Index of Refraction

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Outline

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- Classical Model
- Typical measurement methods
- Application
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Definition of Index of Refraction

In uniform isotropic linear media, the wave equation is:

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

They are satisfied by plane wave

$$\psi = A\, e^{i(\mathbf{k}\,\mathbf{r} - \omega t)} \qquad k = |\mathbf{k}| = \omega\sqrt{\mu\varepsilon}$$

ψ can be any Cartesian components of **E** and **H**

The phase velocity of plane wave travels in the direction of **k** is

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$\nabla\times$

Definition of Index of Refraction

We can define the index of refraction as

$$n = \frac{v}{c} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

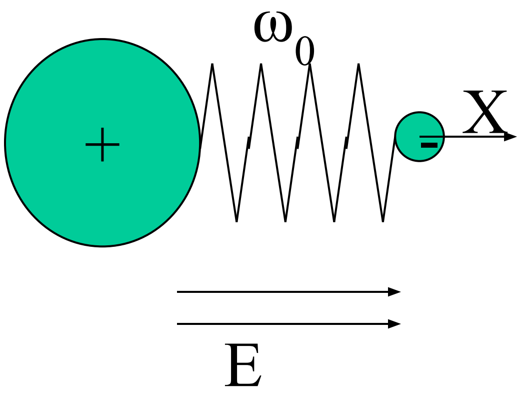
Most media are nonmagnetic and have a magnetic permeability $\mu=\mu_0$, in this case

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$$

In most media, n is a function of frequency.

Classical Electron Model (Lorentz Model)

Let the electric field of optical wave in an atom be



$$\mathbf{E}=\mathbf{E_0}e^{-i\omega t}$$

the electron obeys the following equation of motion

$$m\frac{d^2}{dt^2}\mathbf{X}+m\gamma\frac{d}{dt}\mathbf{X}+m\omega_0{}^2\mathbf{X}=-e\mathbf{E}$$

$\mathbf{X}$ is the position of the electron relative to the atom

m is the mass of the electron

ω_0 is the resonant frequency of the electron motion

γ is the damping coefficient

Classical Electron Model (Lorentz Model)

The solution is $$\mathbf{X} = \frac{-e\mathbf{E_0}}{m(\omega_0{}^2 - \omega^2 - i\omega\gamma)} e^{-i\omega t}$$

The induced dipole moment is

$$\mathbf{p} = -e\mathbf{X} = \frac{e^2}{m(\omega_0{}^2 - \omega^2 - i\omega\gamma)}\mathbf{E} = \alpha\mathbf{E}$$

α is atomic polarizability

$$\alpha = \frac{e^2}{m(\omega_0{}^2 - \omega^2 - i\omega\gamma)}$$

The dielectric constant of a medium depends on the manner in which the atoms are assembled. Let N be the number of atoms per unit volume. Then the polarization can be written approximately as

$$\mathbf{P} = \mathrm{N}\,\mathbf{p} = \mathrm{N}\,\alpha\,\mathbf{E} = \varepsilon_0\,\chi\,\mathbf{E}$$

Classical Electron Model (Lorentz Model)

The dielectric constant of the medium is given by

$$\varepsilon = \varepsilon_0 (1+\chi) = \varepsilon_0 (1+N\alpha/ \varepsilon_0)$$

If the medium is nonmagnetic, the index of refraction is

$$n= (\varepsilon /\varepsilon_0)^{1/2} = (1+N\alpha/ \varepsilon_0)^{1/2}$$

$$n^2 = \frac{\varepsilon}{\varepsilon_0} = 1 + \frac{Ne^2}{\varepsilon_0 m(\omega_0{}^2 - \omega^2 - i\omega\gamma)}$$

If the second term is small enough then

$$n = 1 + \frac{Ne^2}{2\varepsilon_0 m(\omega_0{}^2 - \omega^2 - i\omega\gamma)}$$

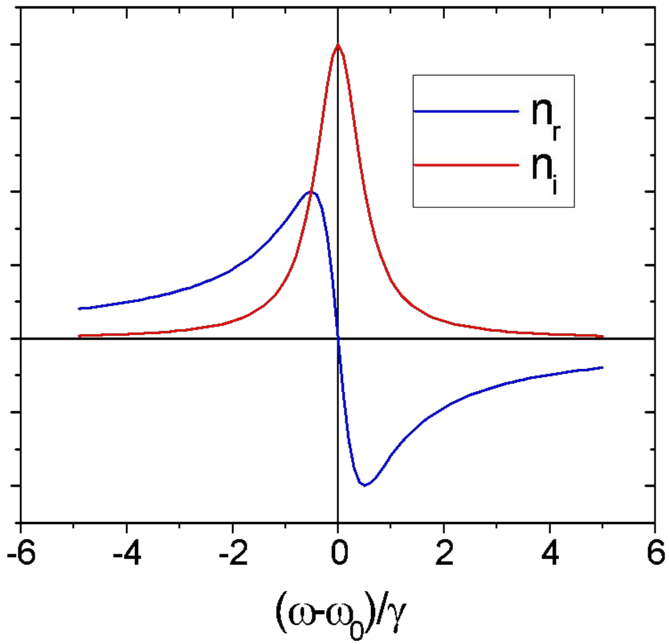
Classical Electron Model (Lorentz Model)

The complex refractive index is

$$n \rightarrow n_r + in_i = 1 + \frac{Ne^2(\omega_0{}^2 - \omega^2)}{2\varepsilon_0 m[(\omega_0{}^2 - \omega^2)^2 + \omega^2\gamma^2]} + i\frac{Ne^2\gamma\omega}{2\varepsilon_0 m[(\omega_0{}^2 - \omega^2)^2 + \omega^2\gamma^2]}$$

at $\omega \sim \omega_0$,

$$n_r + in_i = 1 + \frac{Ne^2(\omega_0 - \omega)}{4\varepsilon_0 m\omega_0[(\omega_0 - \omega)^2 + (\gamma/2)^2]} + i\frac{Ne^2\gamma}{8\varepsilon_0 m\omega_0[(\omega_0 - \omega)^2 + (\gamma/2)^2]}$$



Normalized plot of n-1 and k versus $\omega-\omega_0$

For more than one resonance frequencies for each atom,

$$n^2 = 1 + \frac{Ne^2}{\varepsilon_0 m} \sum_j \frac{f_j}{(\omega_j{}^2 - \omega^2 - i\omega\gamma_j)} \qquad \sum_j f_j = Z$$

Classical Electron Model (Drude model)

If we set ω_0=0, the Lorentz model become Drude model. This model can be used in free electron metals

$$n^2 = 1 - \frac{Ne^2}{\varepsilon_0 m(\omega^2 + i\omega\gamma)}$$

Relation Between Dielectric Constant and Refractive Index

By definition,

$$n^2 = \frac{\varepsilon}{\varepsilon_0}$$

$$n = n_r + in_i$$

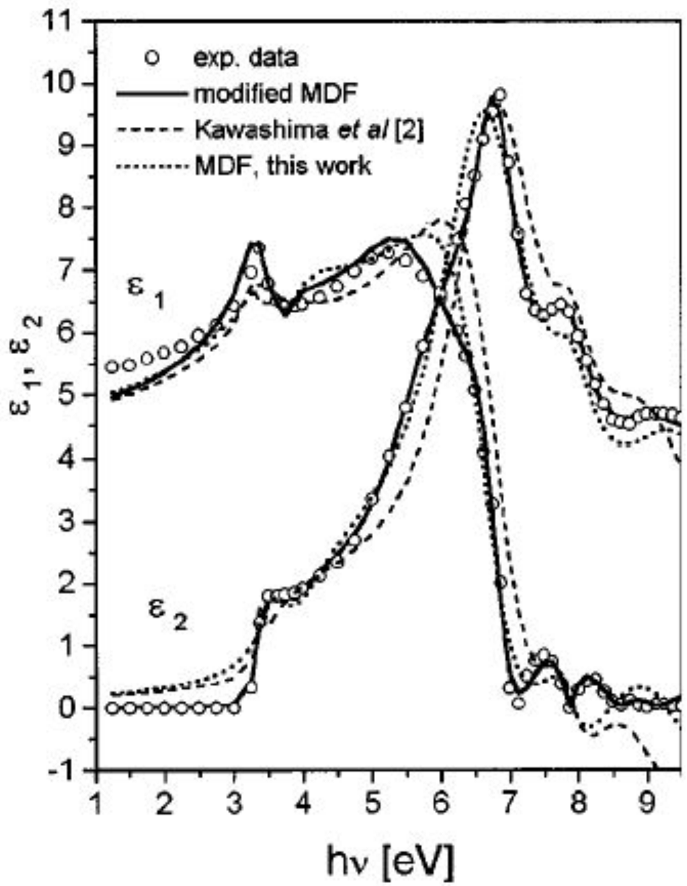
$$\varepsilon = \varepsilon_1 + i\varepsilon_2$$

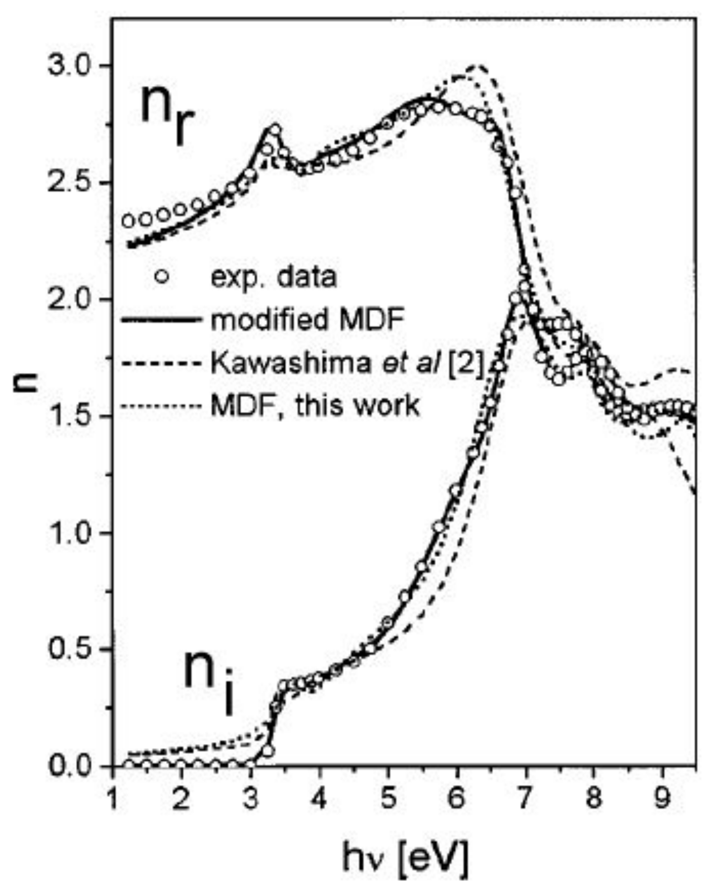
We can easily get:

$$n_r = \{\frac{1}{2}[(\varepsilon_1{}^2 + \varepsilon_2{}^2)^{1/2} + \varepsilon_1]\}^{1/2} / \varepsilon_0$$

$$n_i = \{\frac{1}{2}[(\varepsilon_1{}^2 + \varepsilon_2{}^2)^{1/2} - \varepsilon_1]\}^{1/2} / \varepsilon_0$$

An Example to Calculate Optical Constants





$$\epsilon(E)=\sum_{\alpha=A,B,C}\left(\sum_{n=1}^{\infty}\frac{A_{0\alpha}^{\mathrm{ex}}}{n^3}\frac{1}{E_{0\alpha}-(G_{0\alpha}^{3\mathrm{D}}/n^2)-E-i\Gamma}\right)$$

Real and imaginary part of the index of refraction of GaN vs. energy;

Kramers-Kronig Relation

The real part and imaginary part of the complex dielectric function ε (ω) are not independent. they can connected by Kramers-Kronig relations:

$$\varepsilon_1(\omega) = \varepsilon_0 + \frac{2}{\pi} P \int_0^\infty \frac{\varepsilon_2(\omega')\omega'}{\omega'^2 - \omega^2} d\omega'$$

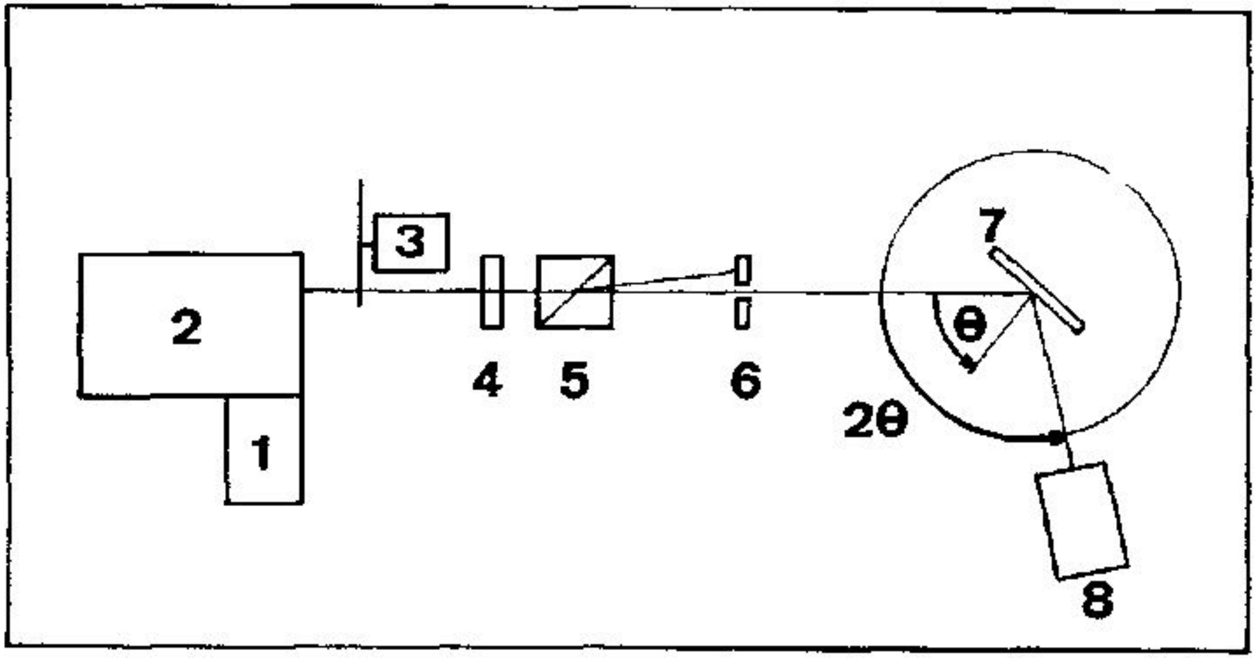
$$\varepsilon_2(\omega) = \frac{2\omega}{\pi} P \int_0^\infty \frac{\varepsilon_1(\omega') - \varepsilon_0}{\omega'^2 - \omega^2} d\omega'$$

P indicates that the integral is a principal value integral.

K-K relation can also be written in other form, like

$$n(\lambda) = \frac{1}{\pi} P \int_0^\infty \frac{\alpha(\lambda')}{1 - (\lambda'/\lambda)^2} d\lambda'$$

A Method Based on Reflection



Typical experimental setup

(1) halogen lamp;

(2) mono-chromator; (3) chopper; (4) filter;

(5) polarizer (get p-polarized light); (6) hole diaphragm;

(7) sample on rotating support (θ); (8) PbS detector(2θ)

Calculation

In this case, $n_1=1$, and $n_2=n_r+i\,n_i$

Snell Law become:

$$k_{1z}=k_{2z}=\frac{2\pi}{\lambda}\sin\theta\equiv\beta$$

Reflection coefficient:

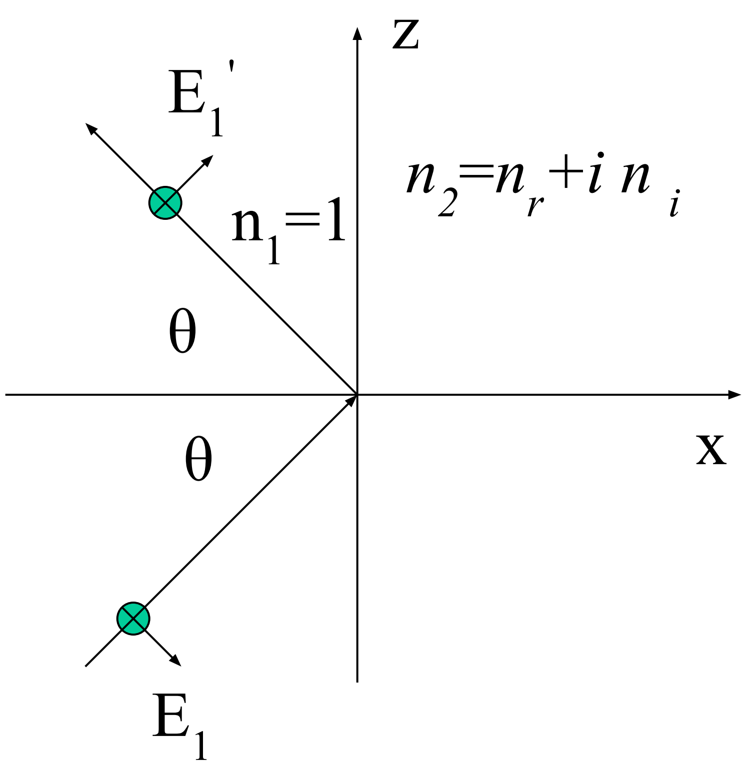
$$r_p=\frac{n_1{}^2k_{2x}-n_2{}^2k_{1x}}{n_1{}^2k_{2x}+n_2{}^2k_{1x}}=\frac{k_{2x}-(n_r+in_i)^2k_{1x}}{k_{2x}+(n_r+in_i)^2k_{1x}}$$

$$k_{1x}=[(\frac{2\pi}{\lambda})^2-\beta^2]^{1/2}$$

$$k_{2x}=[(\frac{2\pi}{\lambda})^2(n_r-in_i)^2-\beta^2]^{1/2}$$

Reflectance:

$R(\theta_1, \lambda, n_r, n_i)=|r_p|^2$



Reflection of p-polarized light

From this measurement, they got R, θ for each wavelength λ,
Fitting the experimental curve, they can get n_r and n_i.

Results Based on Reflection Measurement

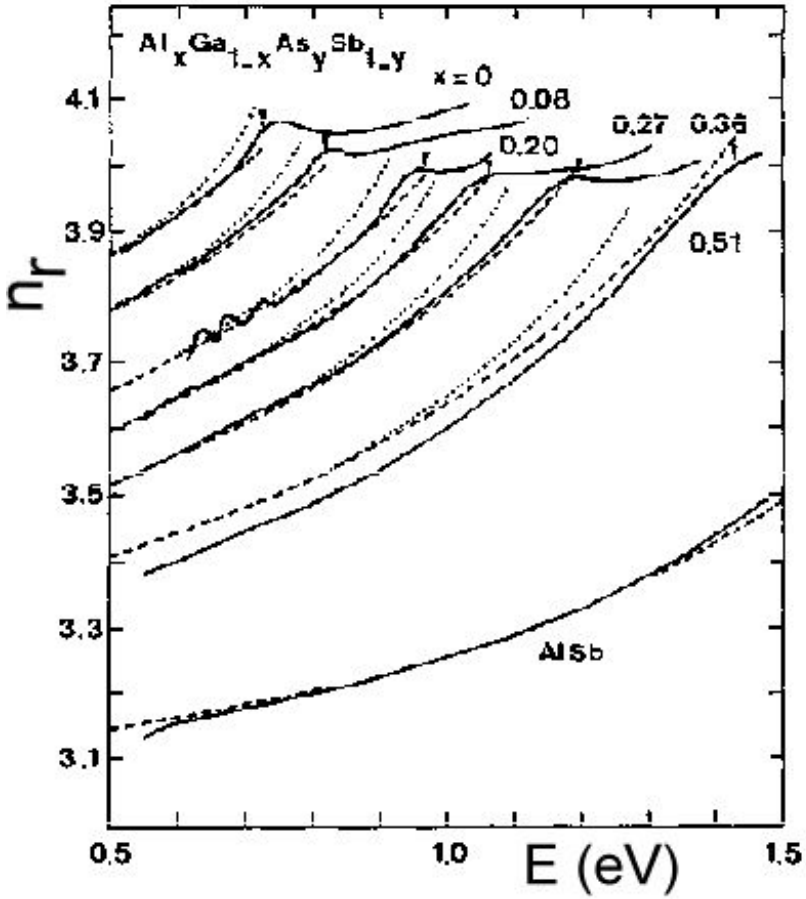


FIG. 2. Measured refractive indices at 300 K vs. photon energy of AlSb and $Al_xGa_{1-x}As_ySb_{1-y}$ layers lattice matched to GaSb (y~0.085 x).

Dashed lines: calculated curves from Eq. (1);

Dotted lines: calculated curves from Eq. (2)

Single effective oscillator model

$$n_r^2 - 1 = \frac{E_0 E_d}{E_0^2 - E^2} \qquad \text{(Eq. 1)}$$

$$n_r^2 - 1 = \frac{E_d}{E_0} + \frac{E^2 E_d}{E_0^3} + \frac{\eta E^4}{\pi} \ln\left(\frac{E_f^2 - E}{E_\Gamma^2 - E^2}\right) \qquad \text{(Eq. 2)}$$

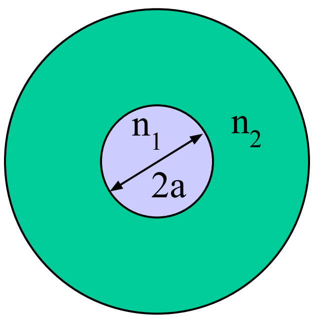
$$E_f^2 = 2E_0^2 - E_\Gamma^2$$

$$\eta = \frac{\pi E_d}{2E_0^3 (E_0^2 - E_\Gamma^2)}$$

E_0: oscillator energy
E_d: dispersion energy
E_Γ: lowest direct band gap energy

Use AFM to Determine the Refractive Index Profiles of Optical Fibers

The basic configuration of optical fiber consists of a hair like, cylindrical, dielectric region (core) surrounded by a concentric layer of somewhat lower refractive index(cladding).



$$a \leq \frac{2.4\lambda_0}{2\pi\left(n_1^2 - n_2^2\right)^{1/2}}$$

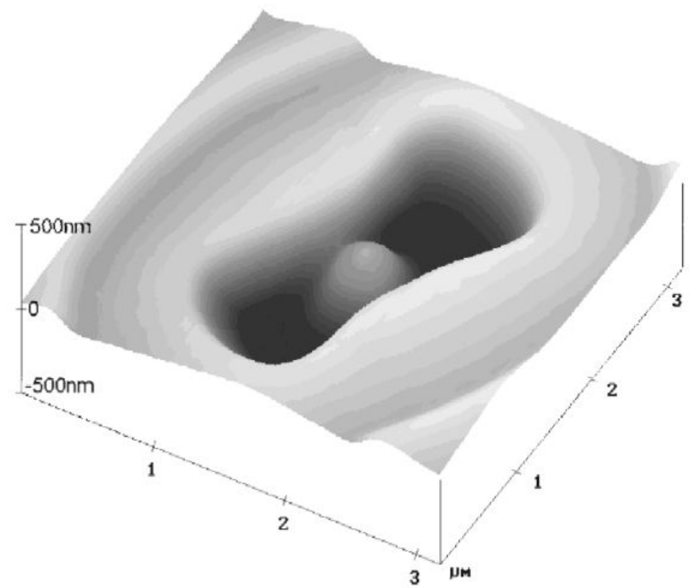


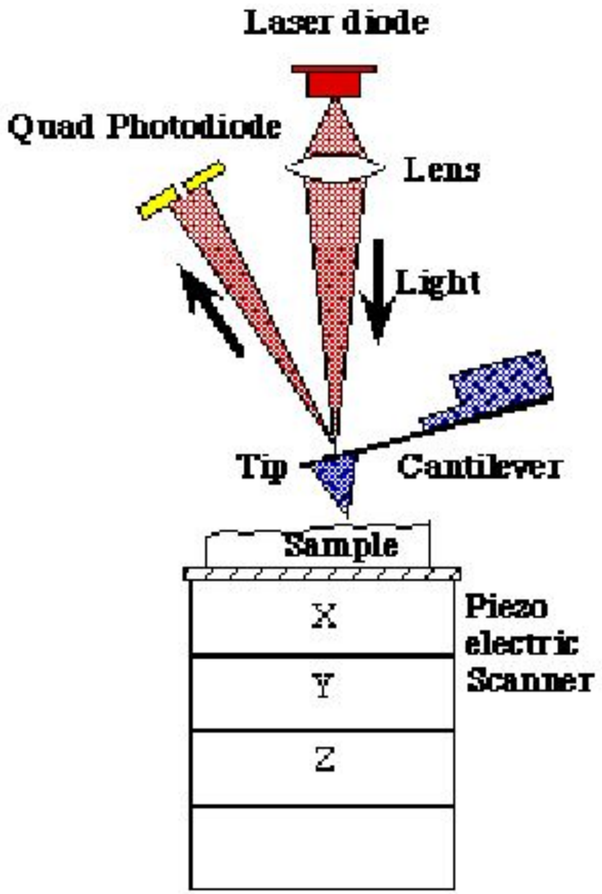
FIG. 3. Typical AFM topography scan of the endface of an etched elliptical core fiber.

There is no way for AFM to measure refractive index directly.
People found fiber material with different refractive index have different etch rate in special solution.

Fiber samples were

- Cleaved and mounted in holder
- Etched with 5% HF solution
- Measured with AFM

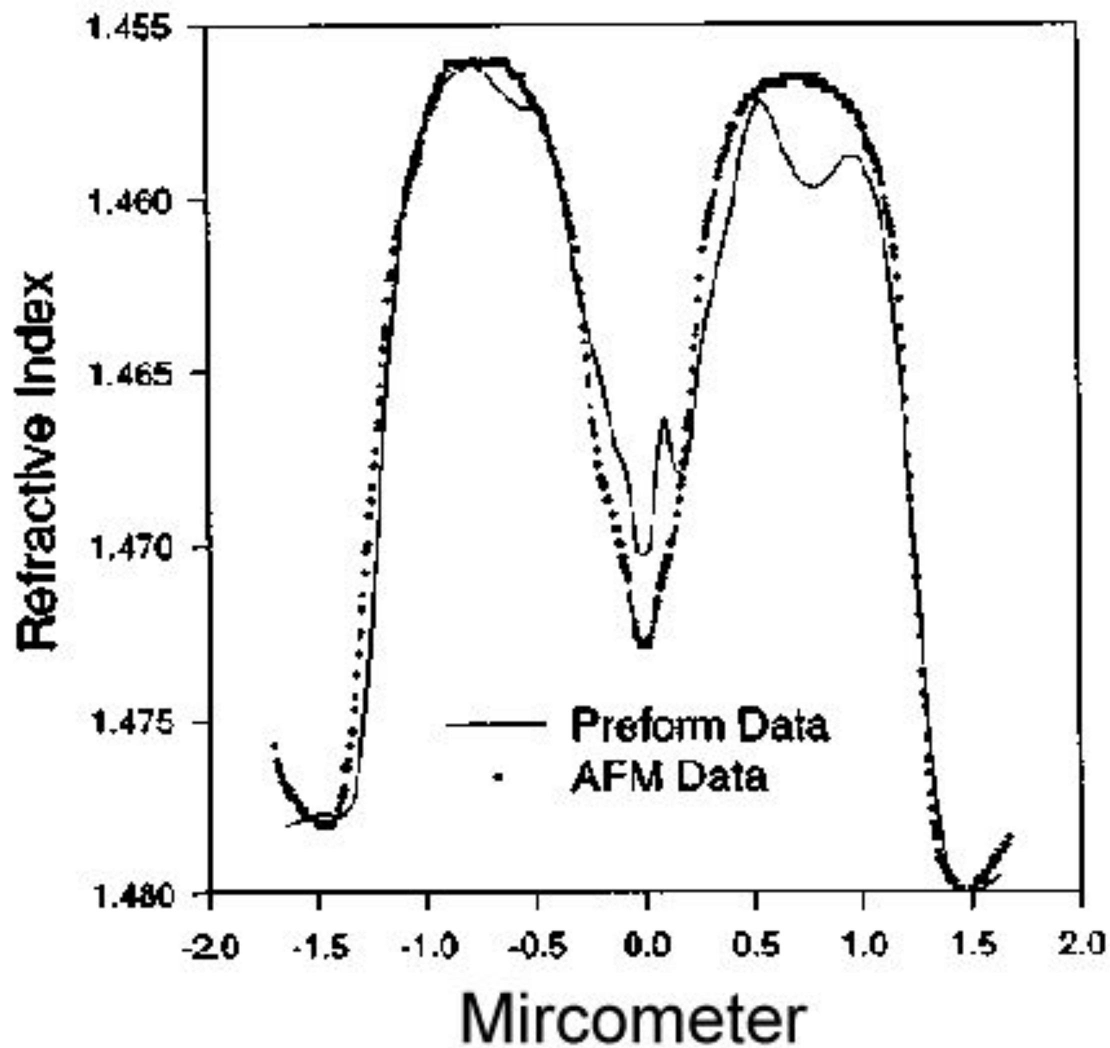
AFM



Atomic Force Microscope

- The optical lever operates by reflecting a laser beam off the cantilever. Angular deflection of the cantilever causes a twofold larger angular deflection of the laser beam.
- The reflected laser beam strikes a position-sensitive photodetector consisting of two side-by-side photodiodes.
- The difference between the two photodiode signals indicates the position of the laser spot on the detector and thus the angular deflection of the cantilever.
- Because the cantilever-to-detector distance generally measures thousands of times the length of the cantilever, the optical lever greatly magnifies motions of the tip.

Result



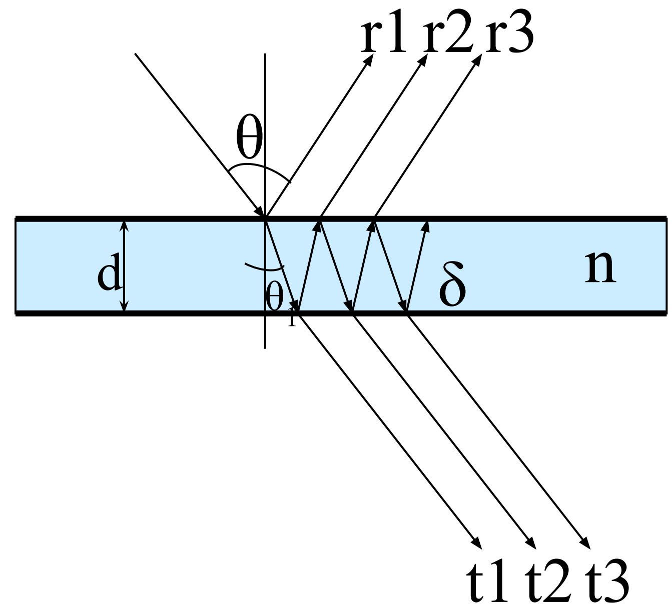
1.455
1.460
1.465
1.470
1.475
1.480
Refractive Index
Preform Data
AFM Data
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
Mircometer

A Method Based on Transmission

For $\theta=0$, input wave function $a\, e^{i\varphi}$,

$t_m = aTT'R'^{2m-1}\, e^{i(\varphi-(2m-1)\delta)}$ (m=1,2…)

$\delta=2\pi dn/\lambda$



The transmission wave function is superposed by all t_m

$$a_T = a\,T\,T'\,e^{i\varphi}\,\Sigma_m (R'^{2m-1}\, e^{-i(2m-1)\delta})$$
$$=(1-R^2)a\,e^{i(\varphi-\delta)}/(1-R^2 e^{-i2\delta})$$

$(TT'=1-R^2\ ;\ R'=-R)$

If $R<<1$, then

$$a_T = a\,e^{i(\varphi-\delta)}$$

maximum condition is $2\delta=2\pi m= 4\pi dn/\lambda$

$$n(\lambda_m)=m\,\lambda_m/2d$$

Result Based on Transmission Measurement

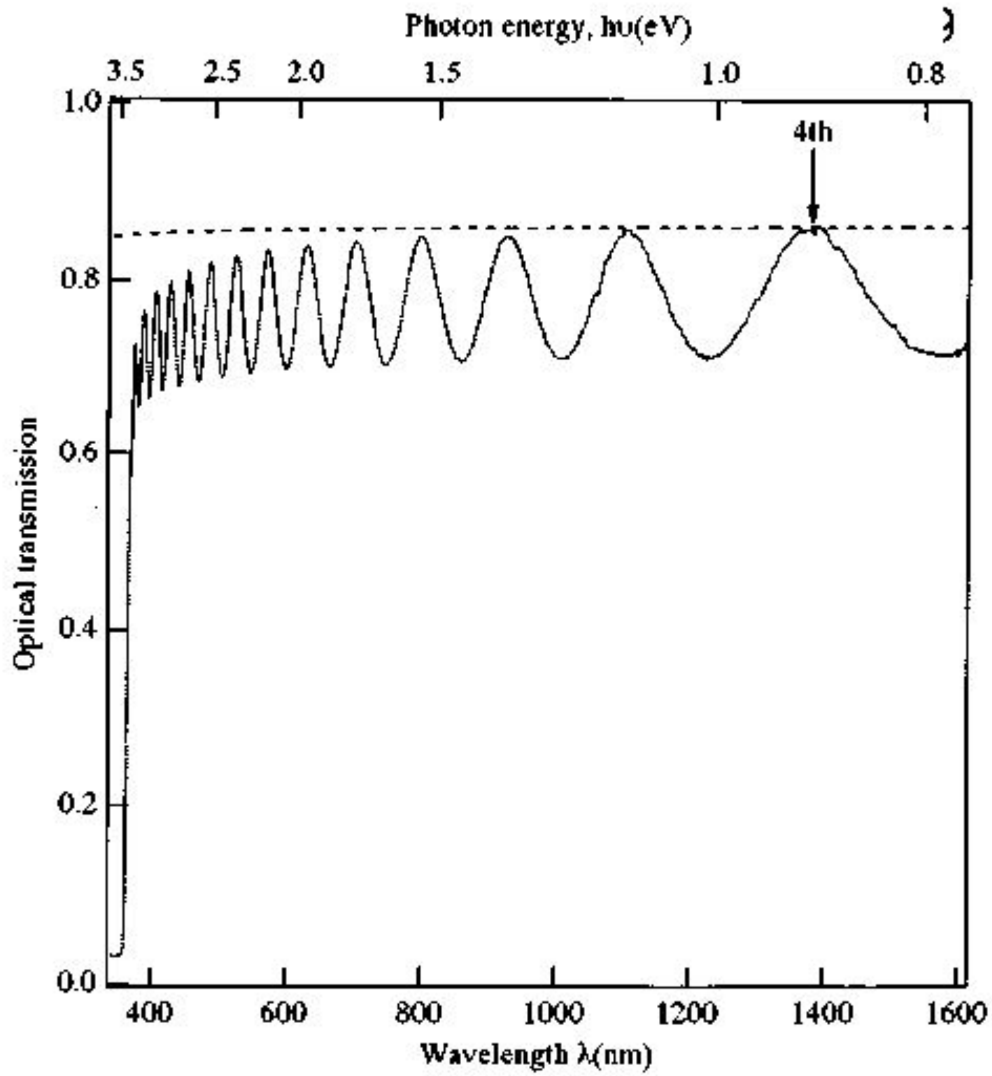


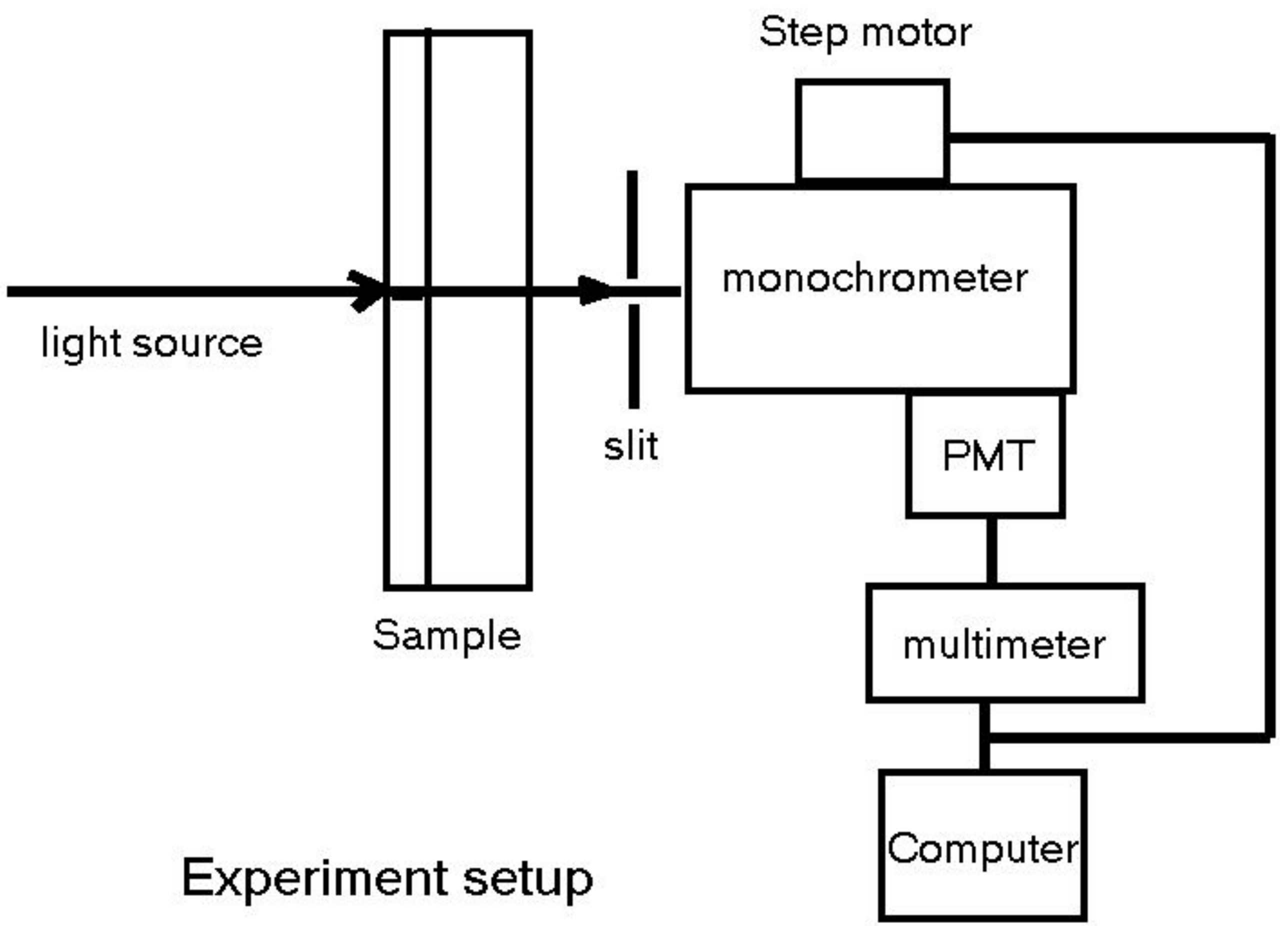
FIG. 2. Optical transmission spectrum of GaN on a sapphire substrate.

G. Yu Appl. Phys. Lett. 70(1997)3209

$$n^2 = 2.27^2 + \frac{304.7^2}{\lambda^2 + 294^2}$$

Application

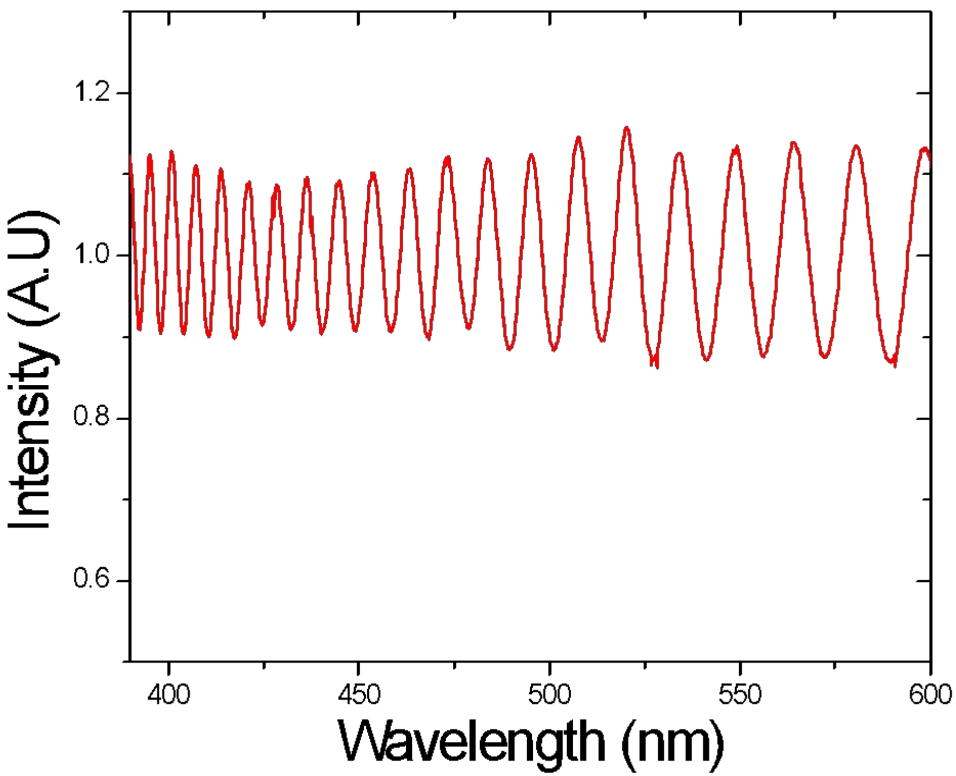
In our lab., we have a simple system to measure the thickness of epitaxial GaN layer.



Experiment setup

Thickness Measurement

$n(\lambda_m)=m\,\lambda_m/2d$



Limit
Minimum thickness:~500/n
Error<λ/2n

Steps to calculate thickness

- Get peak position λ_m
- $d=(\lambda_m \lambda_{m-1})/2/[\lambda_{m-1}\, n(\lambda_m) - \lambda_m\, n(\lambda_{m-1})]$
- Average d
- get $m_{min} = n(\lambda_{max})*2d/\lambda_{max}$
- Calculate d : $d=m\,\lambda_m/2/n(\lambda_m)$ (from m_{min} for each peak)
- Average d again

Reference

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