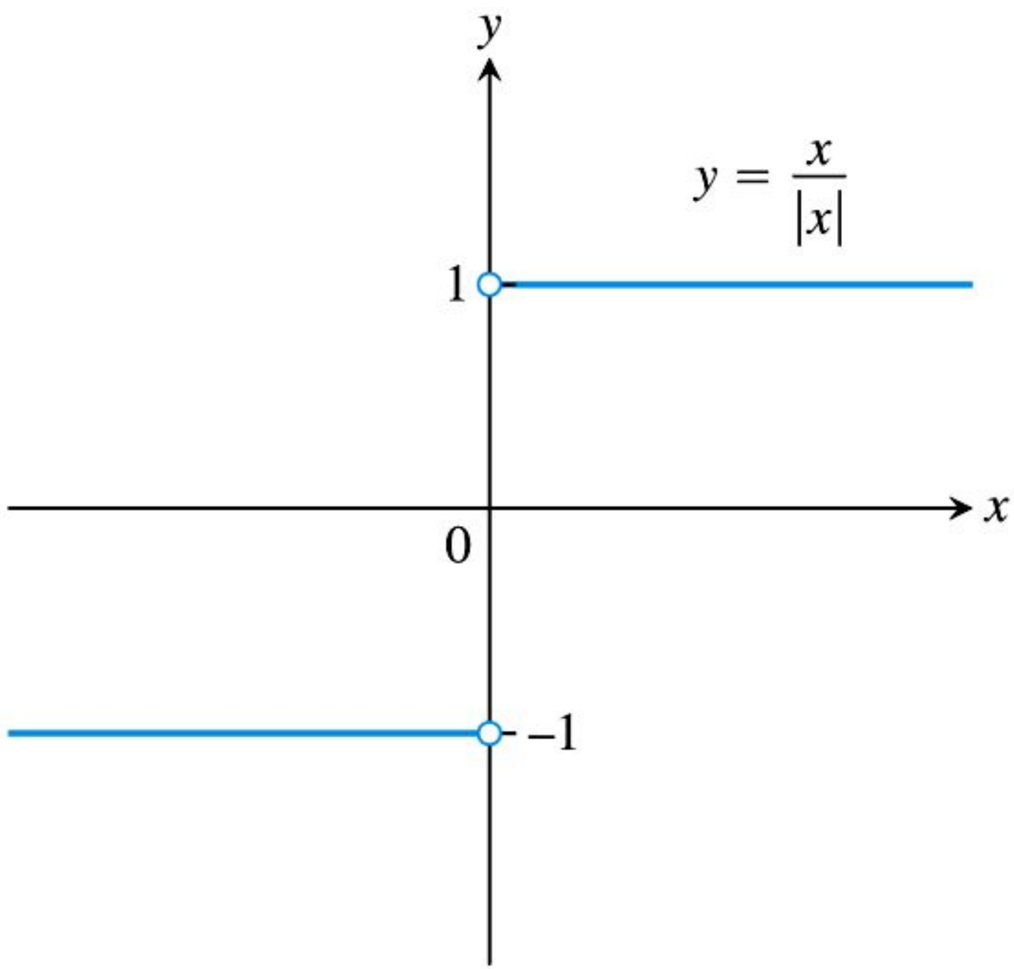


FIGURE 2.21 Different right-hand and left-hand limits at the origin.

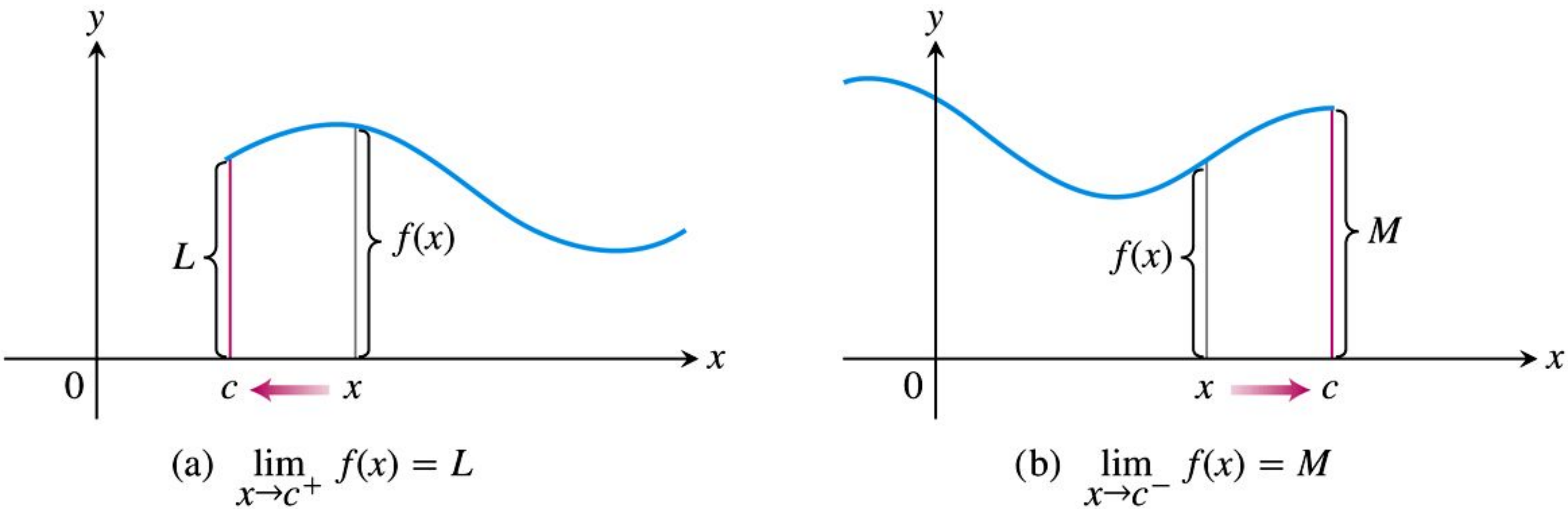


FIGURE 2.22 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

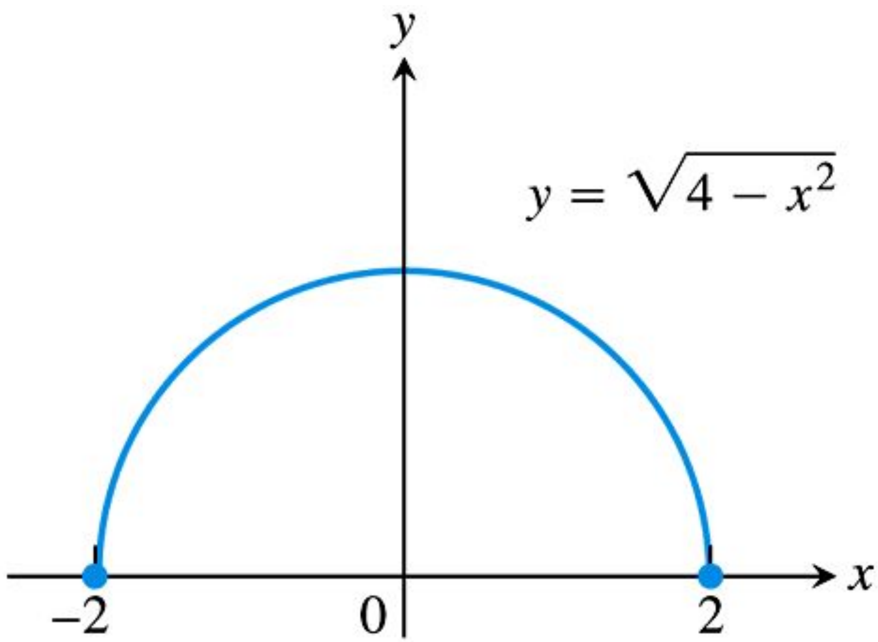


FIGURE 2.23 $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ and $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ (Example 1).

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

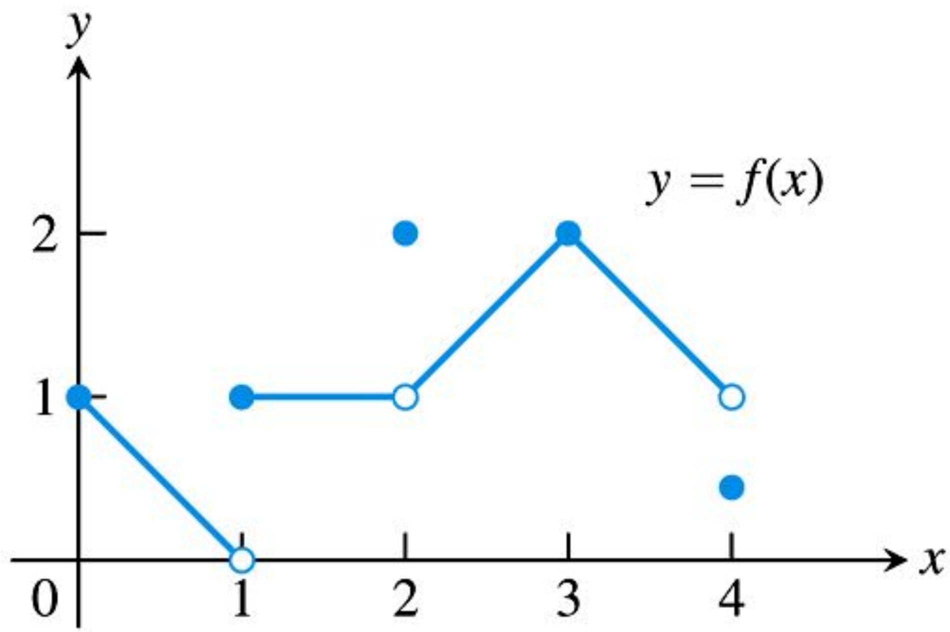


FIGURE 2.24 Graph of the function in Example 2.

DEFINITIONS **Right-Hand, Left-Hand Limits**

We say that $f(x)$ has **right-hand limit L at x_0**, and write

$$\lim_{x \to x_0^+} f(x) = L \qquad \text{(See Figure 2.25)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon .$$

We say that f has **left-hand limit L at x_0**, and write

$$\lim_{x \to x_0^-} f(x) = L \qquad \text{(See Figure 2.26)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \qquad \Rightarrow \qquad |f(x) - L| < \epsilon .$$

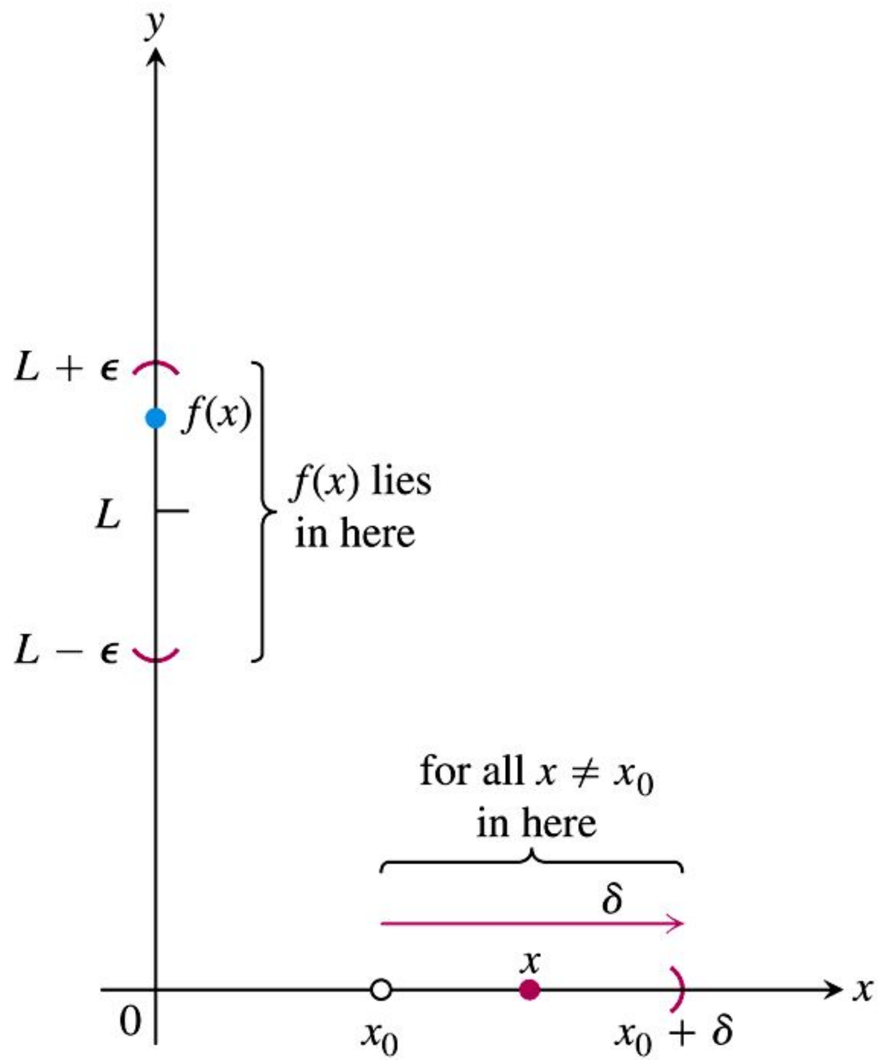


FIGURE 2.25 Intervals associated with the definition of right-hand limit.

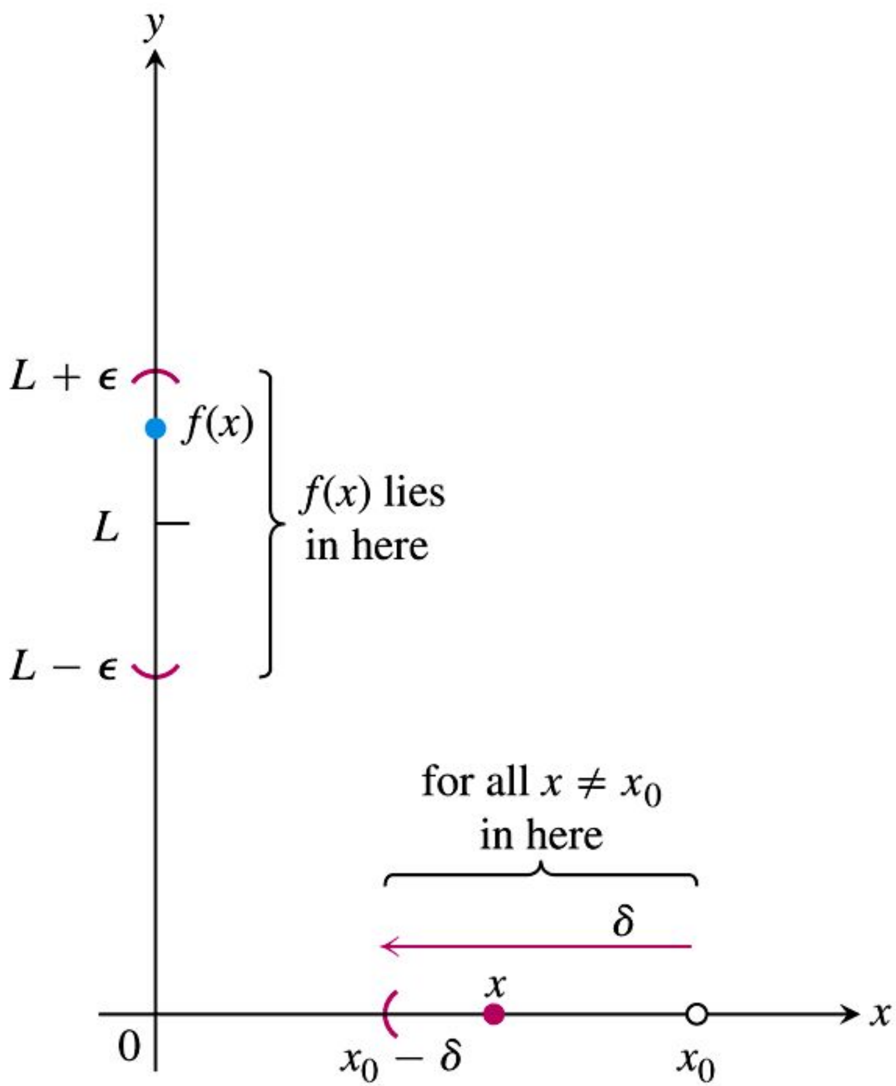


FIGURE 2.26 Intervals associated with the definition of left-hand limit.

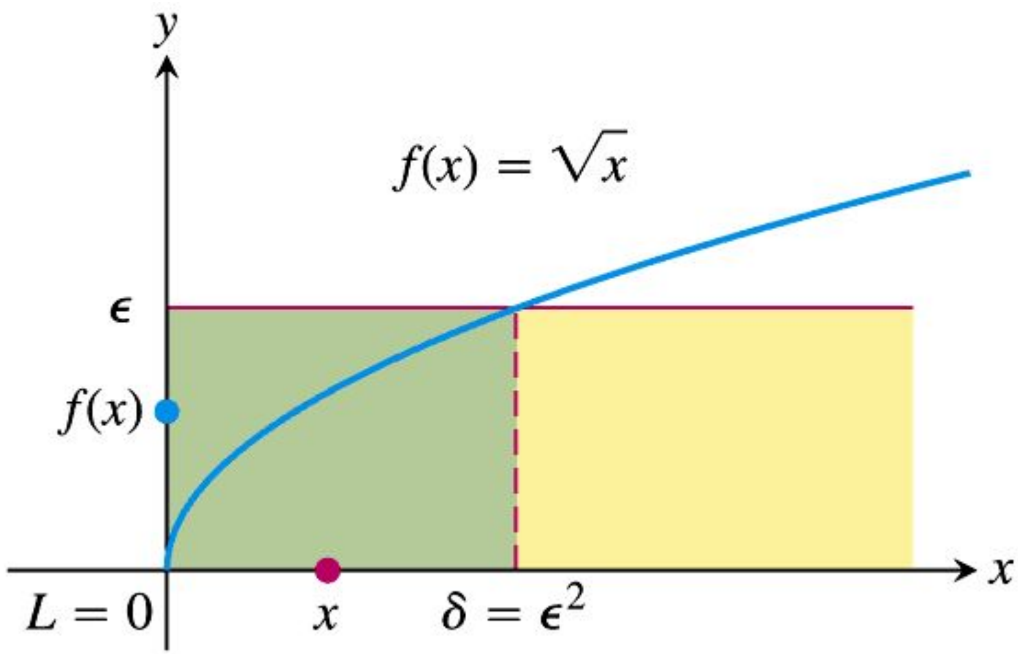


FIGURE 2.27 $\lim_{x \to 0^+} \sqrt{x} = 0$ in Example 3.

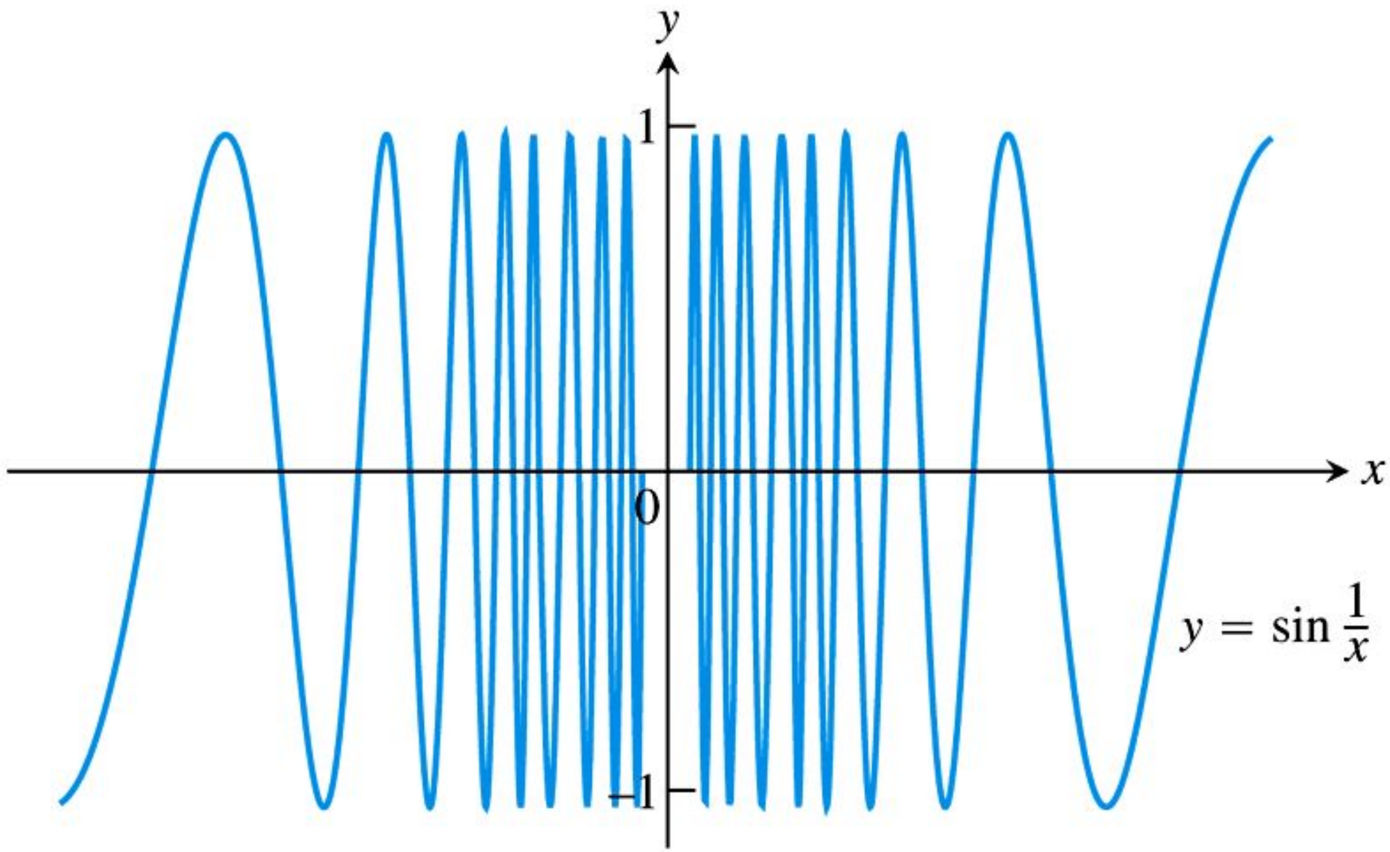


FIGURE 2.28 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4).

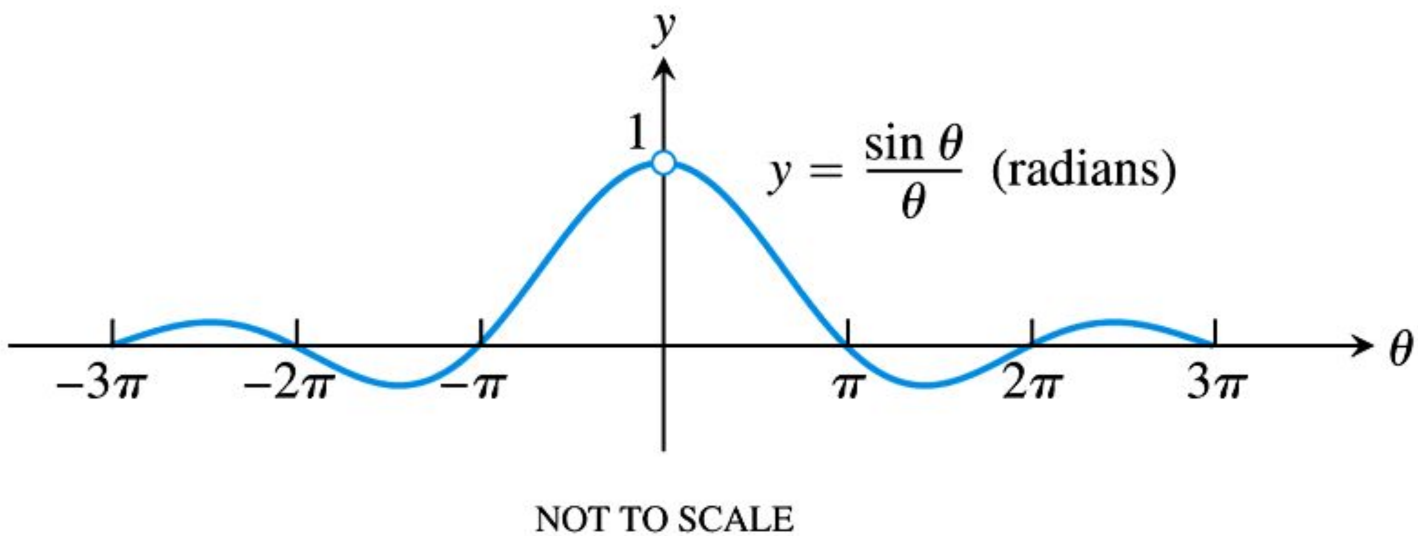


FIGURE 2.29 The graph of $f(\theta) = (\sin\theta)/\theta$.

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

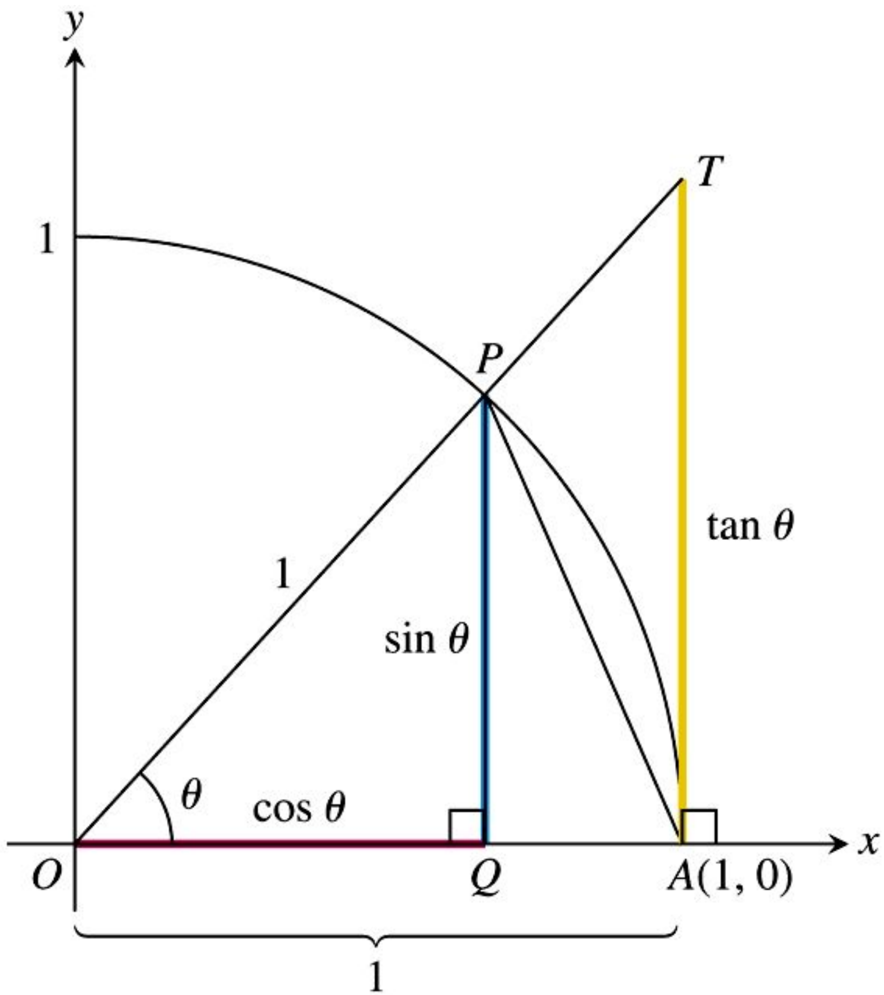


FIGURE 2.30 The figure for the proof of Theorem 7. $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

DEFINITIONS **Limit as x approaches ∞ or $-\infty$**

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \epsilon .$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \qquad \Rightarrow \qquad |f(x) - L| < \epsilon .$$

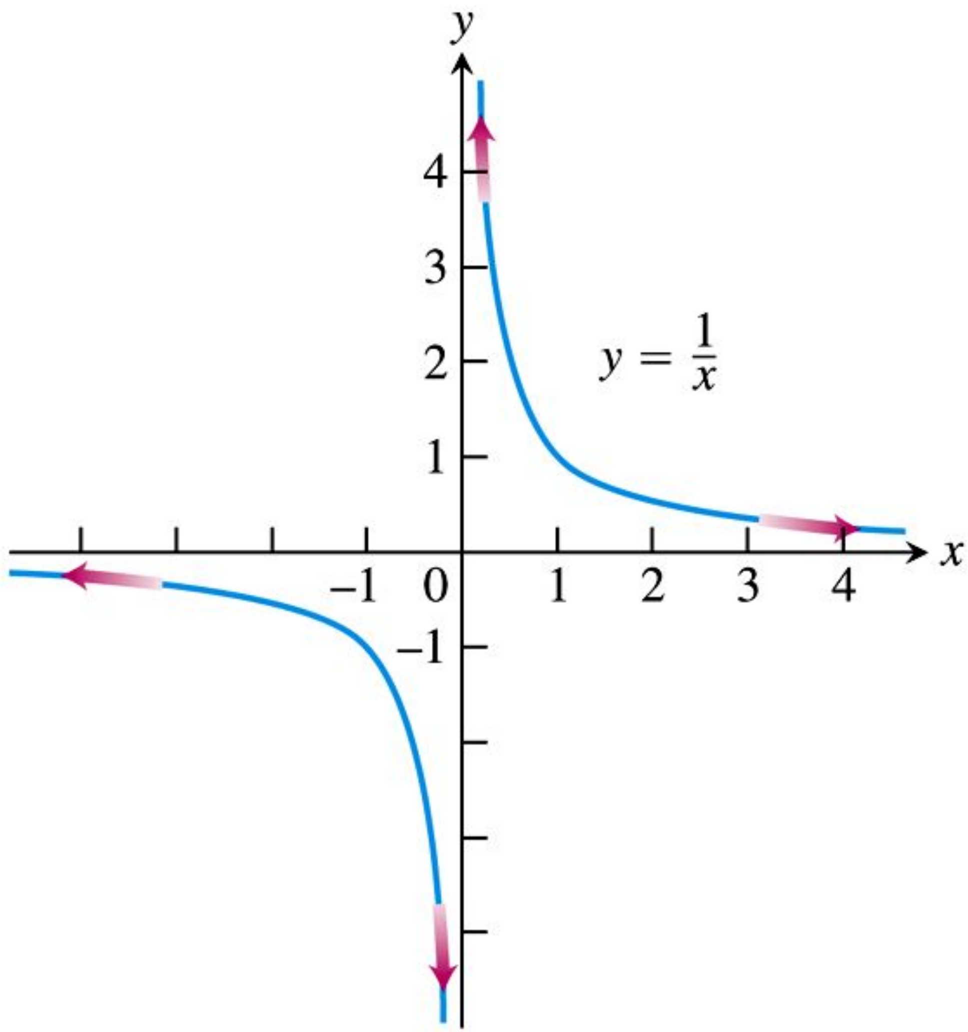


FIGURE 2.31 The graph of $y = 1/x$.

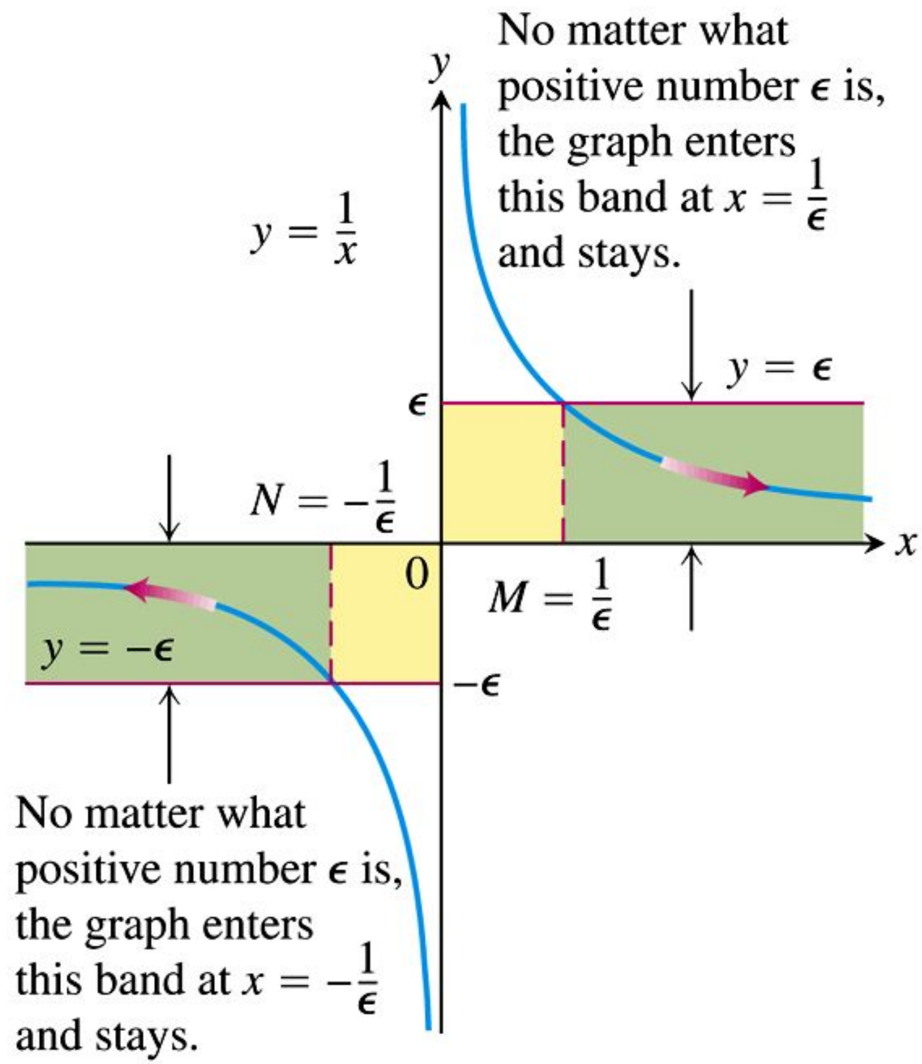


FIGURE 2.32 The geometry behind the argument in Example 6.

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \qquad \text{and} \qquad \lim_{x \to \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

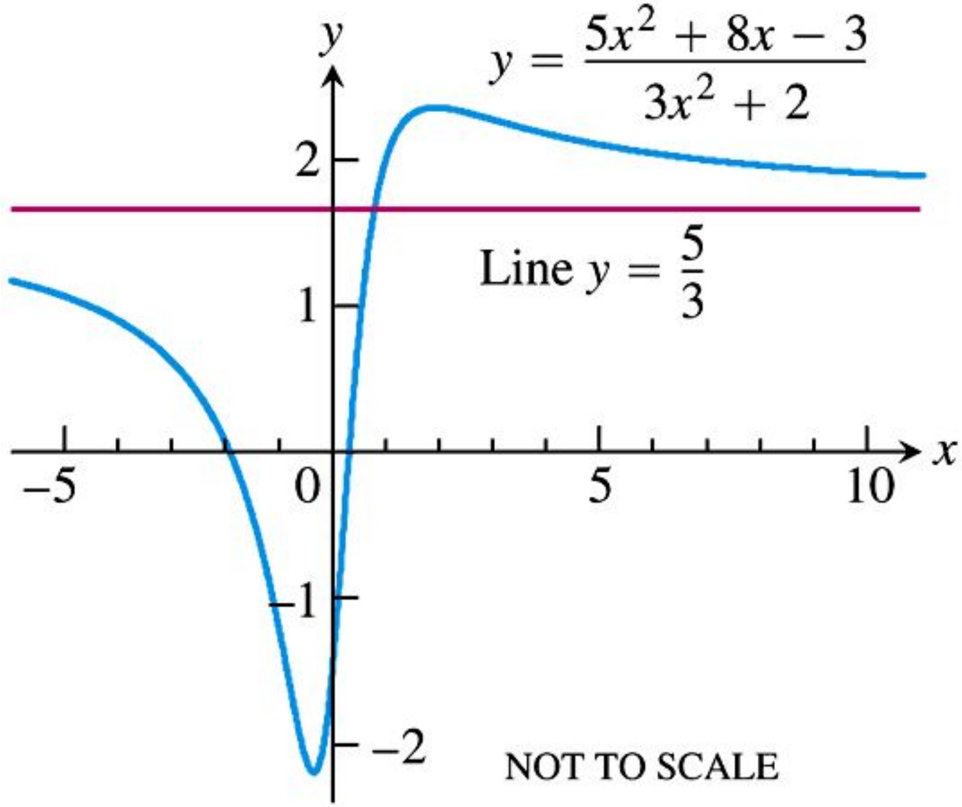


FIGURE 2.33 The graph of the function in Example 8. The graph approaches the line $y = 5/3$ as $|x|$ increases.

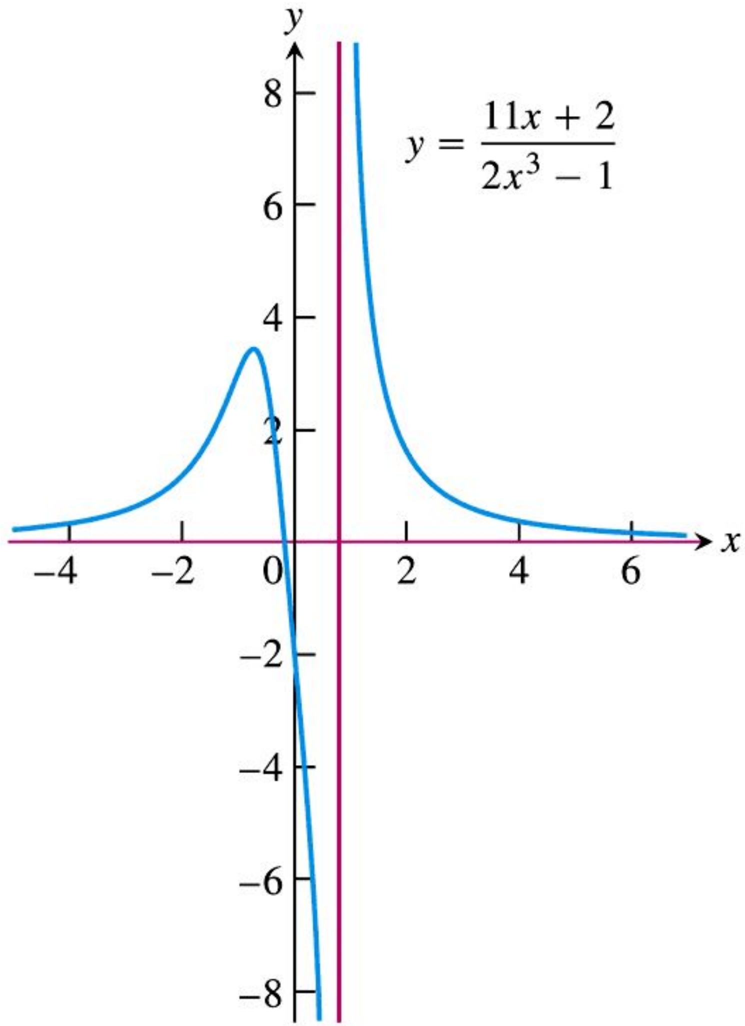


FIGURE 2.34 The graph of the function in Example 9. The graph approaches the x-axis as $|x|$ increases.

DEFINITION Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

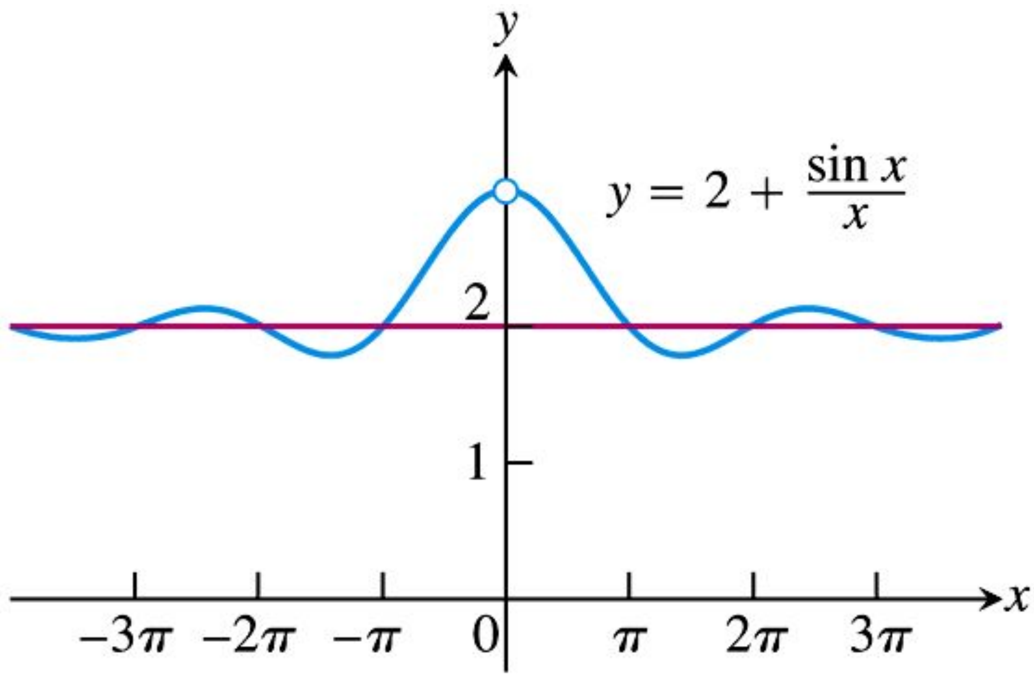


FIGURE 2.35 A curve may cross one of its asymptotes infinitely often (Example 11).

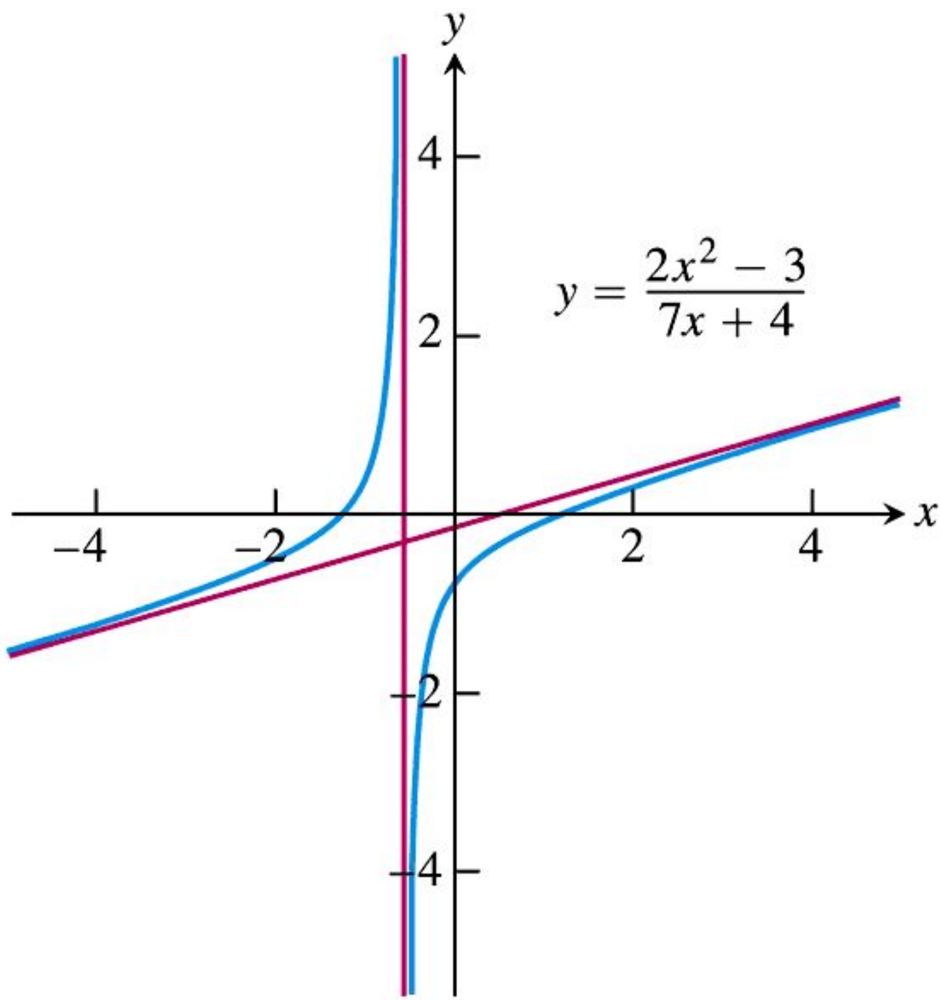


FIGURE 2.36 The function in Example 12 has an oblique asymptote.

2.5

Infinite Limits and Vertical Asymptotes

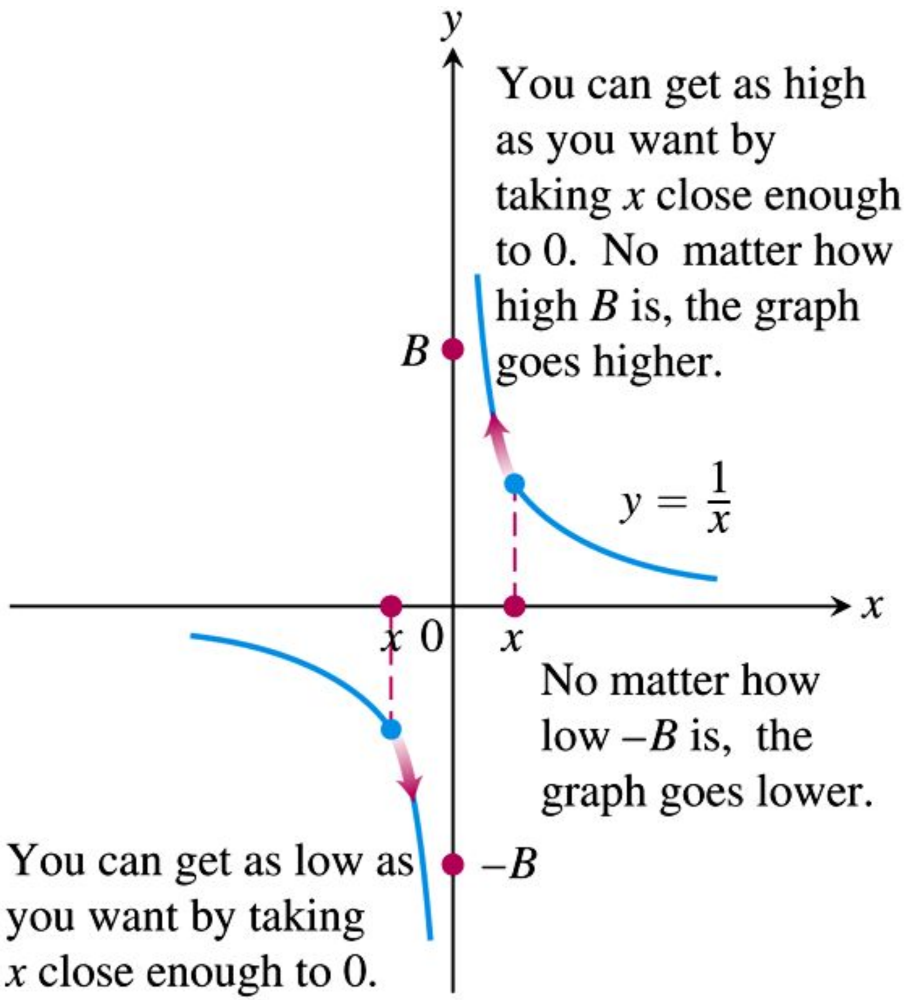


FIGURE 2.37 One-sided infinite limits:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

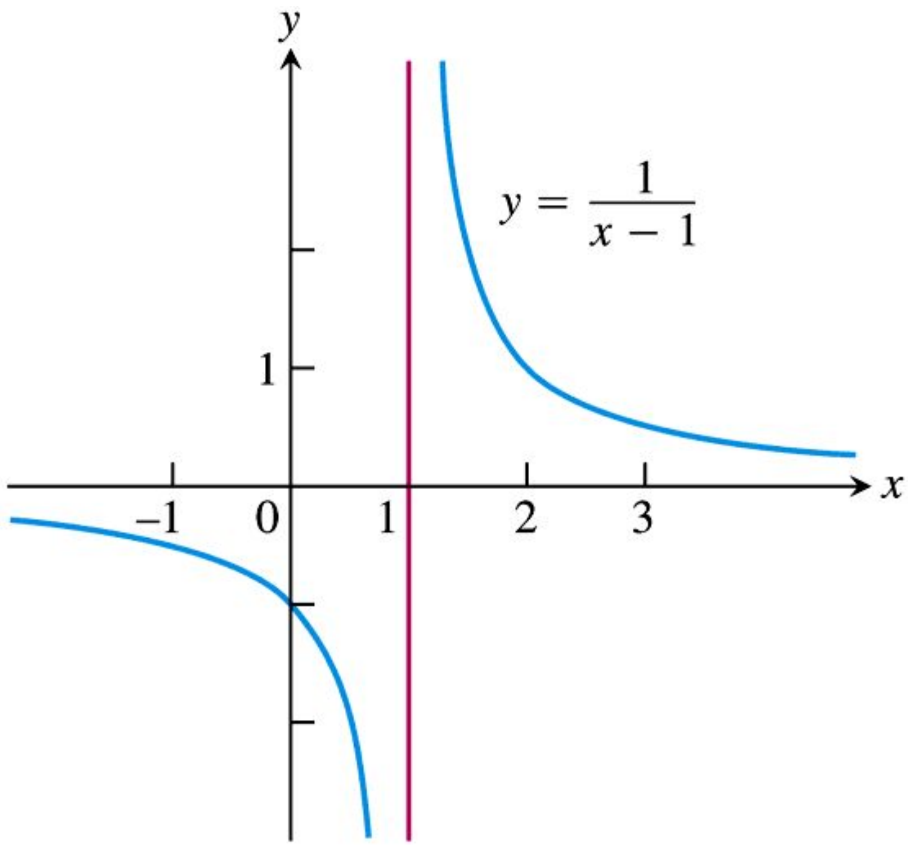
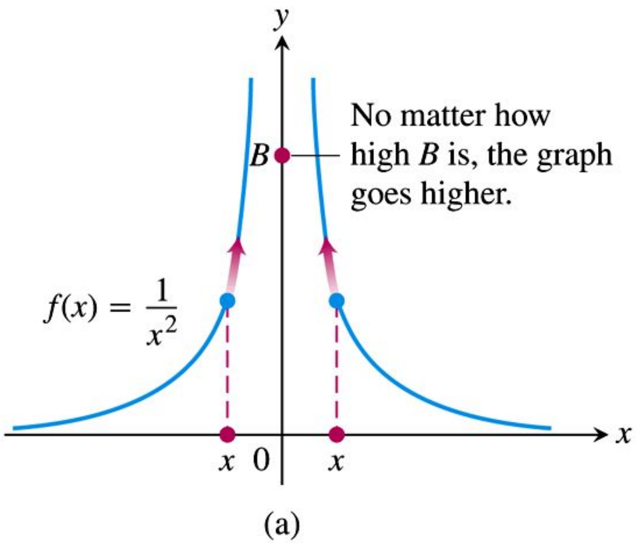


FIGURE 2.38 Near $x = 1$, the function $y = 1/(x-1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 1).



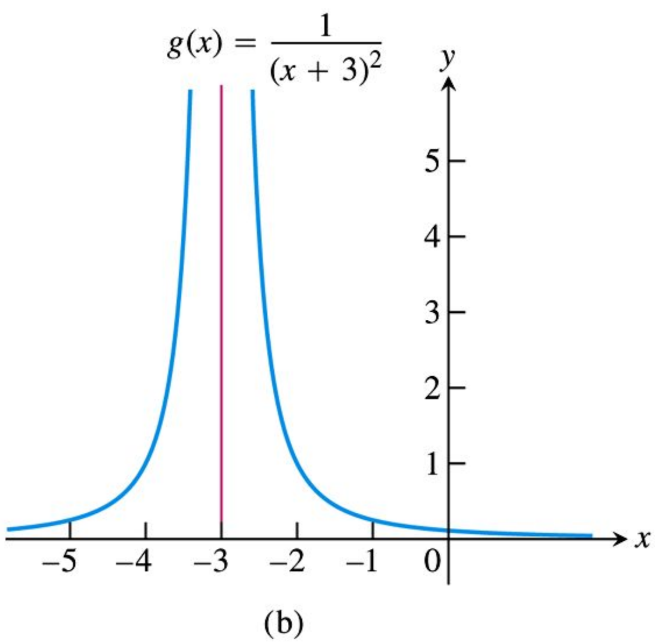


FIGURE 2.39 The graphs of the functions in Example 2. (a) $f(x)$ approaches infinity as $x \to 0$. (b) $g(x)$ approaches infinity as $x \to -3$.

DEFINITIONS **Infinity, Negative Infinity as Limits**

1. We say that **$f(x)$ approaches infinity as x approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that **$f(x)$ approaches negative infinity as x approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B.$$

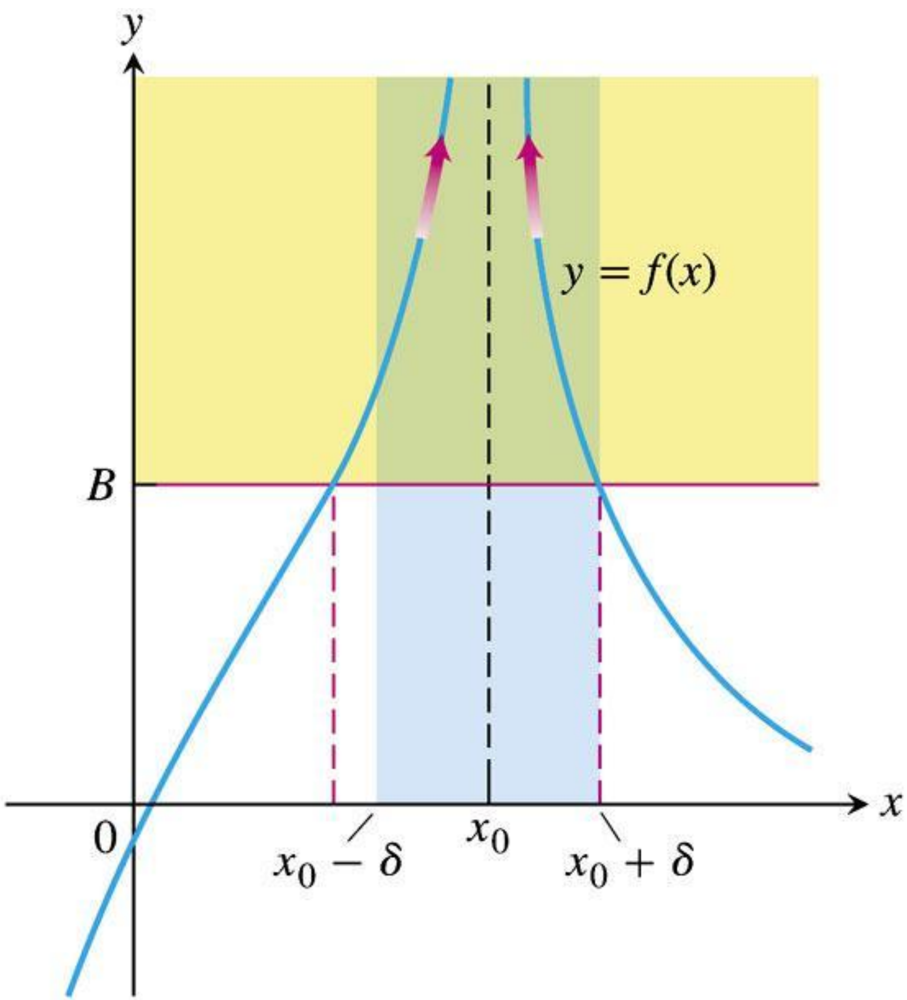


FIGURE 2.40 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies above the line $y = B$.

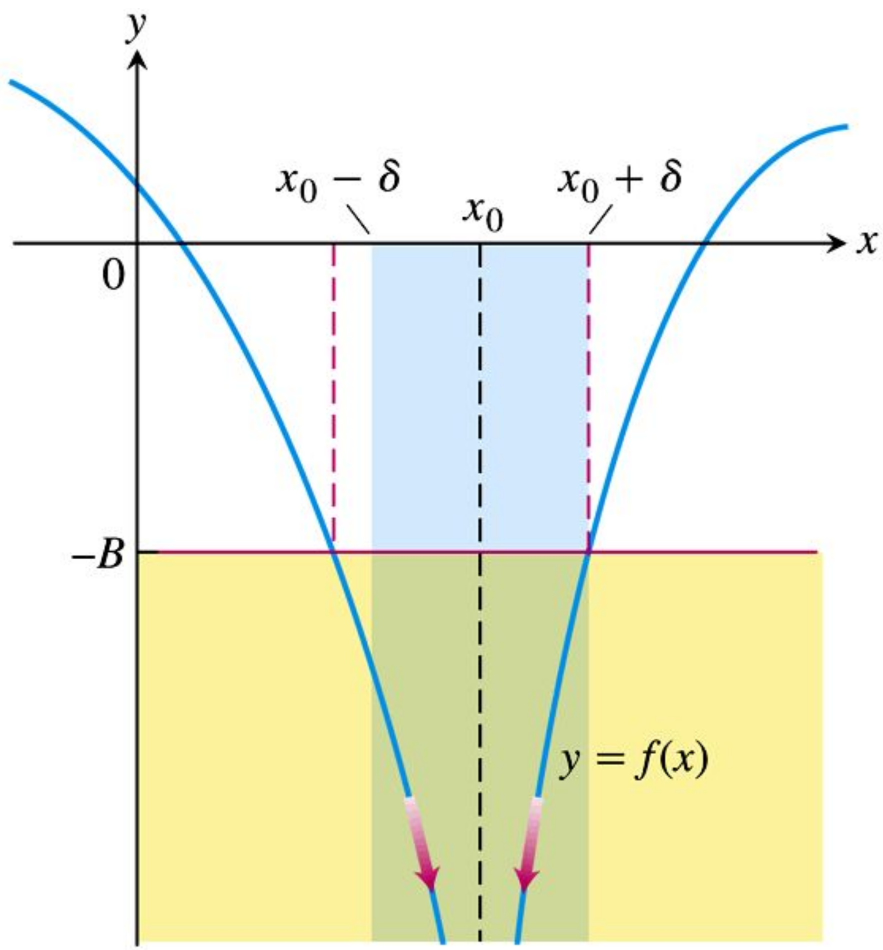


FIGURE 2.41 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

DEFINITION **Vertical Asymptote**

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

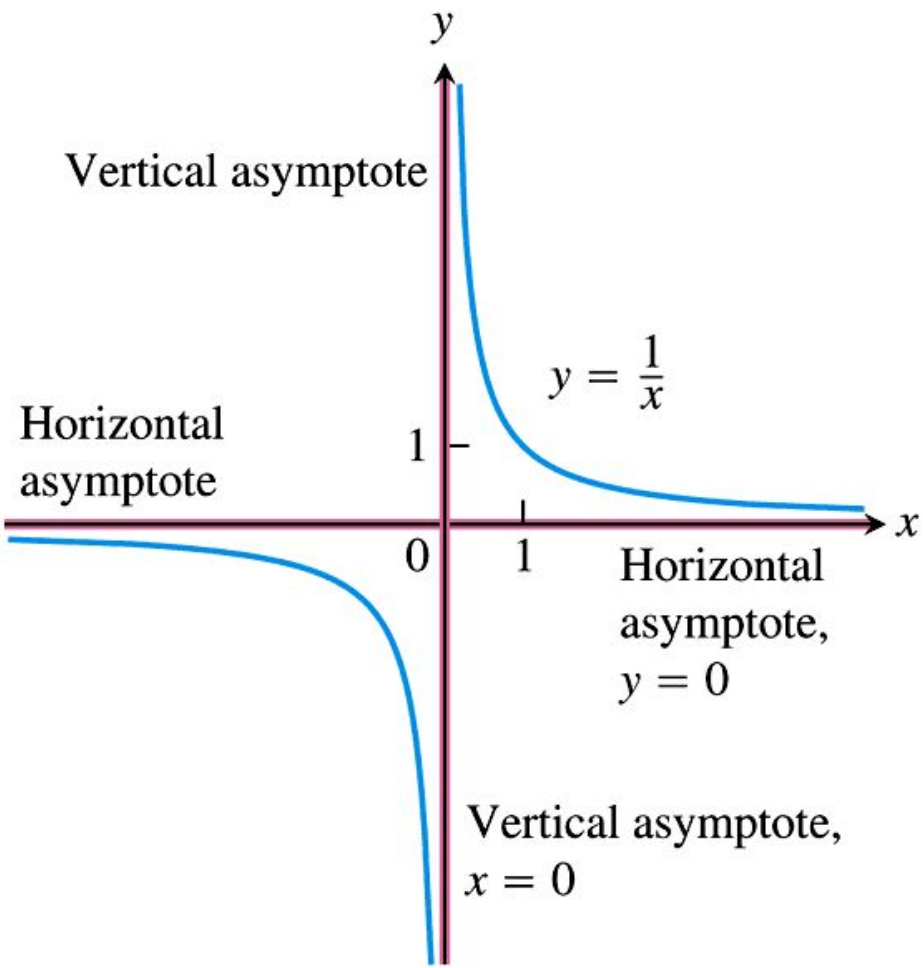


FIGURE 2.42 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

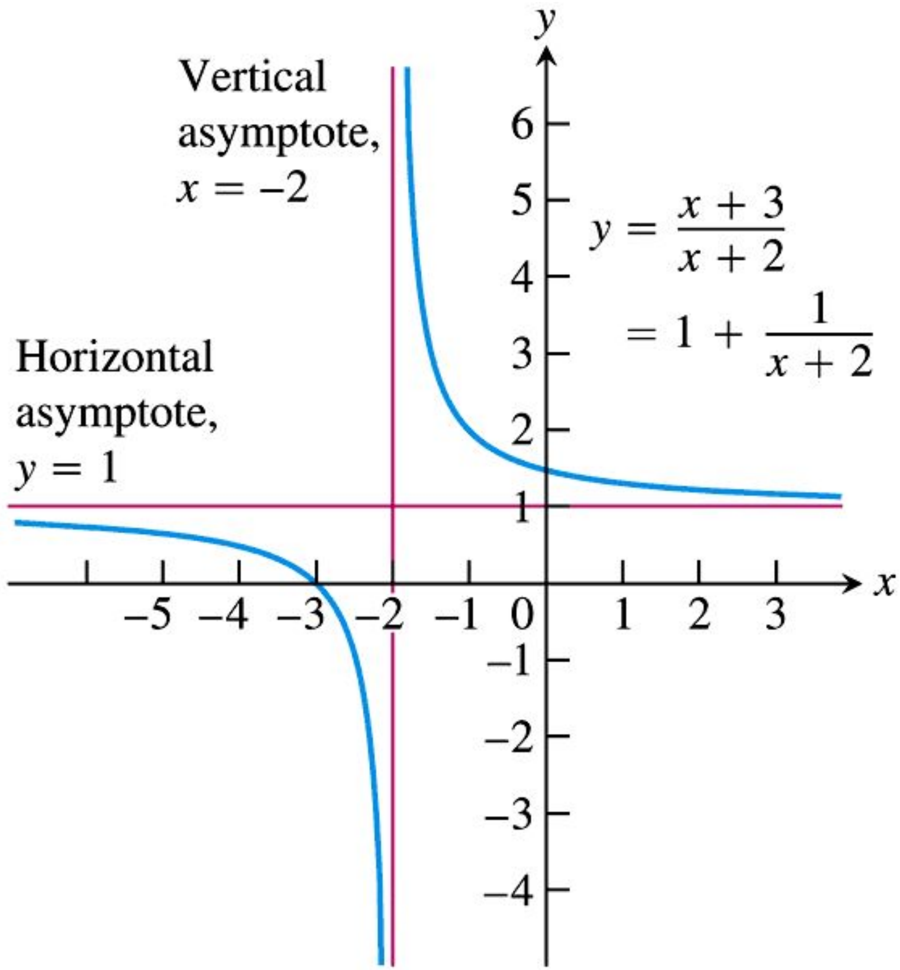


FIGURE 2.43 The lines $y = 1$ and $x = -2$ are asymptotes of the curve $y = (x + 3)/(x + 2)$ (Example 5).

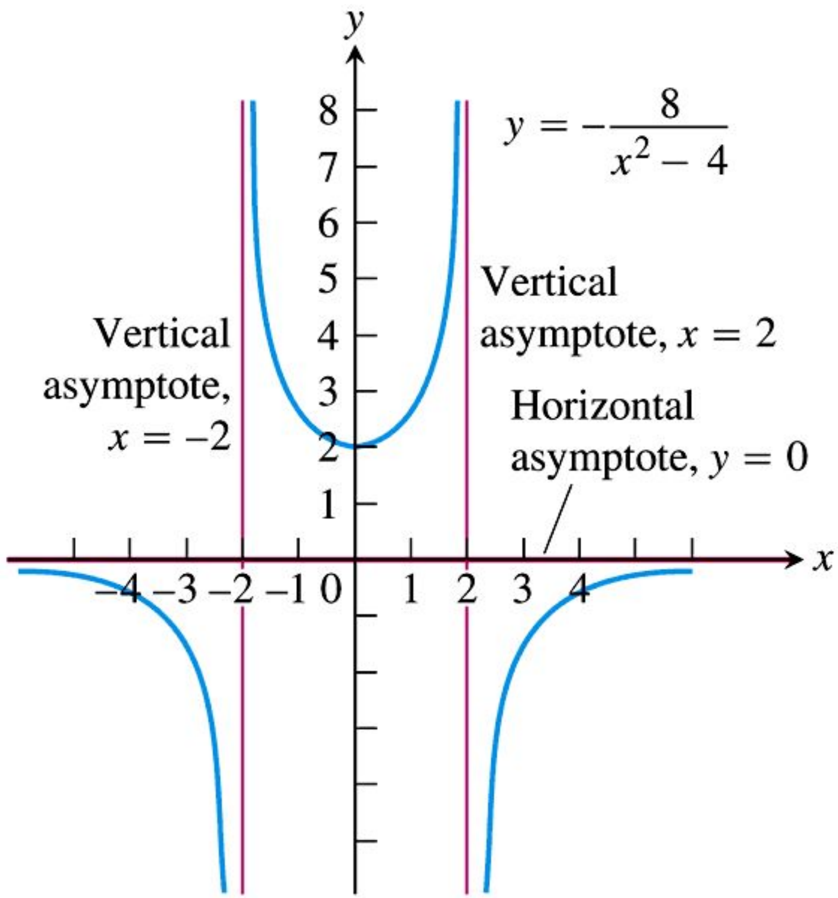


FIGURE 2.44 Graph of $y = -8/(x^2 - 4)$. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided (Example 6).

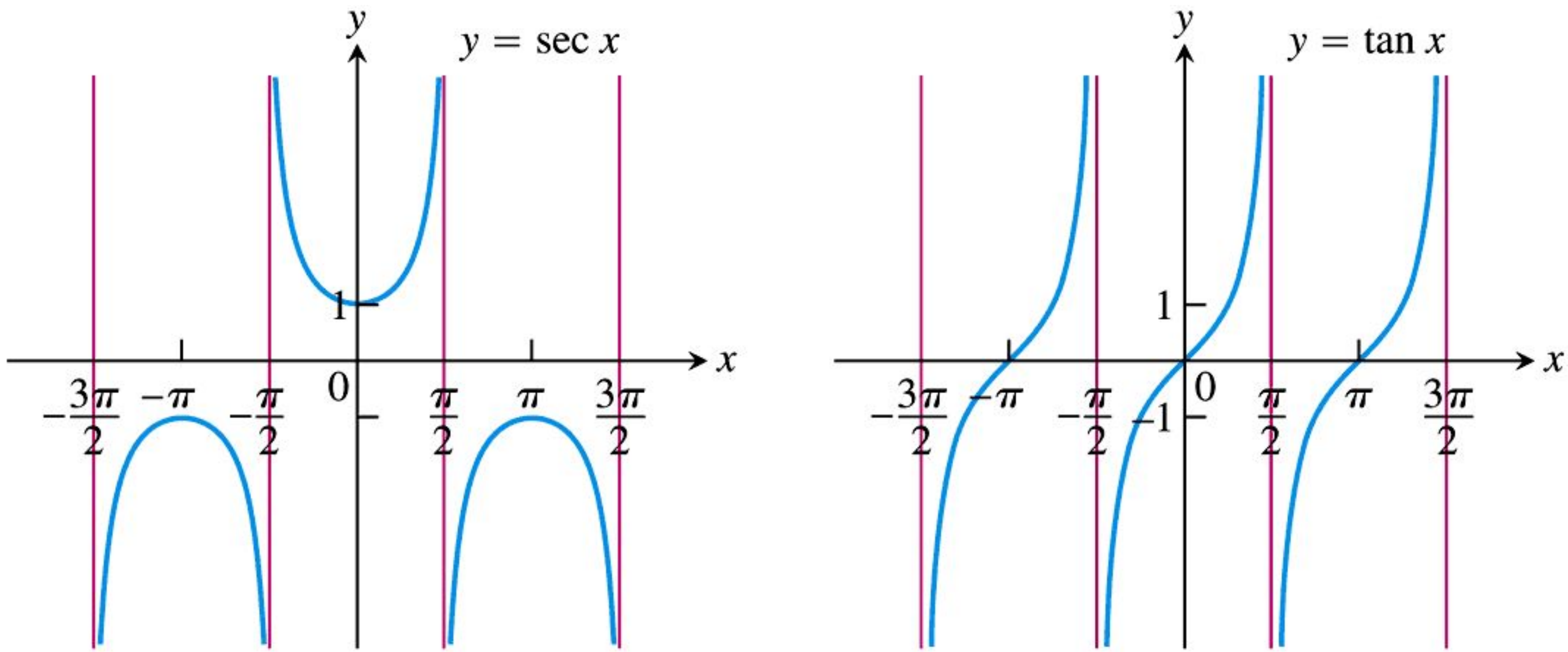


FIGURE 2.45 The graphs of sec x and tan x have infinitely many vertical asymptotes (Example 7).

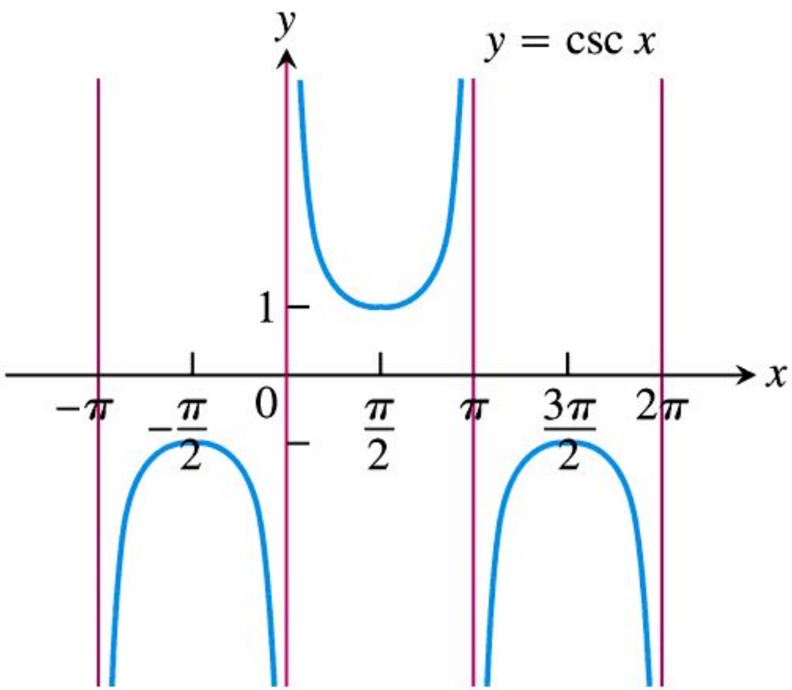


FIGURE 2.46 The graphs of csc x and cot x (Example 7).

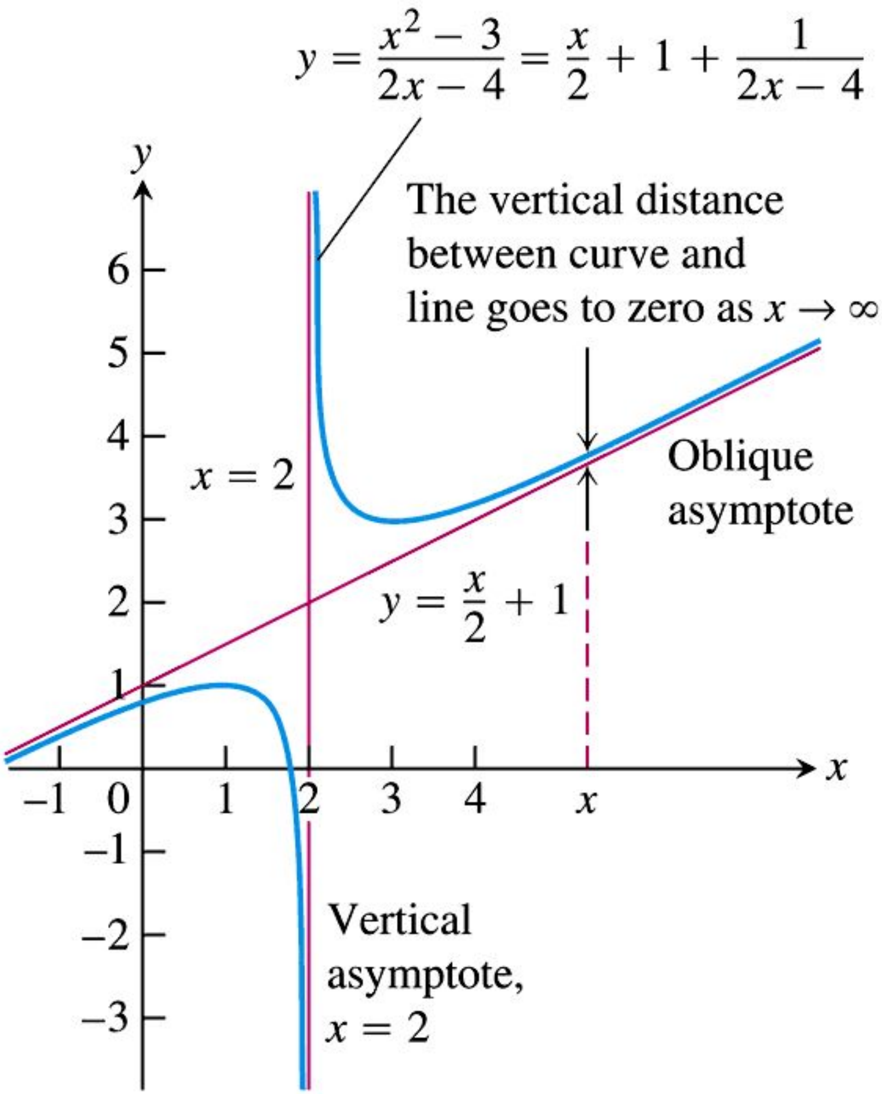


FIGURE 2.47 The graph of $f(x) = (x^2 - 3)/(2x - 4)$ has a vertical asymptote and an oblique asymptote (Example 8).

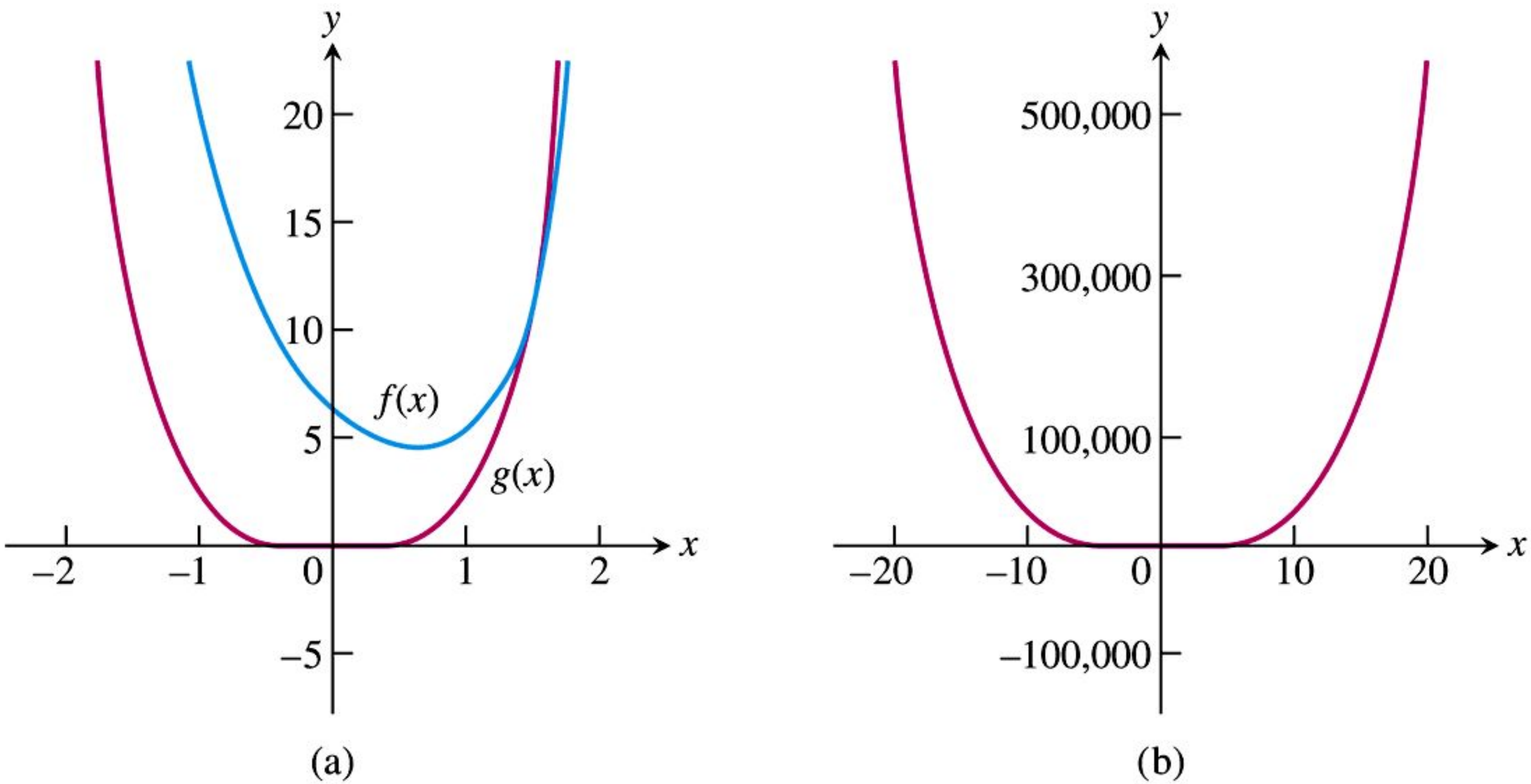


FIGURE 2.48 The graphs of f and g, (a) are distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 9).

2.6

Continuity

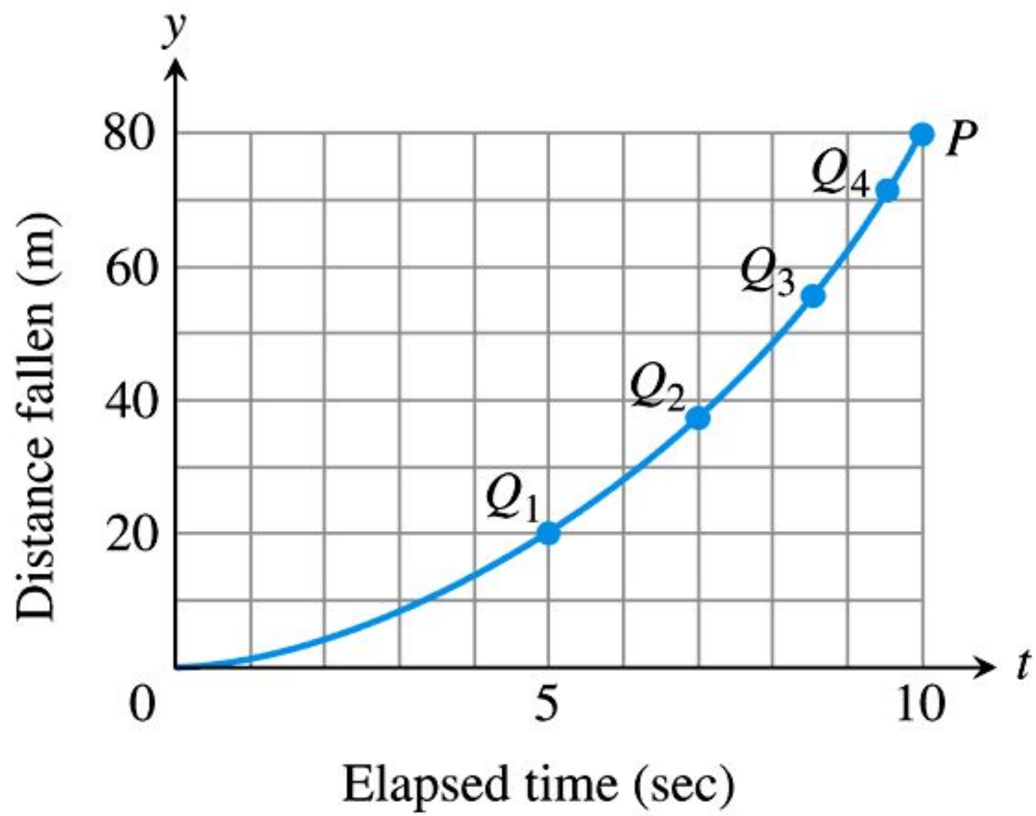


FIGURE 2.49 Connecting plotted points by an unbroken curve from experimental data $Q_1, Q_2, Q_3, \ldots$ for a falling object.

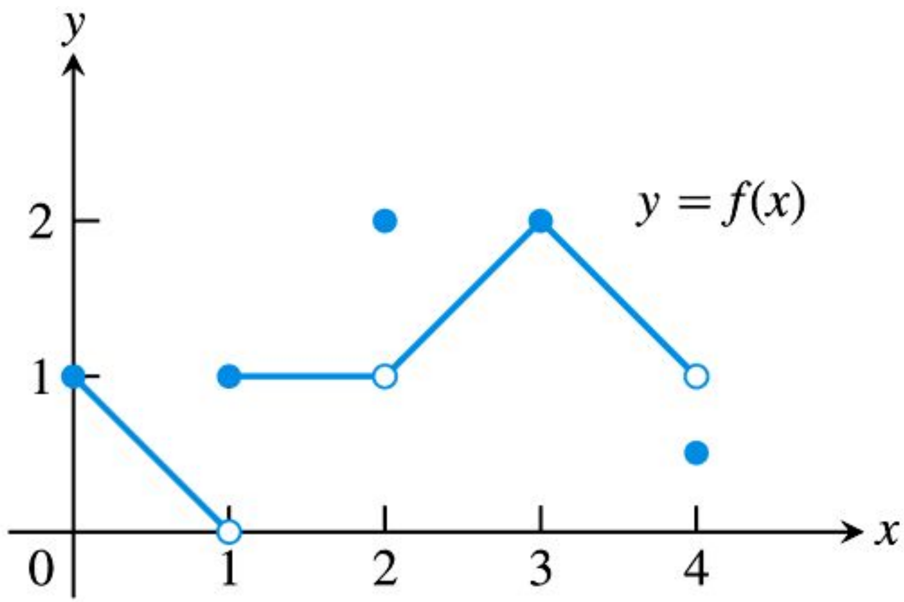


FIGURE 2.50 The function is continuous on [0, 4] except at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

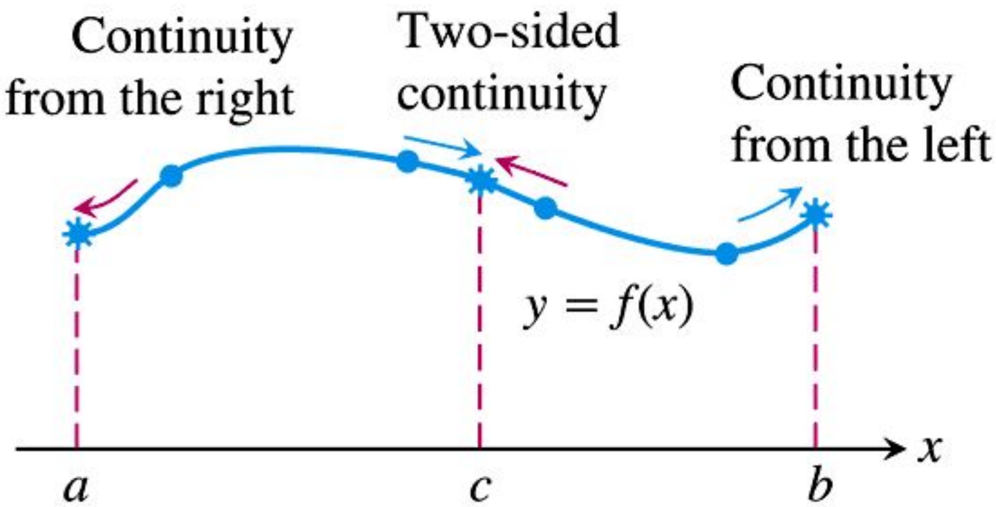


FIGURE 2.51 Continuity at points a, b, and c.

DEFINITION **Continuous at a Point**

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

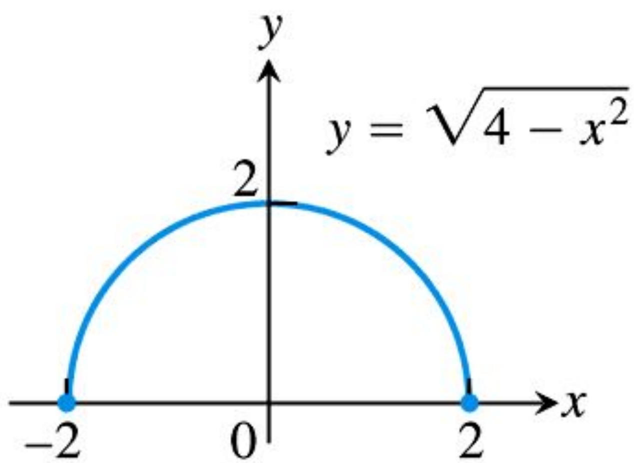


FIGURE 2.52 A function that is continuous at every domain point (Example 2).

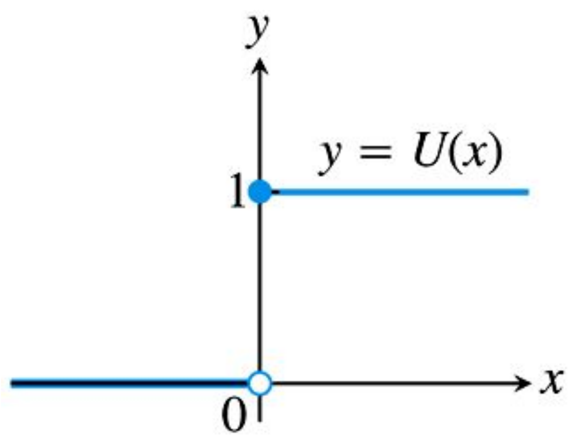


FIGURE 2.53 A function that is right-continuous, but not left-continuous, at the origin. It has a jump discontinuity there (Example 3).

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1.	$f(c)$ exists	(c lies in the domain of f)
2.	$\lim_{x \to c} f(x)$ exists	(f has a limit as $x \to c$)
3.	$\lim_{x \to c} f(x) = f(c)$	(the limit equals the function value)

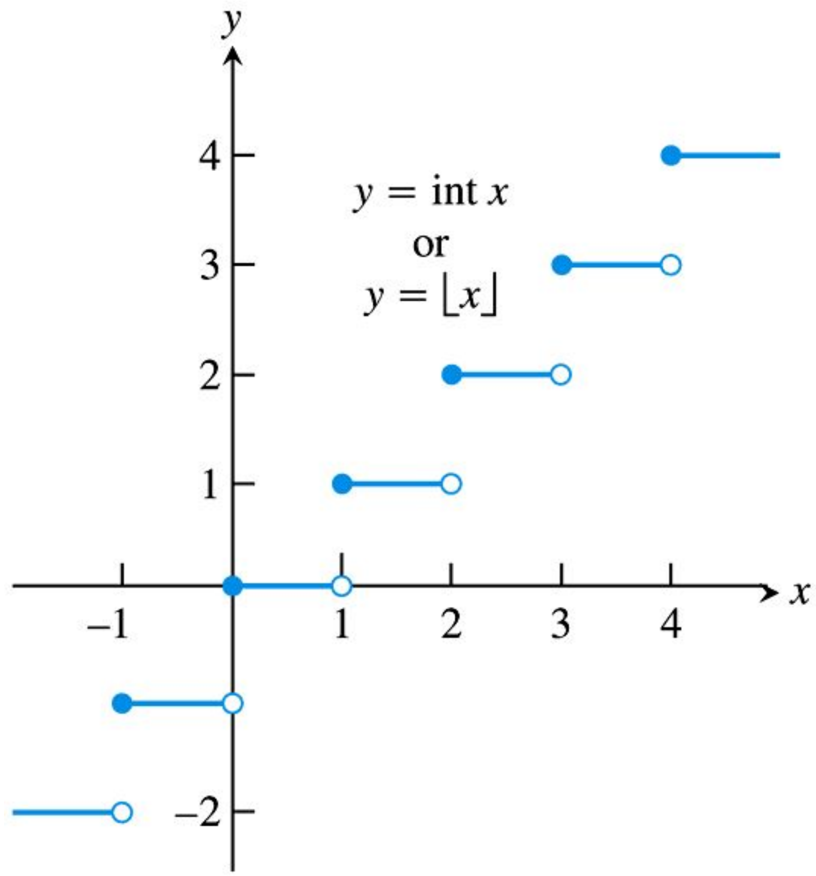


FIGURE 2.54 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

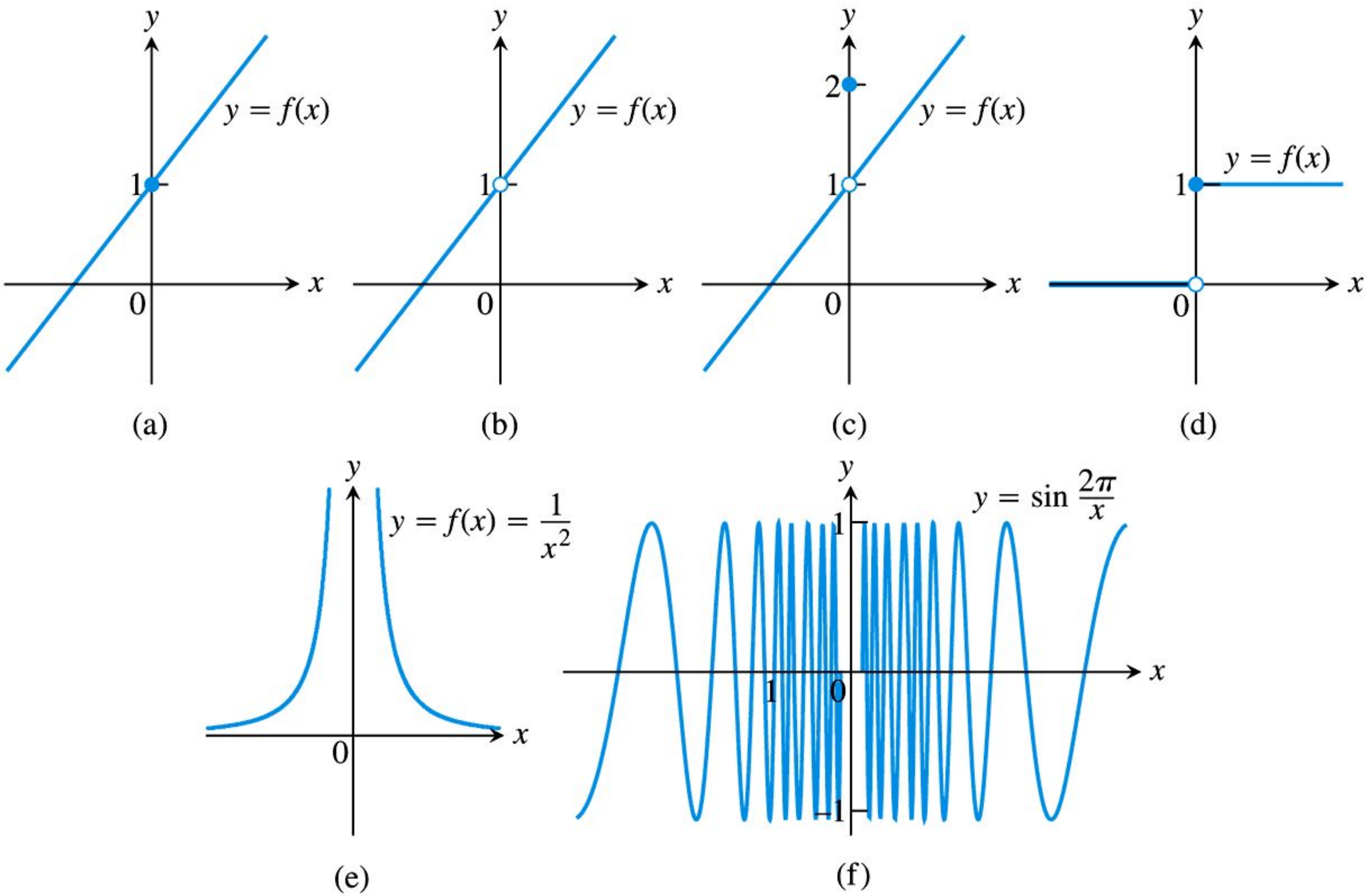


FIGURE 2.55 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

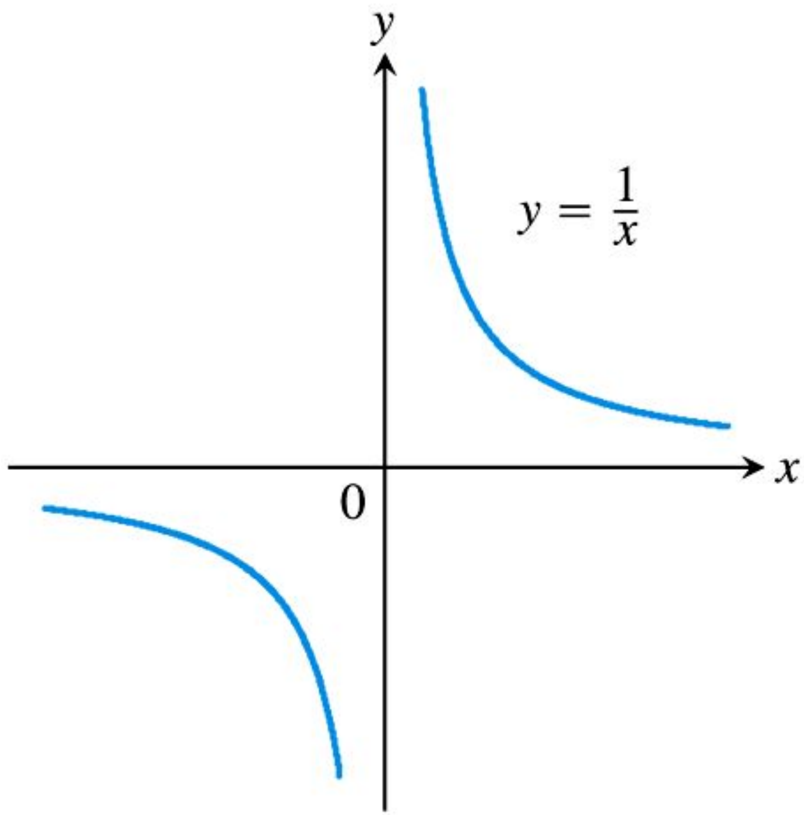


FIGURE 2.56 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$ (Example 5).

THEOREM 9 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

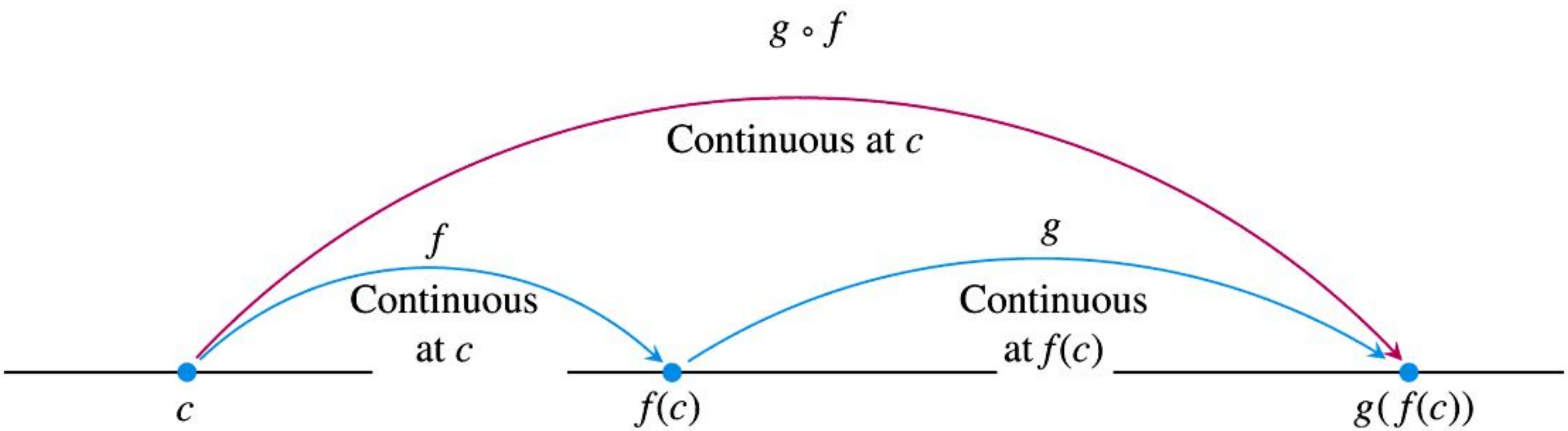


FIGURE 2.57 Composites of continuous functions are continuous.

THEOREM 10 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

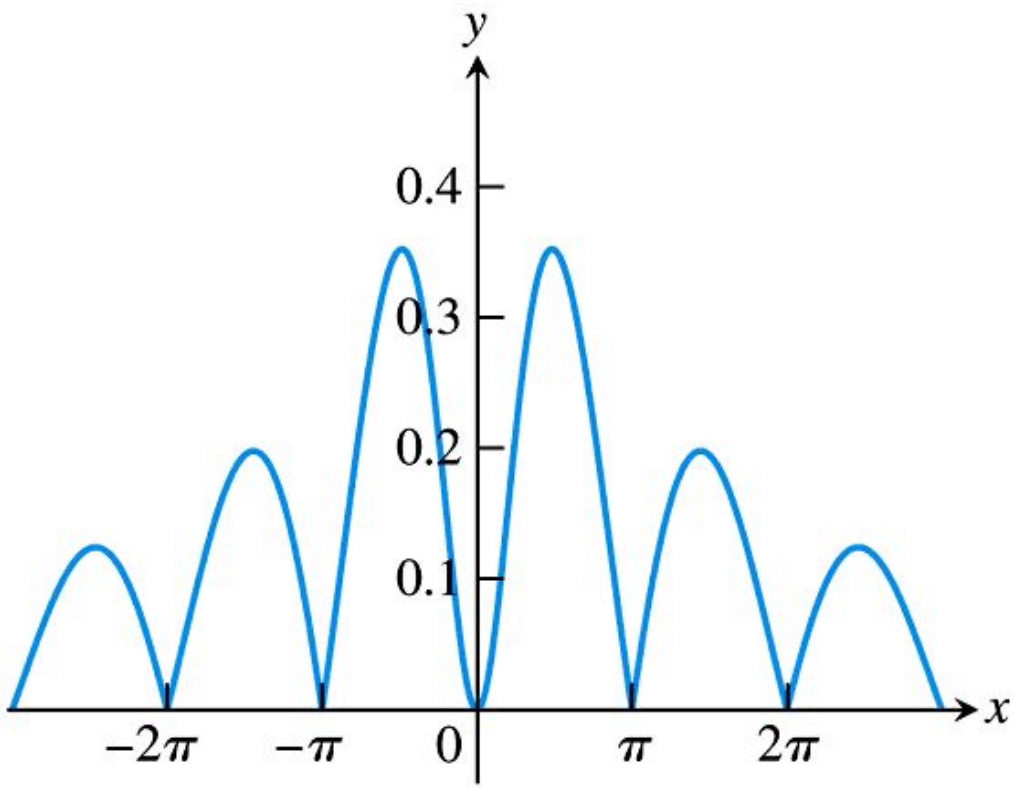


FIGURE 2.58 The graph suggests that $y = \left|(x \sin x)/(x^2 + 2)\right|$ is continuous (Example 8d).

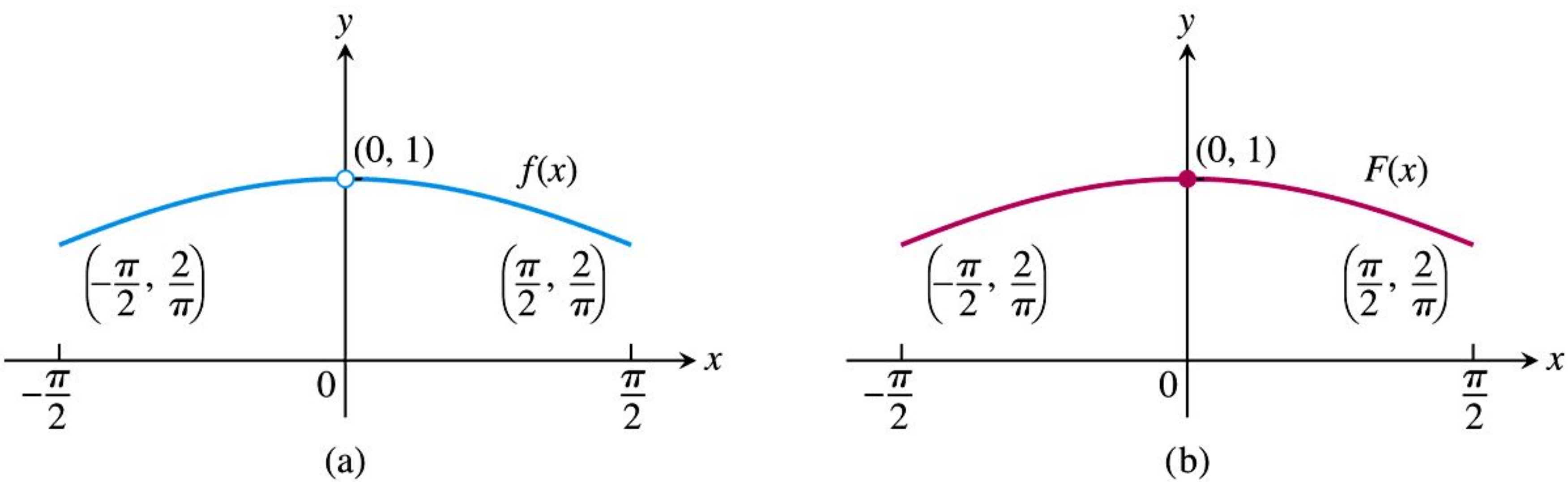
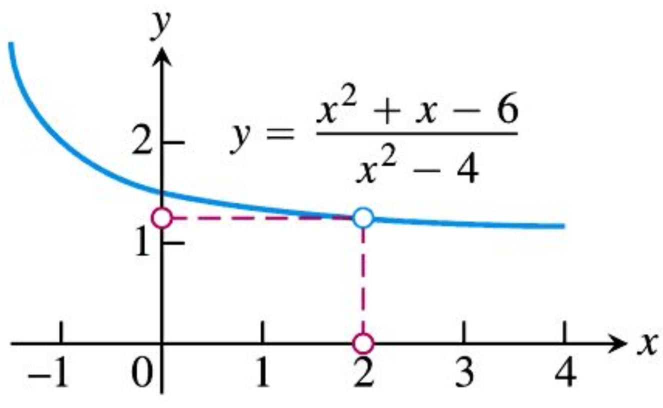


FIGURE 2.59 The graph (a) of $f(x) = (\sin x)/x$ for $-\pi/2 \leq x \leq \pi/2$ does not include the point (0, 1) because the function is not defined at $x = 0$. (b) We can remove the discontinuity from the graph by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \to 0} f(x)$.



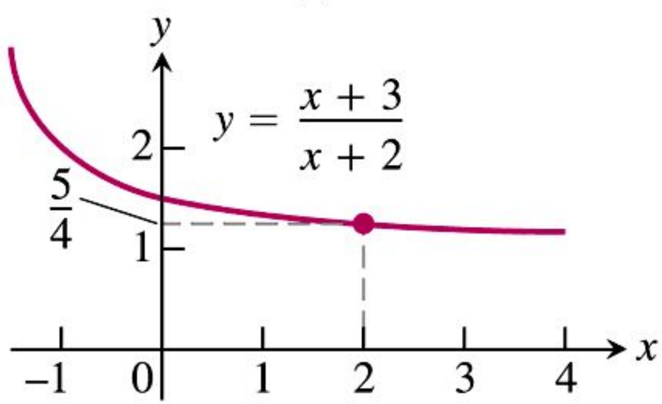
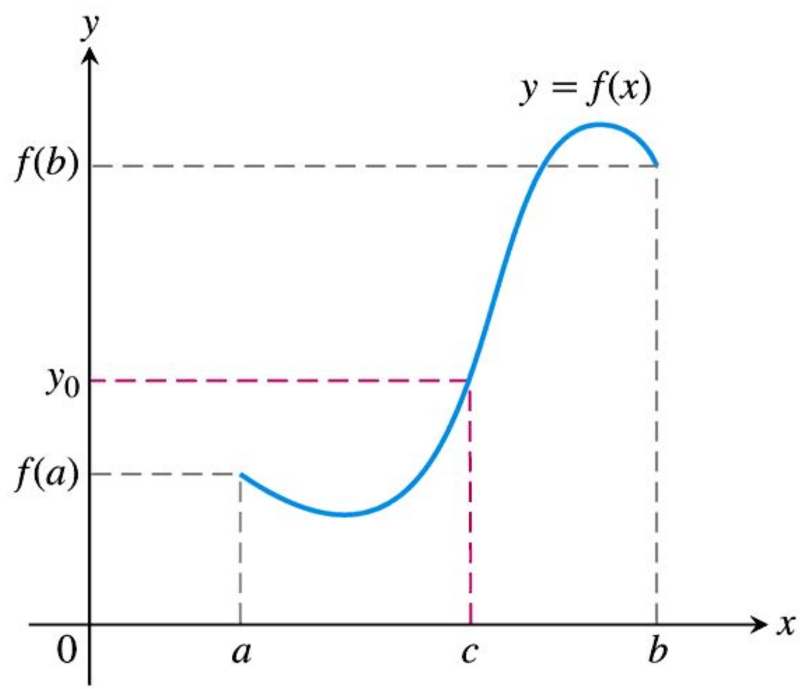


FIGURE 2.60 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 9).

THEOREM 11 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



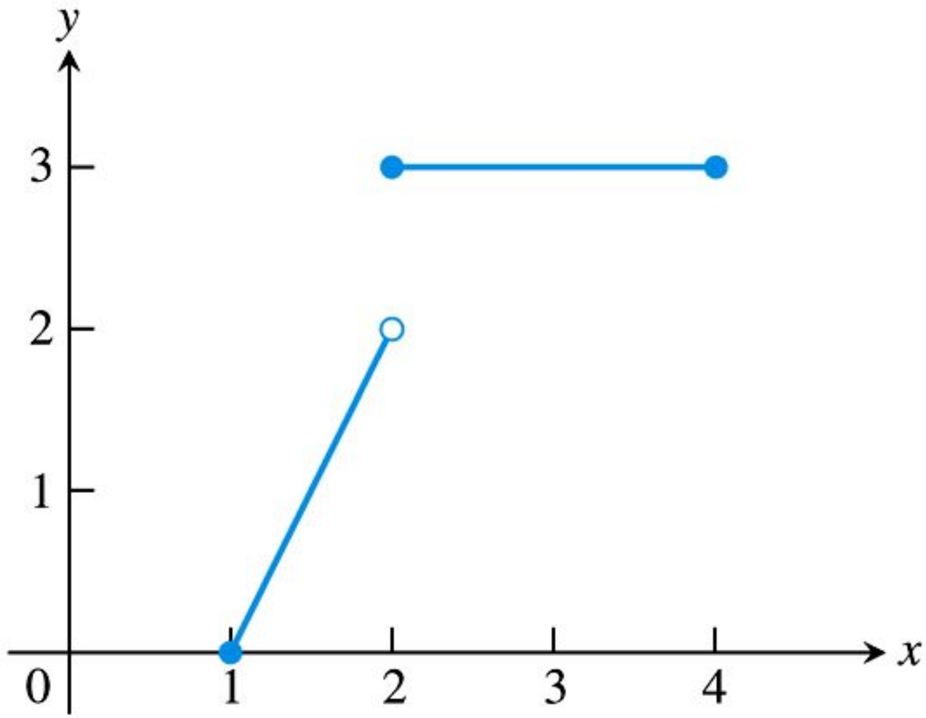


FIGURE 2.61 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

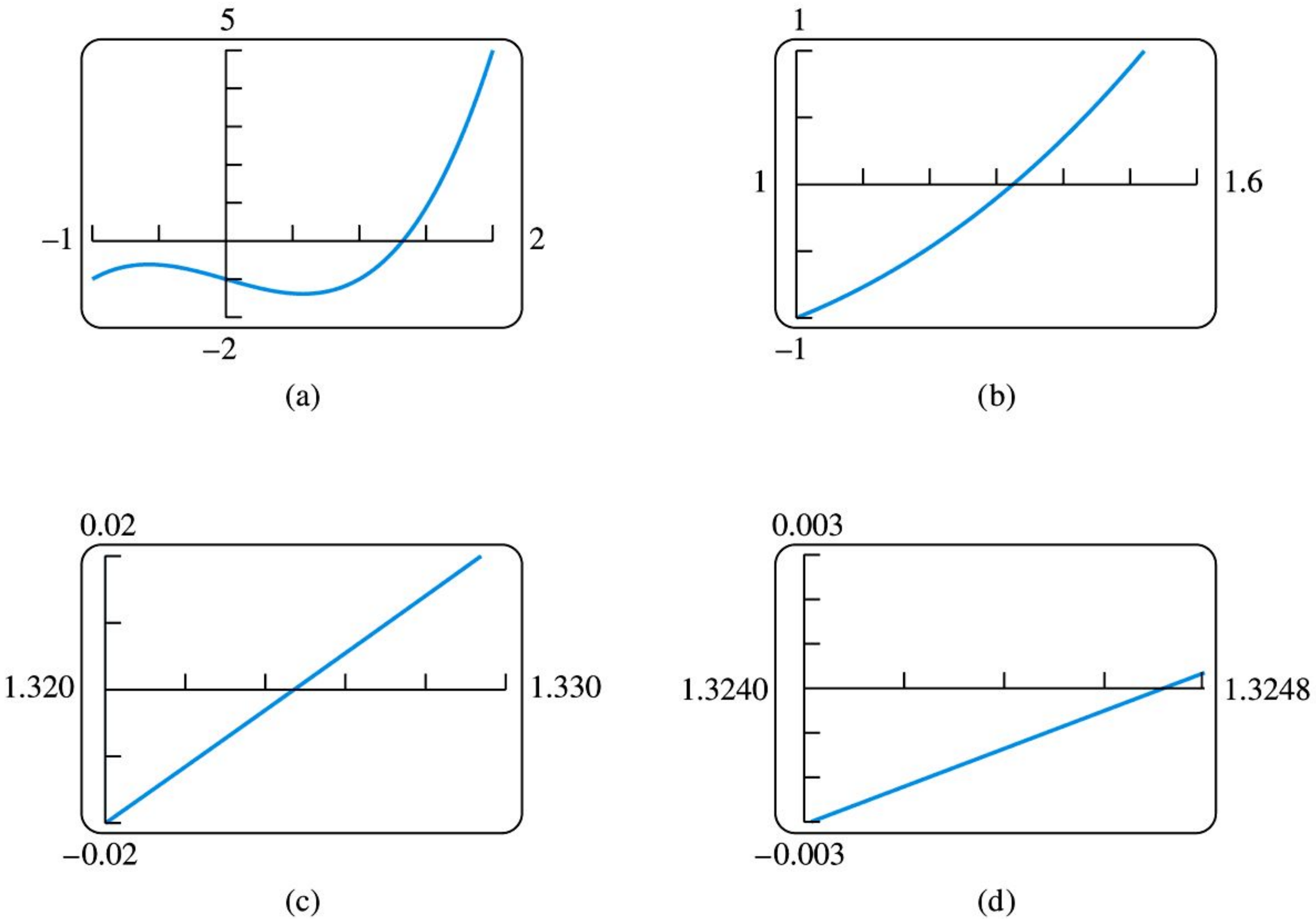


FIGURE 2.62 Zooming in on a zero of the function $f(x) = x^3 - x - 1$. The zero is near $x = 1.3247$.

2.7

Tangents and Derivatives

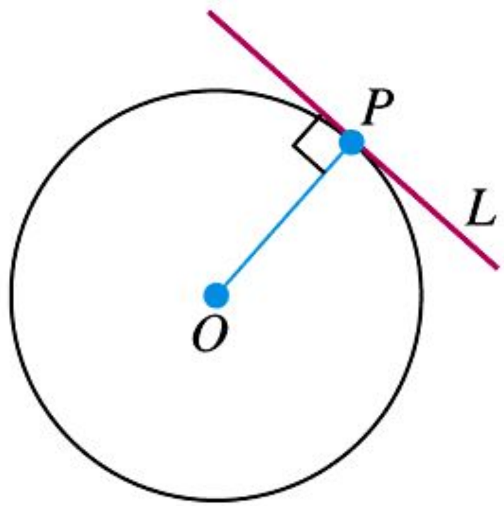
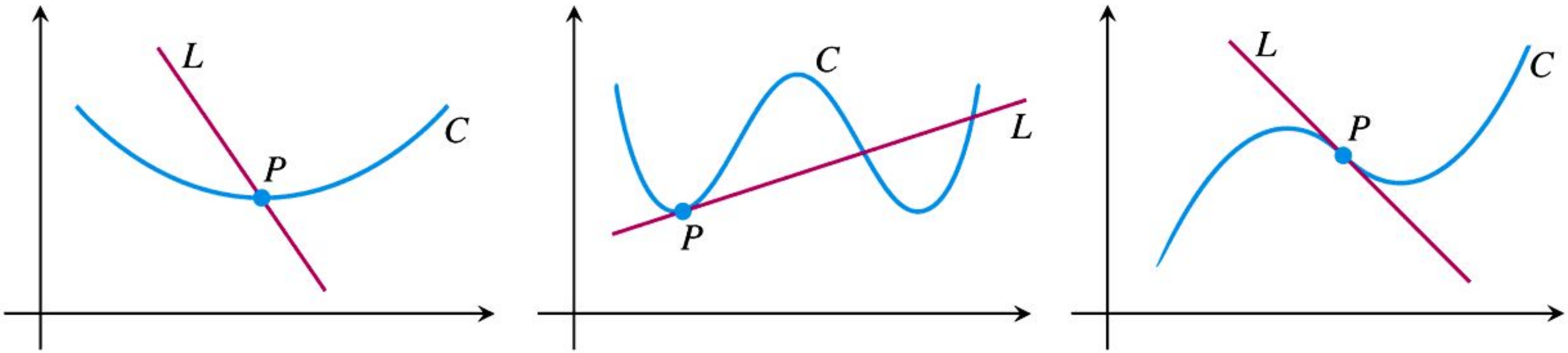


FIGURE 2.63 L is tangent to the circle at P if it passes through P perpendicular to radius OP.



L meets *C* only at *P* but is not tangent to *C*.

L is tangent to *C* at *P* but meets *C* at several points.

L is tangent to *C* at *P* but lies on two sides of *C*, crossing *C* at *P*.

FIGURE 2.64 Exploding myths about tangent lines.

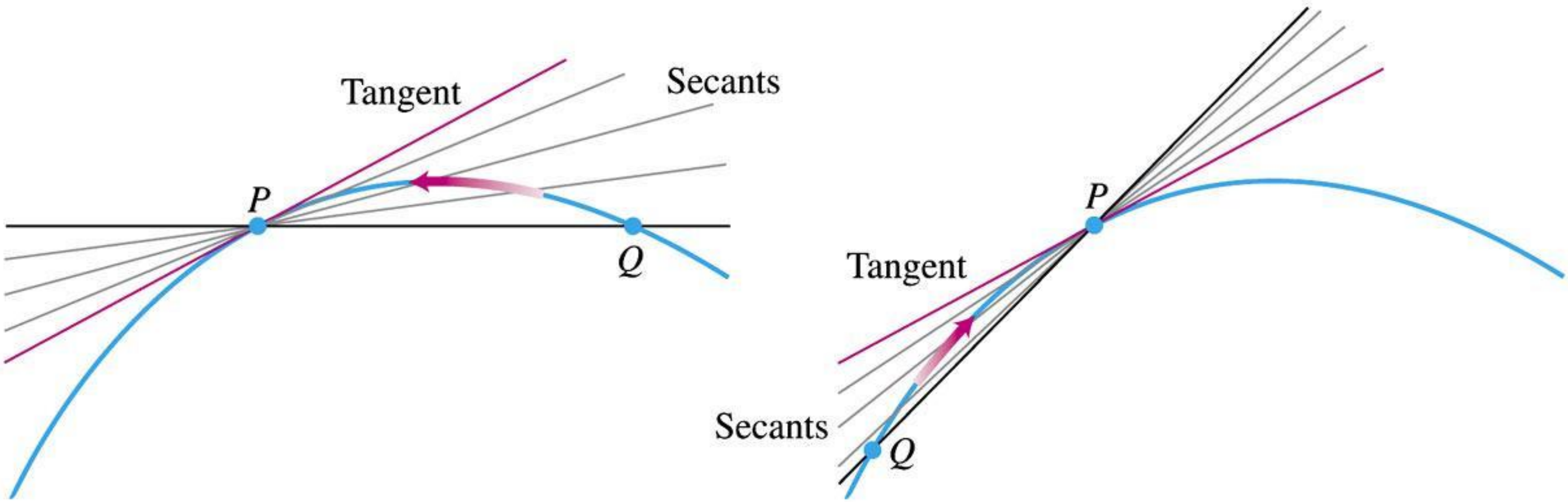


FIGURE 2.65 The dynamic approach to tangency. The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

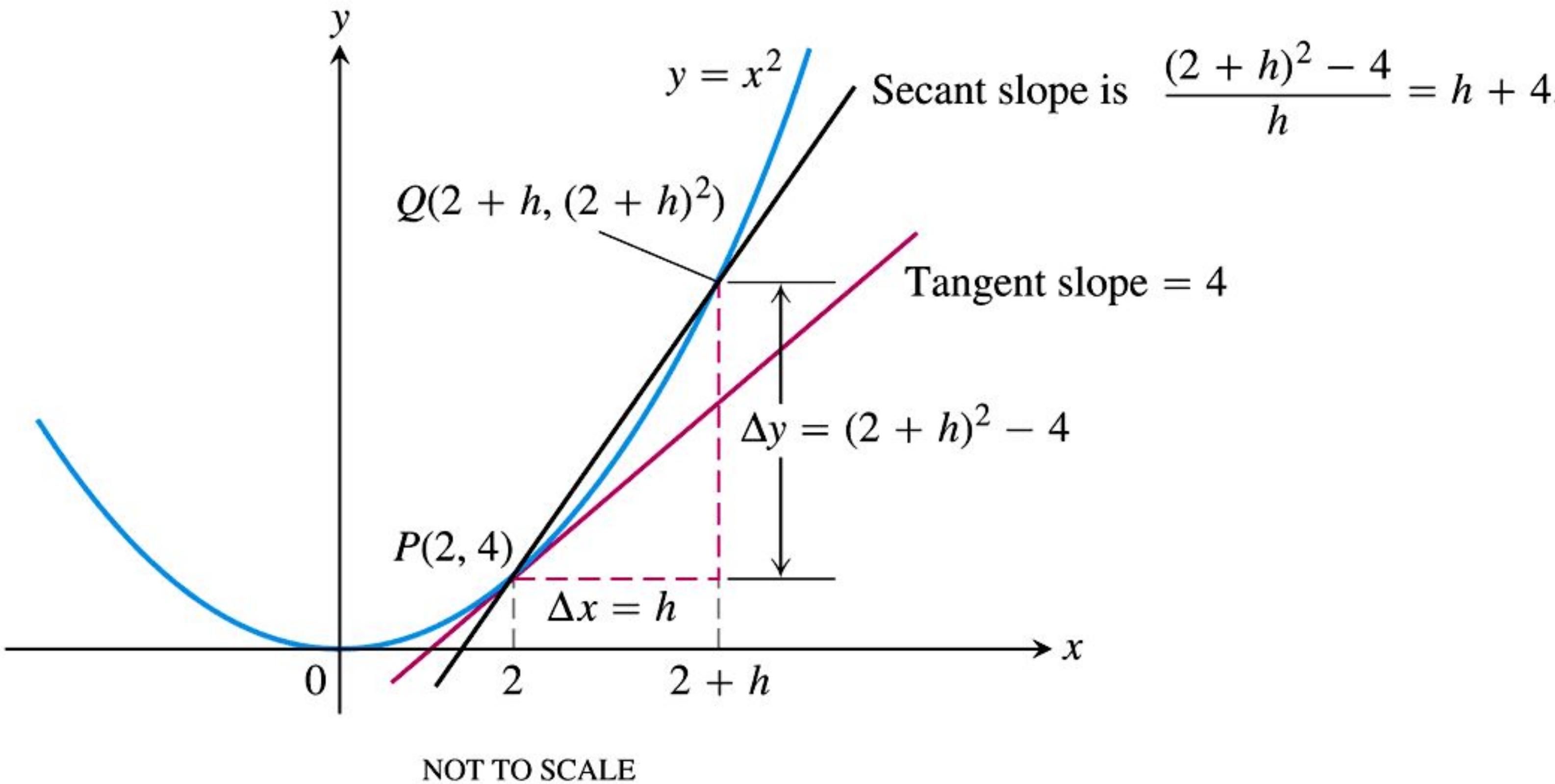


FIGURE 2.66 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ (Example 1).

DEFINITIONS Slope, Tangent Line

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

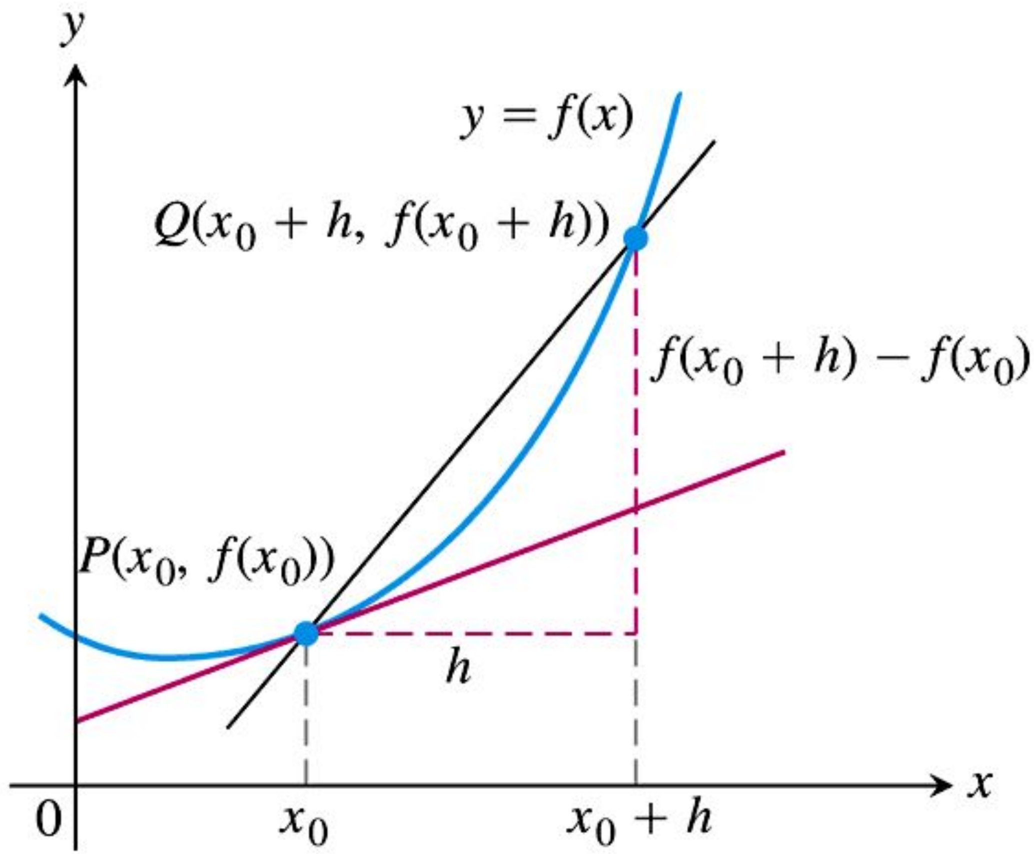


FIGURE 2.67 The slope of the tangent line at P is $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$.

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

3. If the limit exists, find the tangent line as

$$y = y_0 + m(x - x_0).$$

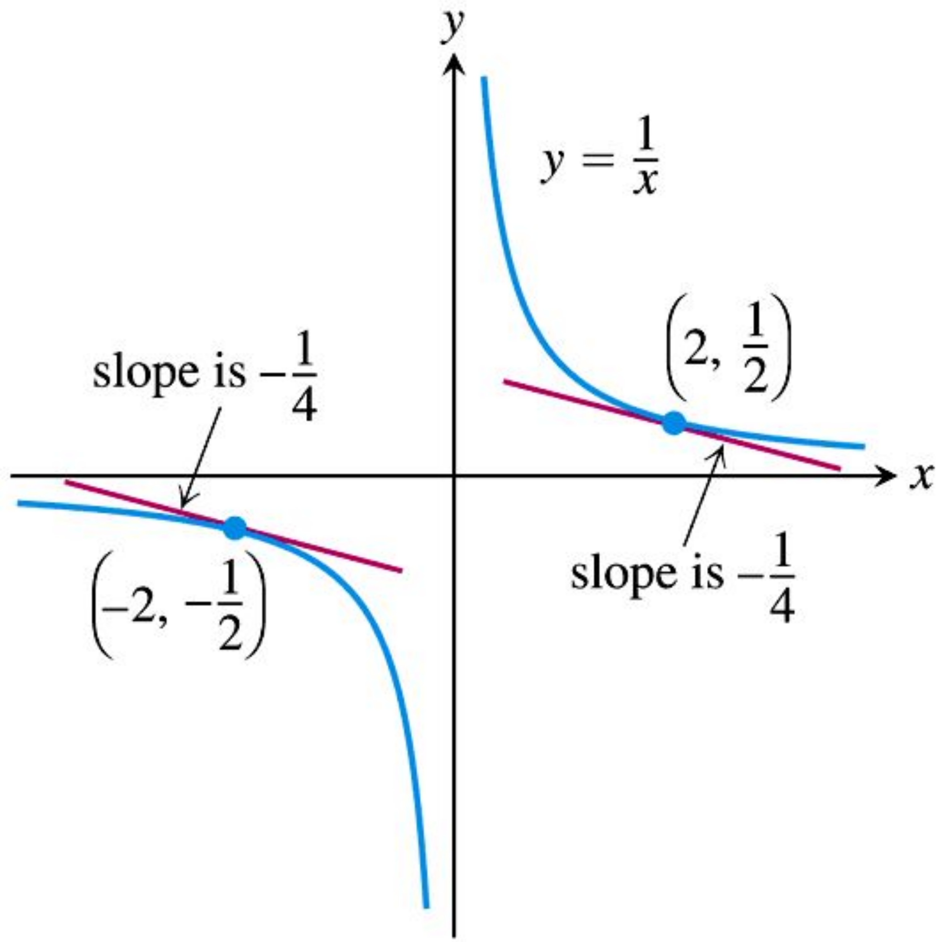


FIGURE 2.68 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 3).

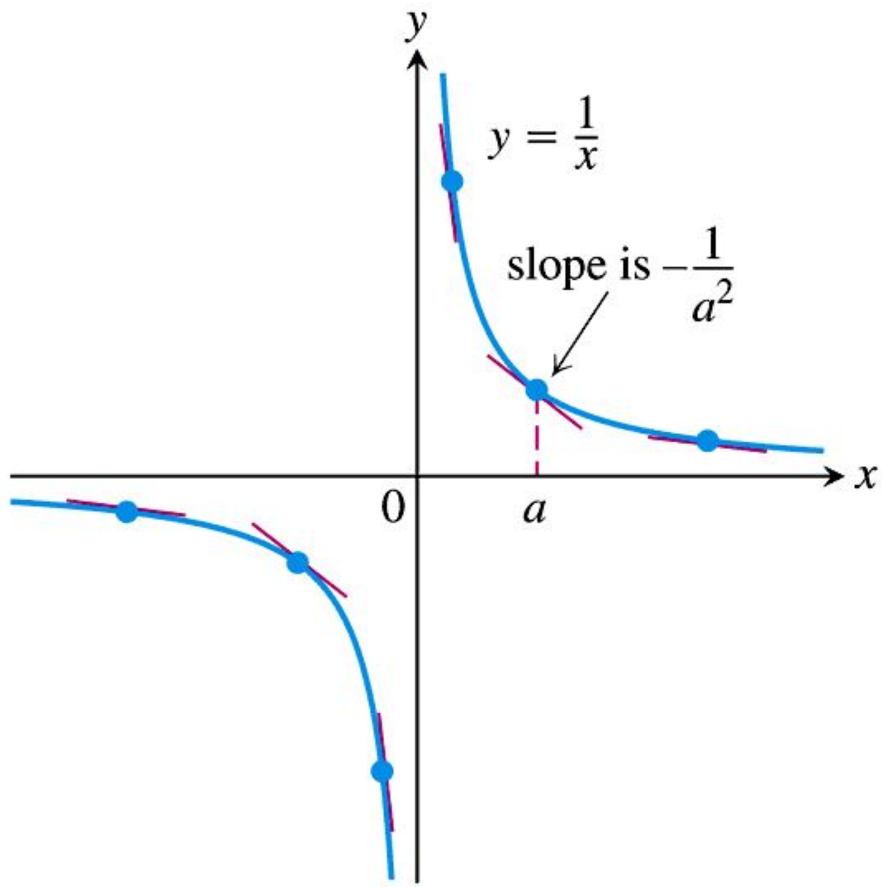


FIGURE 2.69 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away.

1. The slope of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative of f at $x = x_0$
5. The limit of the difference quotient, $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$