WEB GRAPHS

Modeling the Internet and the Web
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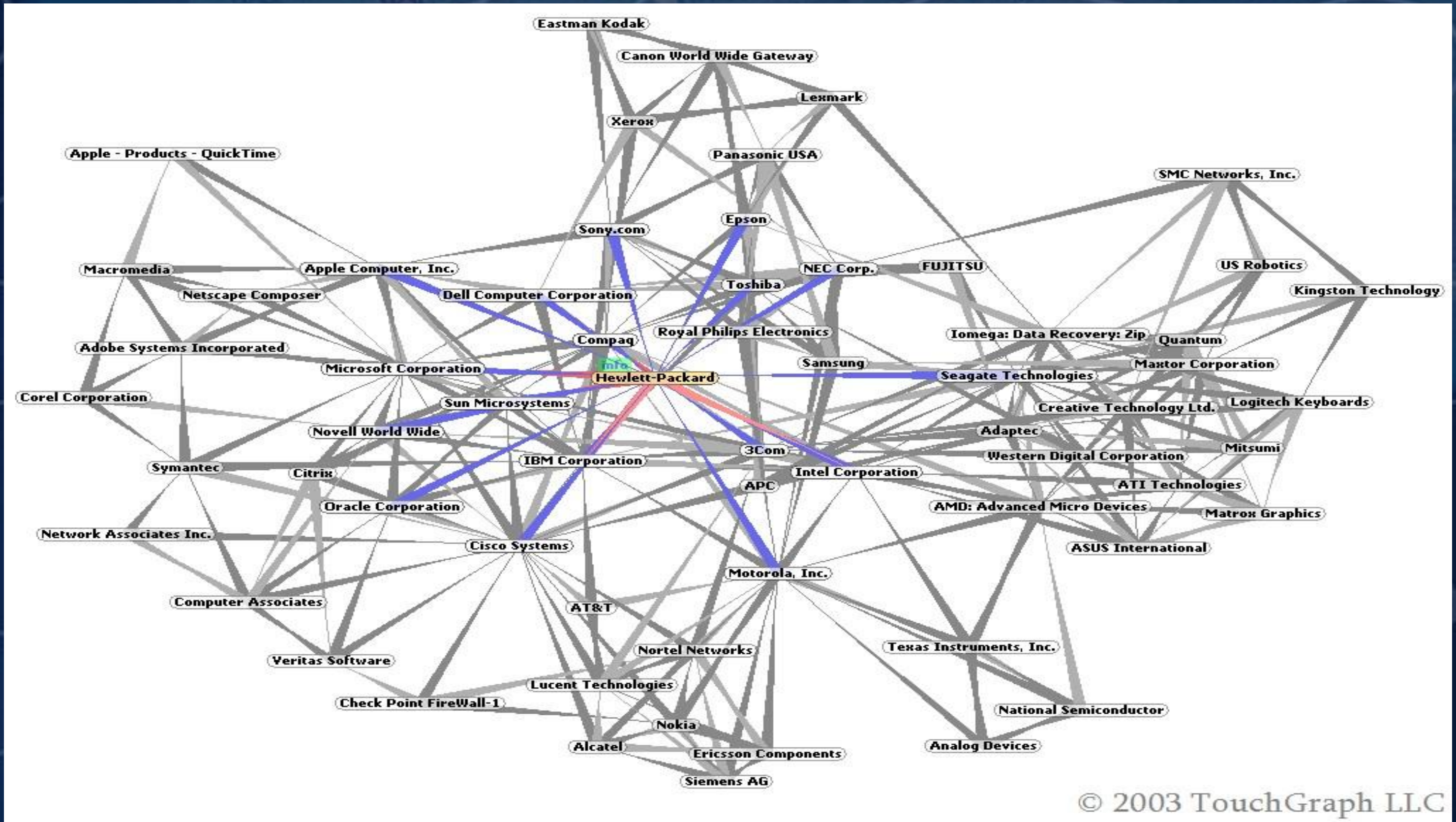
Internet/Web as Graphs

- Graph of the physical layer with routers , computers etc as nodes and physical connections as edges
  - It is limited
  - Does not capture the graphical connections associated with the information on the Internet
- Web Graph where nodes represent web pages and edges are associated with hyperlinks

Web Graph



http://www.touchgraph.com/TGGoogleBrowser.html

Web Graph Considerations

- Edges can be directed or undirected
- Graph is highly dynamic
  - Nodes and edges are added/deleted often
  - Content of existing nodes is also subject to change
  - Pages and hyperlinks created on the fly
- Apart from primary connected component there are also smaller disconnected components

Why the Web Graph?

- Example of a large,dynamic and distributed graph
- Possibly similar to other complex graphs in social, biological and other systems
- Reflects how humans organize information (relevance, ranking) and their societies
- Efficient navigation algorithms
- Study behavior of users as they traverse the web graph (e-commerce)

Statistics of Interest

- Size and connectivity of the graph
- Number of connected components
- Distribution of pages per site
- Distribution of incoming and outgoing connections per site
- Average and maximal length of the shortest path between any two vertices (diameter)

Properties of Web Graphs

- Connectivity follows a power law distribution
- The graph is sparse
  - $|E| = O(n)$ or atleast $o(n^2)$
  - Average number of hyperlinks per page roughly a constant
- A small world graph

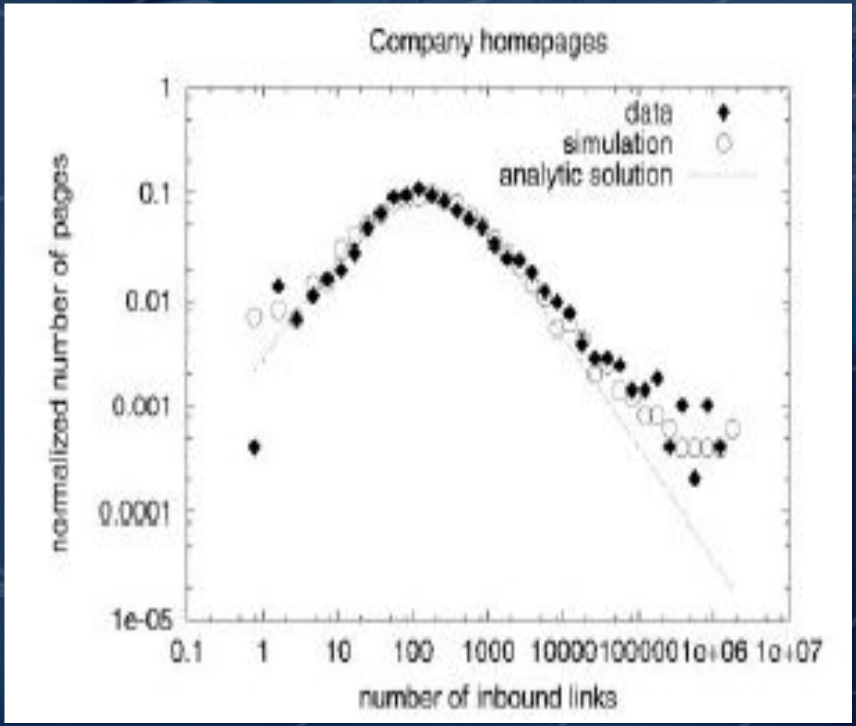
Power Law Size

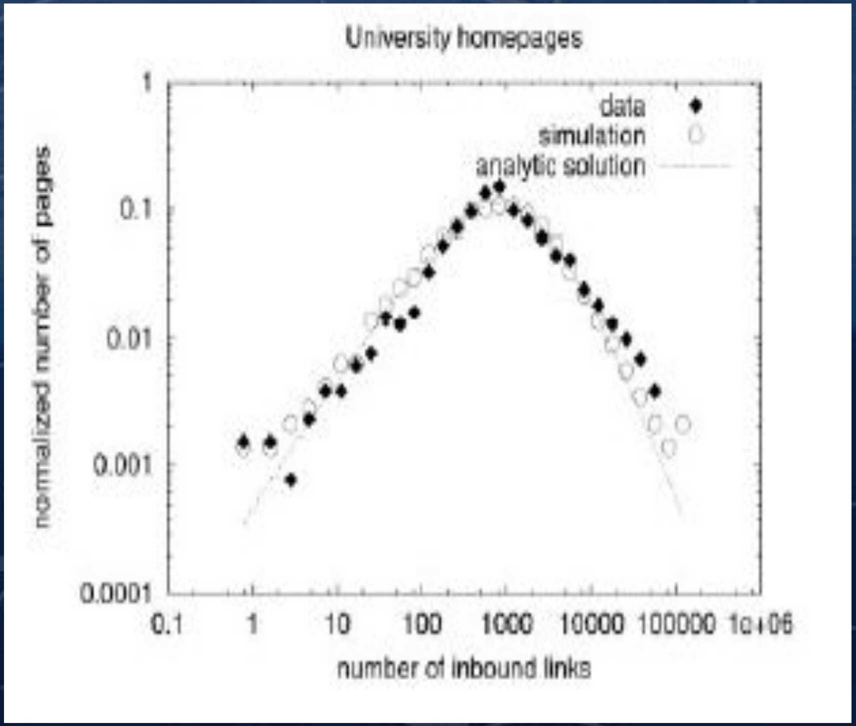
- Simple estimates suggest over a billion nodes
- Distribution of site sizes measured by the *number of pages* follow a power law distribution
- Observed over several orders of magnitude with an exponent γ in the 1.6-1.9 range

Power Law Connectivity

- Distribution of number of connections per node follows a power law distribution
- Study at Notre Dame University reported
  - γ = 2.45 for outdegree distribution
  - γ = 2.1 for indegree distribution
- Random graphs have Poisson distribution if p is large.
  - Decays exponentially fast to 0 as *k* increases towards its maximum value *n-1*

Power Law Distribution -Examples





http://www.pnas.org/cgi/reprint/99/8/5207.pdf

Examples of networks with Power Law Distribution

- Internet at the router and interdomain level
- Citation network
- Collaboration network of actors
- Networks associated with metabolic pathways
- Networks formed by interacting genes and proteins
- Network of nervous system connection in *C. elegans*

Small World Networks

- It is a 'small world'
  - Millions of people. Yet, separated by "six degrees" of acquaintance relationships
  - Popularized by Milgram's famous experiment
- Mathematically
  - Diameter of graph is small (log *N*) as compared to overall size
    - 3. Property seems interesting given 'sparse' nature of graph but …
    - This property is 'natural' in 'pure' random graphs

The small world of WWW

- Empirical study of Web-graph reveals small-world property
  - Average distance (d) in simulated web:

    $d = 0.35 + 2.06 \log (n)$

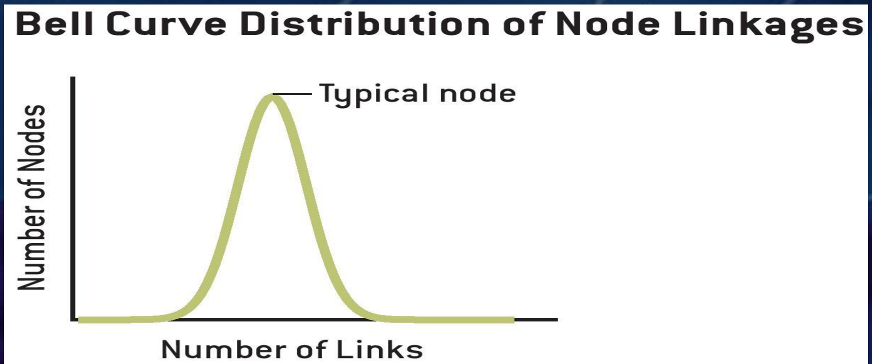
    e.g. $n = 10^9$, $d \sim= 19$
  - Graph generated using power-law model
  - Diameter properties inferred from sampling
    - Calculation of max. diameter computationally demanding for large values of *n*

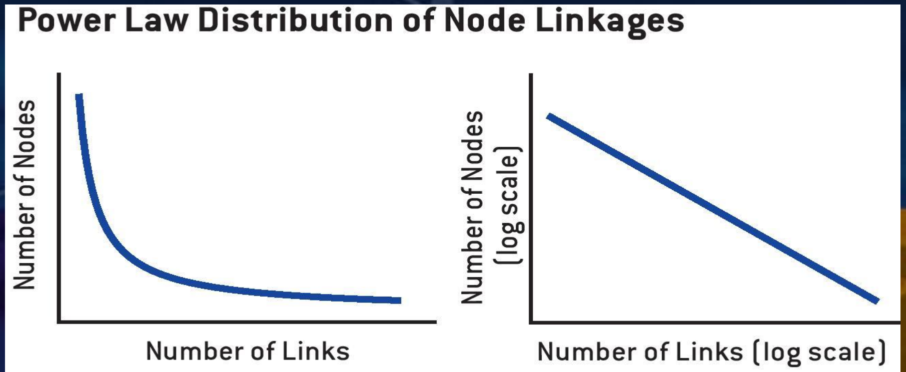
Implications for Web

- Logarithmic scaling of diameter makes future growth of web manageable
  - 10-fold increase of web pages results in only 2 more additional ‘clicks’, but …
  - Users may not take shortest path, may use bookmarks or just get distracted on the way
  - Therefore search engines play a crucial role

Some theoretical considerations

- Classes of small-world networks
  - <u>Scale-free:</u> Power-law distribution of connectivity over entire range
  - <u>Broad-scale:</u> Power-law over "broad range" + abrupt cut-off
  - <u>Single-scale:</u> Connectivity distribution decays exponentially





Power Law of PageRank

- Assess importance of a page relative to a query and rank pages accordingly
  - Importance measured by indegree
  - Not reliable since it is entirely local
- PageRank – proportion of time a random surfer would spend on that page at steady state
- A random first order Markov surfer at each time step travels from one page to another

PageRank contd

- Page rank r(v) of page v is the steady state distribution obtained by solving the system of linear equations given by

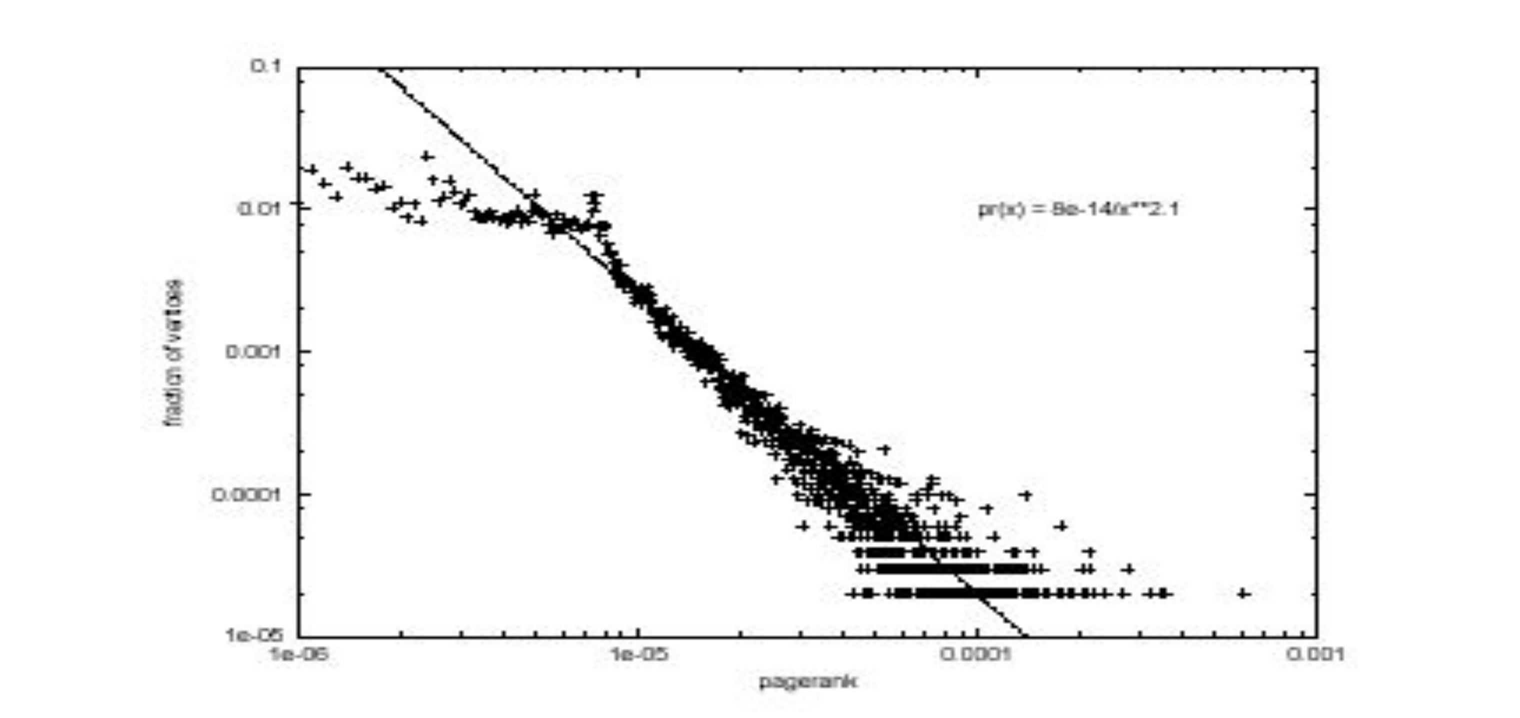
$$r(v) = \frac{1-\epsilon}{n} + \epsilon \sum_{u \in \mathrm{pa}[v]} \frac{r(v)}{|\mathrm{ch}[u]|}$$

Where pa[v] = set of parent nodes

Ch[u] = out degree

Examples

- Log Plot of PageRank Distribution of Brown Domain (*.brown.edu)



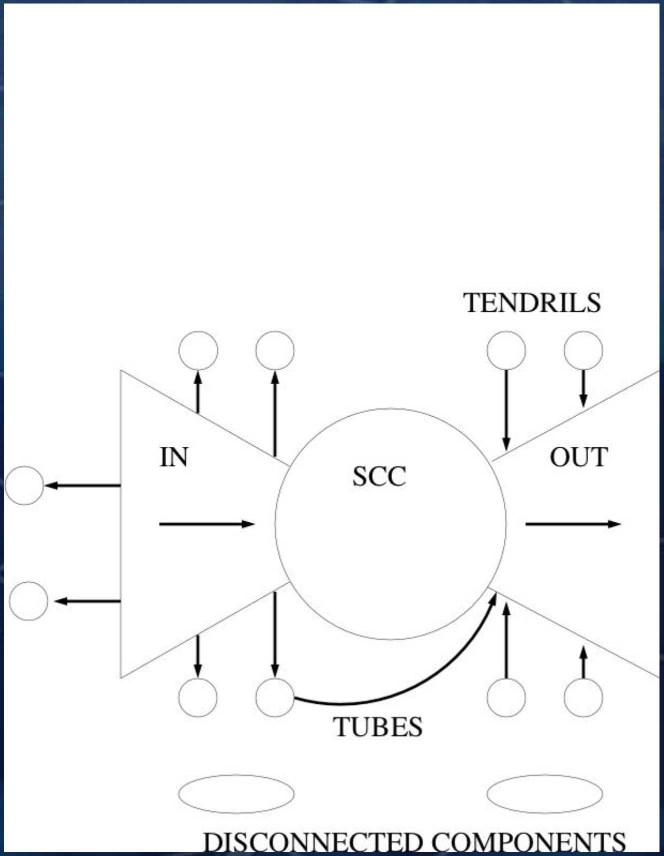
G.Pandurangan, P.Raghavan,E.Upfal,"Using PageRank to characterize Webstructure" ,COCOON 2002

Bow-tie Structure of Web

- A large scale study (Altavista crawls) reveals interesting properties of web
  - Study of 200 million nodes & 1.5 billion links
  - Small-world property not applicable to entire web
    - Some parts unreachable
    - Others have long paths
  - Power-law connectivity holds though
    - Page indegree ($\gamma = 2.1$), outdegree ($\gamma = 2.72$)

Bow-tie Components

- Strongly Connected Component (SCC)
  - Core with small-world property
- Upstream (IN)
  - Core can't reach IN
- Downstream (OUT)
  - OUT can't reach core
- Disconnected (Tendrils)



Component Properties

- Each component is roughly same size
  - ~50 million nodes
- Tendrils not connected to SCC
  - But reachable from IN and can reach OUT
- Tubes: directed paths IN->Tendrils->OUT
- Disconnected components
  - Maximal and average diameter is infinite

Empirical Numbers for Bow-tie

- Maximal minimal (?) diameter
  - 28 for SCC, 500 for entire graph
- Probability of a path between any 2 nodes
  - ~1 quarter (0.24)
- Average length
  - 16 (directed path exists), 7 (undirected)
- Shortest directed path between 2 nodes in SCC: 16-20 links on average

Models for the Web Graph

- Stochastic models that can explain or atleast partially reproduce properties of the web graph
  - The model should follow the power law distribution properties
  - Represent the connectivity of the web
  - Maintain the small world property

Web Page Growth

- Empirical studies observe a power law distribution of site sizes
  - Size includes size of the Web, number of IP addresses, number of servers, average size of a page etc
- A Generative model is being proposed to account for this distribution

Component One of the Generative Model

- The first component of this model is that

  “ *sites have short-term size fluctuations up or down that are proportional to the size of the site* “

- A site with 100,000 pages may gain or lose a few hundred pages in a day whereas the effect is rare for a site with only 100 pages

Component Two of the Generative Model

- There is an overall growth rate α so that the size $S(t)$ satisfies

$$S(t+1) = \alpha(1+\eta_t\beta)S(t)$$

where

- η_t is the realization of a +-1 Bernoulli random variable at time t with probability 0.5

- b is the absolute rate of the daily fluctuations

Component Two of the Generative Model contd

- After T steps

$$S(T) = \alpha^T S(0) \prod_{t=0}^{T-1} (1 + \eta_t \beta)$$

so that

$$\log S(T) = T \log \alpha + \log S(0) + \sum_{t=0}^{T-1} \log(1 + \eta_t \beta)$$

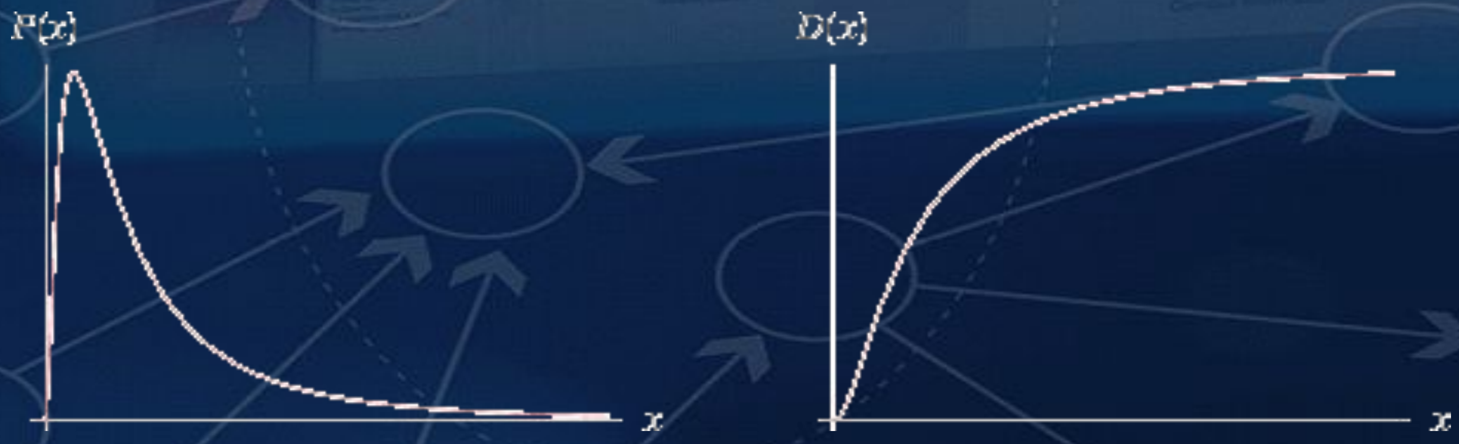
Theoretical Considerations

- Assuming η_t independent, by central limit theorem it is clear that for large values of T, log S(T) is normally distributed
  - The central limit theorem states that given a distribution with a mean μ and variance σ^2, the sampling distribution of the mean approaches a normal distribution with a mean (μ) and a variance σ^2/N as N, the sample size, increases.

http://davidmlane.com/hyperstat/A14043.html

Theoretical Considerations contd

- Log S(*T*) can also be associated with a binomial distribution counting the number of time $\eta_t = +1$
- Hence S(T) has a log-normal distribution



- The probability density and cumulative distribution functions for the log normal distribution

Modified Model

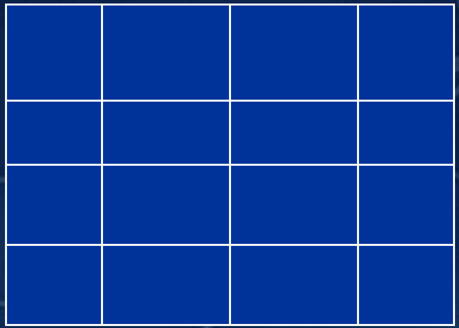
- Can be modified to obey power law distribution
- Model is modified to include the following inorder to obey power law distribution
  - A wide distribution of growth rates across different sites and/or
  - The fact that sites have different ages

Capturing Power Law Property

- Inorder to capture Power Law property it is sufficient to consider that
  - Web sites are being continuously created
  - Web sites grow at a constant rate α during a growth period after which their size remains approximately constant
  - The periods of growth follow an exponential distribution
- This will give a relation $\lambda = 0.8\alpha$ between the rate of exponential distribution λ and α the growth rage when power law exponent $\gamma = 1.08$

Lattice Perturbation (LP) Models

- Some Terms
  - "Organized Networks" (a.k.a Mafia)
    - Each node has same degree k and neighborhoods are entirely local



Probability of Edge (a,b) = { 1 if dist (a,b) = 1; 0 otherwise }

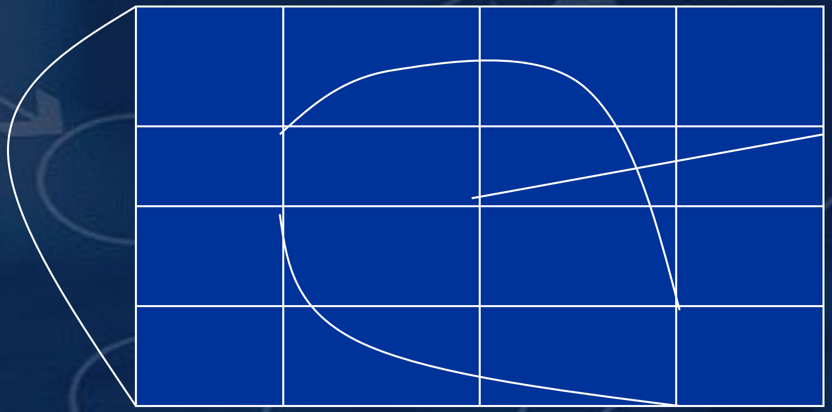
- Note: We are talking about graphs that can be mapped to a Cartesian plane

Terms (Cont'd)

- Organized Networks
  - Are 'cliquish' (Subgraph that is fully connected) in local neighborhood
  - Probability of edges across neighborhoods is almost non existent (*p=0* for fully organized)
- "Disorganized" Networks
  - 'Long-range' edges exist
  - Completely Disorganized <=> Fully Random (Erdos Model) : *p=1*

Semi-organized (SO) Networks

- Probability for long-range edge is between zero and one
- Clustered at local level (cliquish)
- But have long-range links as well



- Leads to networks that
  - Are locally cliquish
  - And have short path lengths

Creating SO Networks

- Step 1:
  - Take a regular network (e.g. lattice)
- Step 2:
  - Shake it up (perturbation)
- Step 2 in detail:
  - For each vertex, pick a local edge
  - ‘Rewire’ the edge into a long-range edge with a probability (*p*)
  - $p=0$: organized, $p=1$: disorganized

Statistics of SO Networks

- <u>Average Diameter (*d*):</u> Average distance between two nodes
- <u>Average Clique Fraction (*c*)</u>
  - Given a vertex *v*, *k(v)*: neighbors of *v*
  - Max edges among *k(v)* = *k(k-1)/2*
  - Clique Fraction (c_v): (Edges present) / (Max)
  - Average clique fraction: average over all nodes
  - Measures: Degree to which “my friends are friends of each other”

Statistics (Cont'd)

- Statistics of common networks:

	n	*k*	*d*	*c*
Actors	225,226	61	3.65	0.79
Power-grid	4,941	2.67	18.7	0.08
C.elegans	282	14	2.65	0.28

Large *k* = large *c*?

Small *c* = large *d*?

Other Properties

- For graph to be sparse but connected:
  - $n >> k >> \log(n) >> 1$
- As p --> 0 (organized)
  - $d \sim= n/2k >> 1$, $c \sim= 3/4$
  - Highly clustered & d grows linearly with n
- As p --> 1 (disorganized)
  - $d \sim= \log(n)/\log(k)$, $c \sim= k/n << 1$
  - Poorly clustered & d grows logarithmically with n

Effect of 'Shaking it up'

- Small shake (*p* close to zero)
  - High cliquishness AND short path lengths
- Larger shake (*p* increased further from 0)
  - *d* drops rapidly (increased small world phenomena_
  - *c* remains constant (transition to small world almost undetectable at local level)
- Effect of long-range link:
  - Addition: non-linear decrease of *d*
  - Removal: small linear decrease of *c*

LP and The Web

- LP has severe limitations
  - No concept of short or long links in Web
    - A page in USA and another in Europe can be joined by one hyperlink
  - Edge rewiring doesn't produce power-law connectivity!
    - Degree distribution bounded & strongly concentrated around mean value
- Therefore, we need other models …