Solving linear recurrence relations

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- Linear homogeneous recurrence relations with constant coefficients
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Linear homogeneous recurrence relations with constant coefficients

Definition 1

A **linear homogeneous recurrence relation of degree k with constant coefficients** is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

Linear homogeneous recurrence relations with constant coefficients

- $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

A sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions:

$$\boldsymbol{a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}}.$$

Linear homogeneous recurrence relations with constant coefficients

- Example 1

The recurrence relation

$$P_n = (1.11)P_{n-1}$$

is a linear homogeneous recurrence relation of degree one.

Linear homogeneous recurrence relations with constant coefficients

Example 2

The recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

is a linear homogeneous recurrence relation of degree two.

The sequence of **Fibonacci numbers** satisfies this recurrence relation $f_n = f_{n-1} + f_{n-2}$ and also satisfies the initial conditions $f_0 = 0, f_1 = 1$.

Linear homogeneous recurrence relations with constant coefficients

<u>Example 3</u>

The recurrence relation

$$a_n = a_{n-5}$$

is a linear homogeneous recurrence relation of degree five.

Linear homogeneous recurrence relations with constant coefficients

Example 4

The recurrence relation

$$a_n = a_{n-1} + a_{n-2}^2$$

is not linear.

Linear homogeneous recurrence relations with constant coefficients

Example 5

The recurrence relation

$$H_n = 2H_{n-1} + 1$$

is not homogeneous.

Linear homogeneous recurrence relations with constant coefficients

• Example 6

The recurrence relation

$$B_n = nB_{n-1}$$

does not have constant coefficients.

Solving linear homogeneous recurrence relations with constant coefficients

- The basic approach for solving linear homogeneous recurrence relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is to look for solutions of the form

$$\boldsymbol{a_n = r^n},$$

where r is a constant.

Solving linear homogeneous recurrence relations with constant coefficients

- Note that

$$a_n = r^n$$

is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

Solving linear homogeneous recurrence relations with constant coefficients

- $$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

When both sides of this equation are divided by r^{n-k}, we obtain the equation

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_{k-1} r + c_k.$$

When the right-hand side is subtracted from the left we obtain the equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0.$$

Solving linear homogeneous recurrence relations with constant coefficients

- Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution of linear homogeneous recurrence relations with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \quad (*)$$

is a solution **if and only if**

r is a solution of this last equation

$$\boldsymbol{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0}.$$

We call this the **characteristic equation** of the recurrence relation $(*)$.

The solutions of this equation are called the **characteristic roots** of the recurrence relation $(*)$.

Solving linear homogeneous recurrence relations with constant coefficients

As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

- Theorem 1

Let c_1 and c_2 be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if $\boldsymbol{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n}$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof of theorem 1

? • If r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, α_1 and α_2 are constants then the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$.

$$r_1^2 = c_1 r_1 + c_2, \quad r_2^2 = c_1 r_2 + c_2$$

$$\Downarrow$$

$c_1 a_{n-1} + c_2 a_{n-2} =$
$c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) =$
$\alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) =$
$\alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 =$
$\alpha_1 r_1^n + \alpha_2 r_2^n =$
a_n ■

Proof of theorem 1

- If the sequence $\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, where r_1 and r_2 are distinct roots of $r^2 - c_1 r - c_2 = 0$.

Let $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

and the initial conditions $a_0 = C_0, a_1 = C_1$ hold.

Proof of theorem 1

- If the sequence $\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, where r_1 and r_2 are distinct roots of $r^2 - c_1 r - c_2 = 0$.

It will be shown that there are constants α_1 and α_2 such that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies these same initial conditions $a_0 = C_0, a_1 = C_1$.

Proof of theorem 1

- If the sequence $\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, where r_1 and r_2 are distinct roots of $r^2 - c_1 r - c_2 = 0$.

This requires that

$$a_0 = C_0 = \alpha_1 + \alpha_2,$$
$$a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2,$$

We can solve these two equations for α_1 and α_2:

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}, \alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2}.$$

Proof of theorem 1

- If the sequence $\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, where r_1 and r_2 are distinct roots of $r^2 - c_1 r - c_2 = 0$.

Hence, with these values for

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}, \alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2},$$

the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$,
satisfies the two initial conditions $a_0 = C_0, a_1 = C_1$.

Proof of theorem 1

- If the sequence $\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, where r_1 and r_2 are distinct roots of $r^2 - c_1 r - c_2 = 0$.

We know that $\{a_n\}$ and $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ are both solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and both satisfy the initial conditions when $n = 0$ and $n = 1$.

Because there is a unique solution of a linear homogeneous recurrence relation of degree two with two initial conditions, it follows that the two solutions are the same, that is, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$. ■

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

Example 7

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution

The characteristic equation of the recurrence relation is

$$r^2 - r - 2 = 0.$$

Its roots are

$$r = 2 \text{ and } r = -1.$$

By theorem 1, $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$.

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

Example 7

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$$

$a_0 = 2 = \alpha_1 + \alpha_2,$

$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1), \Rightarrow$

$\alpha_1 = 3, \alpha_2 = -1, \Rightarrow$

$a_n = 3 \cdot 2^n - (-1)^n.$

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

Example 8 (Fibonacci numbers)

What is the solution of the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

with $f_0 = 0$ and $f_1 = 1$?

Solution

The characteristic equation of the recurrence relation is

$$r^2 - r - 1 = 0.$$

Its roots are

$$r = (1 + \sqrt{5})/2 \text{ and } r = (1 - \sqrt{5})/2.$$

By theorem 1,

$$f_n = \alpha_1\left((1 + \sqrt{5})/2\right)^n + \alpha_2\left((1 - \sqrt{5})/2\right)^n.$$

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

- <u>Example 8</u> (Fibonacci numbers)

What is the solution of the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

with $f_0 = 0$ and $f_1 = 1$?

<u>Solution</u>

$$f_n = \alpha_1\left((1+\sqrt{5})/2\right)^n + \alpha_2\left((1-\sqrt{5})/2\right)^n$$

$f_0 = 0 = \alpha_1 + \alpha_2,$

$f_1 = 1 = \alpha_1\left((1+\sqrt{5})/2\right) + \alpha_2\left((1-\sqrt{5})/2\right), \Rightarrow$

$\alpha_1 = 1/\sqrt{5}, \alpha_2 = -1/\sqrt{5}, \Rightarrow$

$f_n = 1/\sqrt{5}\left((1+\sqrt{5})/2\right)^n + (-1/\sqrt{5})\left((1-\sqrt{5})/2\right)^n.$

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

- Theorem 2

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has only one root r_0.

A sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if $\boldsymbol{a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n}$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

Example 9

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with $a_0 = 1$ and $a_1 = 6$?

Solution

The characteristic equation of the recurrence relation is

$$r^2 - 6r + 9 = 0.$$

Its root is

$$r = 3.$$

By theorem 2, $a_n = \alpha_1 3^n + \alpha_2 n 3^n$.

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

- Example 9

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with $a_0 = 1$ and $a_1 = 6$?

Solution

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

$a_0 = 1 = \alpha_1,$

$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3, \Rightarrow$

$\alpha_1 = 1, \alpha_2 = 1, \Rightarrow$

$a_n = 3^n + n3^n.$

Solving linear homogeneous recurrence relations with constant coefficients of **degree three**

- Theorem 3

Let $c_1, c_2, \dots, c_k$ be real numbers.

Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

has k distinct roots $r_1, r_2, \dots, r_k$.

A sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if $\boldsymbol{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n}$

for $n = 0, 1, 2, \dots$, where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

Solving linear homogeneous recurrence relations with constant coefficients of **degree three**

Example 10

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with $a_0 = 2, a_1 = 5, a_2 = 15.$

Solution

The characteristic equation of the recurrence relation is

$$r^3 - 6r^2 + 11r - 6 = 0.$$

Its roots are

$$r_1 = 1, r_2 = 2, r_3 = 3.$$

By theorem 3, $a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$.

Solving linear homogeneous recurrence relations with constant coefficients of **degree three**

Example 10

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with $a_0 = 2, a_1 = 5, a_2 = 15.$

Solution

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$$

$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3,$

$a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3,$

$a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9, \Rightarrow$

$\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 2, \Rightarrow$

$a_n = 1 - 2^n + 2 \cdot 3^n.$

Linear nonhomogeneous recurrence relations with constant coefficients

Definition 2

A linear nonhomogeneous recurrence relation with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, $c_k \neq 0$; $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the **associated homogeneous recurrence relation**.

It plays an important role in the solution of the nonhomogeneous recurrence relation.

Linear nonhomogeneous recurrence relations with constant coefficients

• Example 11

The recurrence relation

$$a_n = a_{n-1} + 2^n$$

is a linear nonhomogeneous recurrence relation with constant coefficients.

The associated linear homogeneous recurrence relation is

$$a_n = a_{n-1}.$$

Linear nonhomogeneous recurrence relations with constant coefficients

- Example 12

The recurrence relation

$$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$$

is a linear nonhomogeneous recurrence relation with constant coefficients.

The associated linear homogeneous recurrence relation is

$$a_n = a_{n-1} + a_{n-2}.$$

Linear nonhomogeneous recurrence relations with constant coefficients

Example 13

The recurrence relation

$$a_n = 3a_{n-1} + n3^n$$

is a linear nonhomogeneous recurrence relation with constant coefficients.

The associated linear homogeneous recurrence relation is

$$a_n = 3a_{n-1}.$$

Linear nonhomogeneous recurrence relations with constant coefficients

Example 14

The recurrence relation

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

is a linear nonhomogeneous recurrence relation with constant coefficients.

The associated linear homogeneous recurrence relation is

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

Linear nonhomogeneous recurrence relations with constant coefficients

<u>Theorem 4</u>

If $\left\{a_n^{(p)}\right\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$,

then every solution is of the form $\left\{a_n^{(p)} + a_n^{(h)}\right\}$,

where $\left\{a_n^{(h)}\right\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Proof of theorem 4

- Because $\left\{a_n^{(p)}\right\}$ is a particular solution of the nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

we know that

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n).$$

Proof of theorem 4

- Now suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

so that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n).$$

Proof of theorem 4

So,

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n),$$

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} =$$

$$= c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + \cdots + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\left\{b_n - a_n^{(p)}\right\}$ is a solution of the associated homogeneous linear recurrence relation, say, $\left\{a_n^{(h)}\right\}$.

Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ ■

Linear nonhomogeneous recurrence relations with constant coefficients

Example 15

Find all solutions of the recurrence relation

$$a_n = 3a_{n-1} + 2n.$$

Solution

This is a linear nonhomogeneous recurrence relation.

The solutions of its associated homogeneous recurrence relation

$$a_n = 3a_{n-1}$$

are $a_n^{(h)} = \alpha \cdot 3^n$.

Linear nonhomogeneous recurrence relations with constant coefficients

Example 15

Find all solutions of the recurrence relation

$$a_n = 3a_{n-1} + \mathbf{2n}.$$

Solution

We now find a particular solution.

Suppose that $p_n = \boldsymbol{cn + d}$, where c and d are constants, such a solution.

$$cn + d = 3(c(n-1) + d) + 2n,$$

$$(2 + 2c)n + (2d - 3c) = 0.$$

Linear nonhomogeneous recurrence relations with constant coefficients

Example 15

Find all solutions of the recurrence relation

$$a_n = 3a_{n-1} + \mathbf{2n}.$$

Solution

$$(2 + 2c)n + (2d - 3c) = 0,$$

$$\begin{cases} 2 + 2c = 0, \\ 2d - 3c = 0, \end{cases} \Rightarrow$$

$$c = -1, d = -\frac{3}{2}, \Rightarrow$$

$$a_n^{(p)} = -n - 3/2, \Rightarrow$$

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha \cdot 3^n.$$

Linear nonhomogeneous recurrence relations with constant coefficients

Example 16

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

Solution

This is a linear nonhomogeneous recurrence relation.

The solutions of its associated homogeneous recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

are $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$.

Linear nonhomogeneous recurrence relations with constant coefficients

Example 16

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + \mathbf{7^n}.$$

Solution

We now find a particular solution.

Suppose that $F(n) = \boldsymbol{C \cdot 7^n}$, where C is a constant, such a solution.

$$C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n,$$

$$49C = 35C - 6C + 49,$$

$$C = 49/20.$$

Linear nonhomogeneous recurrence relations with constant coefficients

Example 16

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + \mathbf{7^n}.$$

Solution

$a_n^{(p)} = (49/20)7^n, \Rightarrow$

$a_n = a_n^{(p)} + a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$

Generating functions

Definition 3

The **generating function** for the sequence

$$a_0, a_1, \ldots, a_k, \ldots$$

of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \ \ldots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Generating functions

Example 17

The generating function for the sequence

$$\{a_k\}, a_k = 3,$$

is

$$3 + 3x + \ ... + 3x^k + \cdots = \sum_{k=0}^{\infty} 3x^k$$

Generating functions

Example 18

The generating function for the sequence

$$\{a_k\}, a_k = k+1,$$

is

$$1 + 2x + \ ... + (k+1)x^k + \cdots = \sum_{k=0}^{\infty} (k+1)x^k$$

Generating functions

Example 19

The generating function for the sequence

$$\{a_k\}, a_k = 2^k,$$

is

$$1 + 2x + \ \ldots + 2^k x^k + \cdots = \sum_{k=0}^{\infty} 2^k x^k$$

Generating functions

- We can define generating functions for finite sequences of real numbers by extending a finite sequence

$$a_0, a_1, \ldots, a_n,$$

into an infinite sequence by setting

$$a_{n+1} = 0, a_{n+2} = 0, \text{ and so on.}$$

The generating function of this infinite sequence is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + \ldots + a_n x^n.$$

Generating functions

Example 20

The generating function of

$$1, 1, 1, 1, 1, 1$$

is

$$1 + x + x^2 + x^3 + x^4 + x^5.$$

We have

$$(x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$$

when $x \neq 1$.

Consequently, $G(x) = (x^6 - 1)/(x - 1)$ is the generating function of the sequence $1, 1, 1, 1, 1, 1$.

Generating functions

Example 21

Let m be a positive integer.

The generating function $G(x)$ for the sequence

$$\{a_k\}, a_k = C(m, k) \text{ with } k = 0, 1, 2, \ldots, m$$

is

$$C(m, 0) + C(m, 1)x + C(m, 2)x^2 + \cdots + C(m, m)x^m.$$

The binomial theorem shows that $G(x) = (1 + x)^m$.

Generating functions

Example 22

The function

$$f(x) = 1/(1-x)$$

is the generating function of the sequence

$$1, 1, 1, 1, \dots,$$

because

$$1/(1-x) = 1 + x + x^2 + \cdots$$

for $|x| < 1$.

Generating functions

Example 23

The function

$$f(x) = 1/(1 - ax)$$

is the generating function of the sequence

$$1, a, a^2, a^3, \dots,$$

because

$$1/(1 - ax) = 1 + ax + a^2x^2 + \cdots$$

for $|ax| < 1$.

Using generating functions to solve recurrence relations

Example 24

Solve the recurrence relation

$$a_k = 3a_{k-1}, \text{for } k = 1, 2, 3, \ldots$$

and initial condition $a_0 = 2$.

$a_k = 3a_{k-1}, a_0 = 2$

Solution of example 24

Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is,

$$G(x) = \sum_{k=0}^{\infty} a_k x^k .$$

First note that

$$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

$a_k = 3a_{k-1}, a_0 = 2$

Solution of example 24

$$G(x) - 3xG(x) = \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1} x^k$$

$$= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1})\, x^k$$

$$= 2 + \sum_{k=0}^{\infty} 0 \cdot x^k$$

$$= 2$$

$a_k = 3a_{k-1}, a_0 = 2$

- Solution of example 24

$$G(x) - 3xG(x) = 2$$

$$\Downarrow$$

$$(1 - 3x)G(x) = 2$$

$$\Downarrow$$

$$G(x) = 2/(1 - 3x)$$

$$\Downarrow$$

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

$a_k = 3a_{k-1}, a_0 = 2$

- Solution of example 24

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

$$\Downarrow$$

$$a_k = 2 \cdot 3^k$$

Using generating functions to solve recurrence relations

<u>Example 25</u>

Solve the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1} \text{ for } n = 2, 3, 4, \ldots$$

and initial condition $a_1 = 9$.

Suppose that a valid codeword is an n-digit number in decimal notation containing an even number of 0s. Let a_n denote the number of valid codewords of length n.

Proof that a_n satisfies the recurrence relation $a_n = 8a_{n-1} + 10^{n-1}$ and the initial condition $a_1 = 9$.

Use generating functions to find an explicit formula for a_n.

$$a_n = 8a_{n-1} + 10^{n-1}, a_1 = 9$$

Solution of example 25

To make our work with generating functions simpler, we extend this sequence by setting $a_0 = 1$ and use the recurrence relation, we have

$$a_1 = 8a_{1-1} + 10^{1-1} = 8a_0 + 10^0 = 8 + 1 = 9,$$

which is consistent with our original initial condition.

(It also makes sense because there is one code word of length 0 – the empty string.)

$a_n = 8a_{n-1} + 10^{n-1}, a_0 = 1$

Solution of example 25

$$a_n = 8a_{n-1} + 10^{n-1}$$
$$\Downarrow$$
$$a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n$$

Let

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function of the sequence $a_0, a_1, \dots, a_n, \dots$.

$$a_n = 8a_{n-1} + 10^{n-1}, a_0 = 1$$

Solution of example 25

$$\boldsymbol{a_n x^n = 8a_{n-1} x^n + 10^{n-1} x^n}, G(x) = \sum_{n=0}^{\infty} a_n x^n$$

We sum both sides of the last equation starting with $n = 1$, to find that

$$G(x) - 1 = \sum_{n=1}^{\infty} \boldsymbol{a_n x^n} = \sum_{n=1}^{\infty} (\boldsymbol{8a_{n-1} x^n + 10^{n-1} x^n})$$

$$a_n = 8a_{n-1} + 10^{n-1}, a_0 = 1$$

Solution of example 25

$$G(x) - 1 = \sum_{n=1}^{\infty} (\mathbf{8a_{n-1}x^n + 10^{n-1}x^n})$$

$$= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1}$$

$$a_n = 8a_{n-1} + 10^{n-1}, a_0 = 1$$

Solution of example 25

$$G(x) - 1 = 8x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1} x^{n-1}$$

$$= 8x \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} 10^n x^n$$

$$= 8xG(x) + x/(1 - 10x)$$

$$a_n = 8a_{n-1} + 10^{n-1}, a_0 = 1$$

- Solution of example 25

$$G(x) - 1 = 8xG(x) + x/(1 - 10x)$$

$$\Downarrow$$

$$G(x) = \frac{1 - 9x}{(1 - 8x)(1 - 10x)}$$

Expanding the right-hand side of this equation into partial fractions gives

$$\frac{1 - 9x}{(1 - 8x)(1 - 10x)} = \frac{1}{2}\left(\frac{1}{1 - 8x} + \frac{1}{1 - 10x}\right)$$

$$a_n = 8a_{n-1} + 10^{n-1}, a_0 = 1$$

<u>Solution of example 25</u>

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)} = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n) x^n$$

$$a_n = 8a_{n-1} + 10^{n-1}, a_0 = 1$$

Solution of example 25

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)\, x^n$$

$$\Downarrow$$

$$a_n = \frac{1}{2}(8^n + 10^n)$$