NUFYP Mathematics

4.3 Limits

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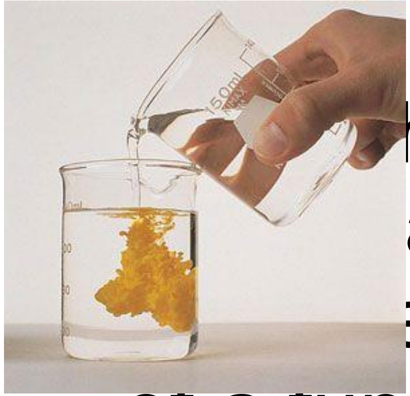
Lecture Outline

- One-sided Limits
- Calculating Limits
  - Numerically (previous lecture)
  - **Graphically**
  - **Algebraically**

Introduction

Air resistance prevents the velocity of a skydiver from increasing indefinitely. The velocity approaches the "terminal velocity".

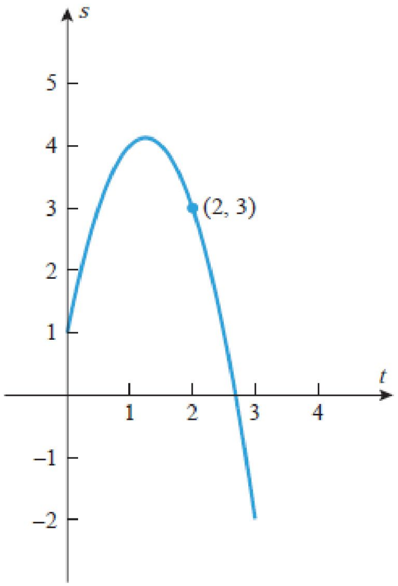


hen two compounds are combined a beaker to form a new compound, e amount of new compound is the limit of a function as time goes to infinity.

Computing limits graphically

- When you know how a certain function looks like, any limit can easily be determined by following the graph of the function.

Example 1: The function $s(t) = 1 + 5t - 2t^2$ has the graph on the right. Compute the limits:



$a)\lim_{t\to 2} s(t)$

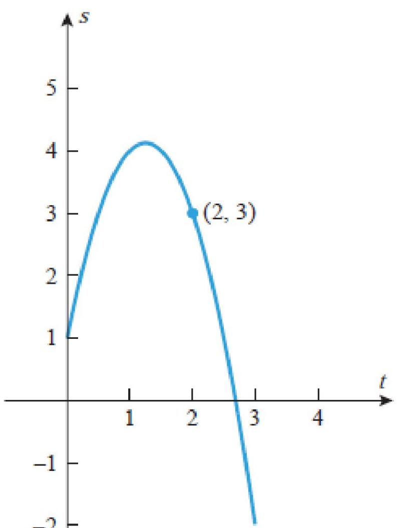
$b)\lim_{t\to +\infty} s(t)$

Computing limits graphically

Solution: We will follow the graph to find the limits

a) No matter from which side you approach 2, the function will approach 3. Hence,

$$\lim_{t \to 2} s(t) = 3$$



$s(t) = 1 + 5t - 2t^2$

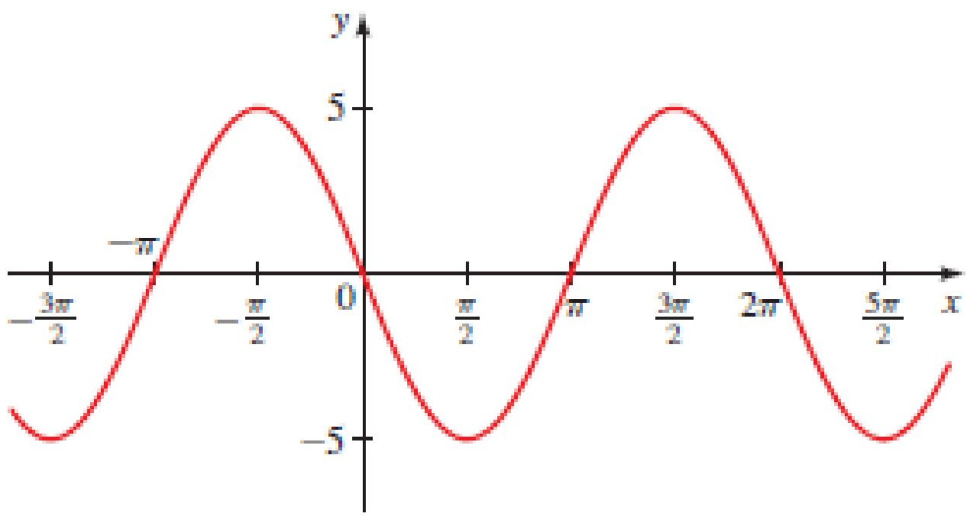
b) As you know, a parabola which is concave down will decrease **without bound**. Hence, $\lim_{t \to +\infty} s(t) = -\infty$

Example 2: Compute the limits of the transformed sine function $y(x)$

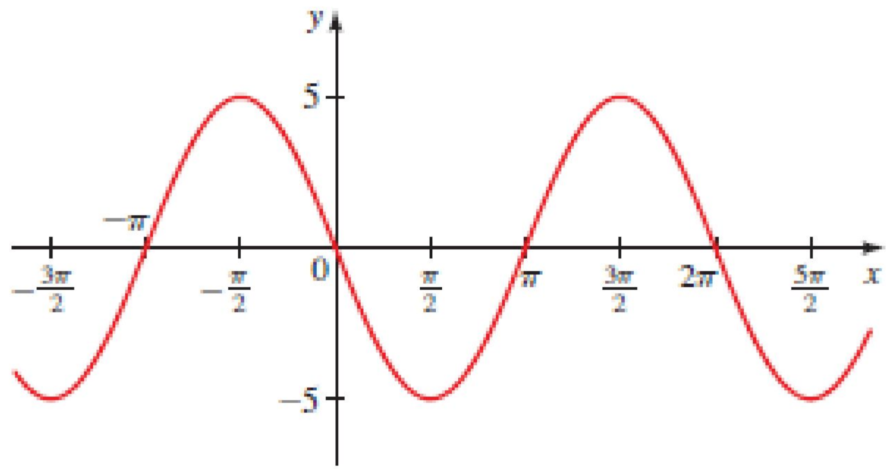
a) $\lim_{x \to \pi} y(x)$

b) $\lim_{x \to \frac{3\pi}{2}} y(x)$

c) $\lim_{x \to -\infty} y(x)$



Solution:



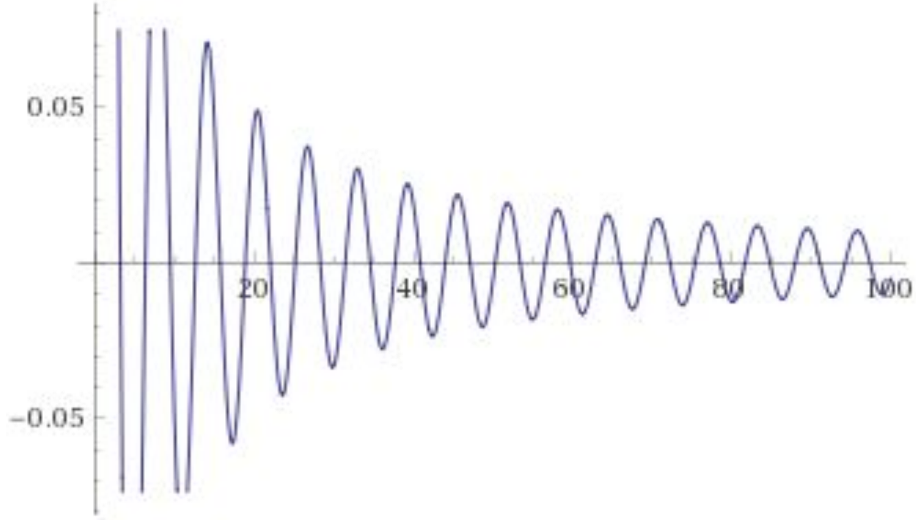
a) $\lim_{x \to \pi} y(x) = 0$

b) $\lim_{x \to \frac{3\pi}{2}} y(x) = 5$

c) $\lim_{x \to -\infty} y(x)$ does not exist, because sine keeps oscilating between -5 and 5, and it will not tend to a specific value.

Your turn!

Compute the limit at positive infinity of the function $y(x)$ sketched below.

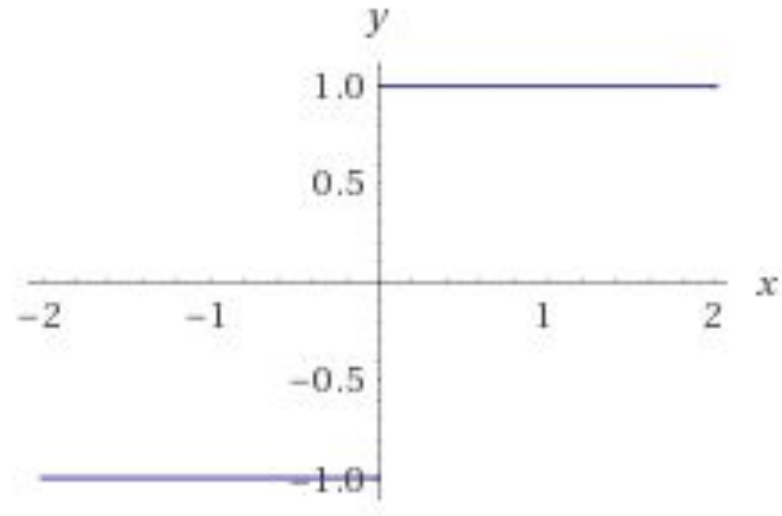


Solution: Even though this function also keeps oscillating, the values between which it varies become smaller and smaller (positive and negative), approaching 0. Thus, $\lim_{x \to +\infty} y(x) = 0$

One-sided limits

Example 3: Consider the function

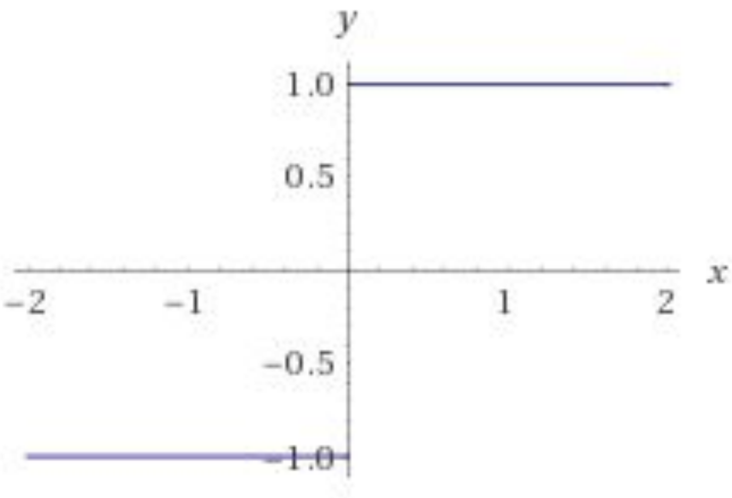
$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$$



If we ask what is the limit when x approaches 0, we get different answers depending on whether we follow the graph from the right or from the left.

Example 3:

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$$



$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad \text{but} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

The **superscript +** in 0^+, indicates a limit from the **right** and the **superscript -** in 0^-, indicates a limit from the **left**. This leads to the idea of a one-sided limit.

One-sided limits

If the values of $f(x)$ can be made as close as we like to *L* by taking values of x sufficiently close to *a*, but greater than *a*, then we write

$$\lim_{x \to a^+} f(x) = L$$

If the values of $f(x)$ can be made as close as we like to *L* by taking values of x sufficiently close to *a*, but less than *a*, then we write

$$\lim_{x \to a^-} f(x) = L$$

We read "the limit of $f(x)$ as x approaches *a* from the right (resp. left) is *L*."

One-sided and two-sided limits

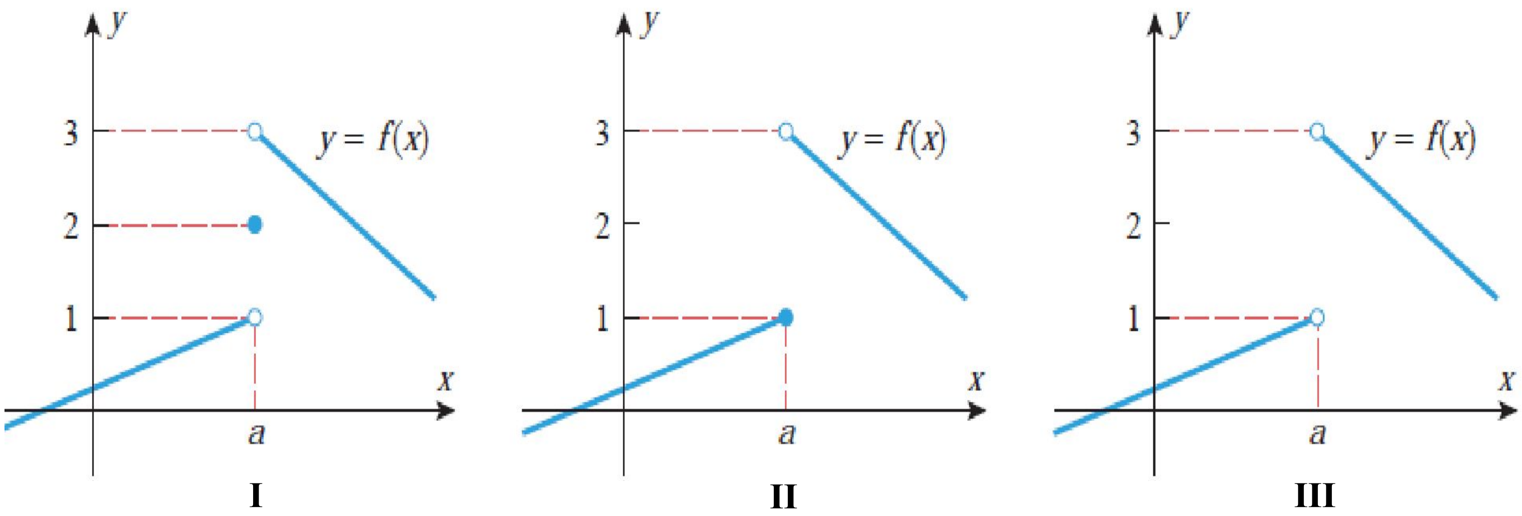
- The two-sided limit of a function $f(x)$ (often called just limit) exists at a if and only if **both** of the one-sided limits exist at a **and** have the same value:

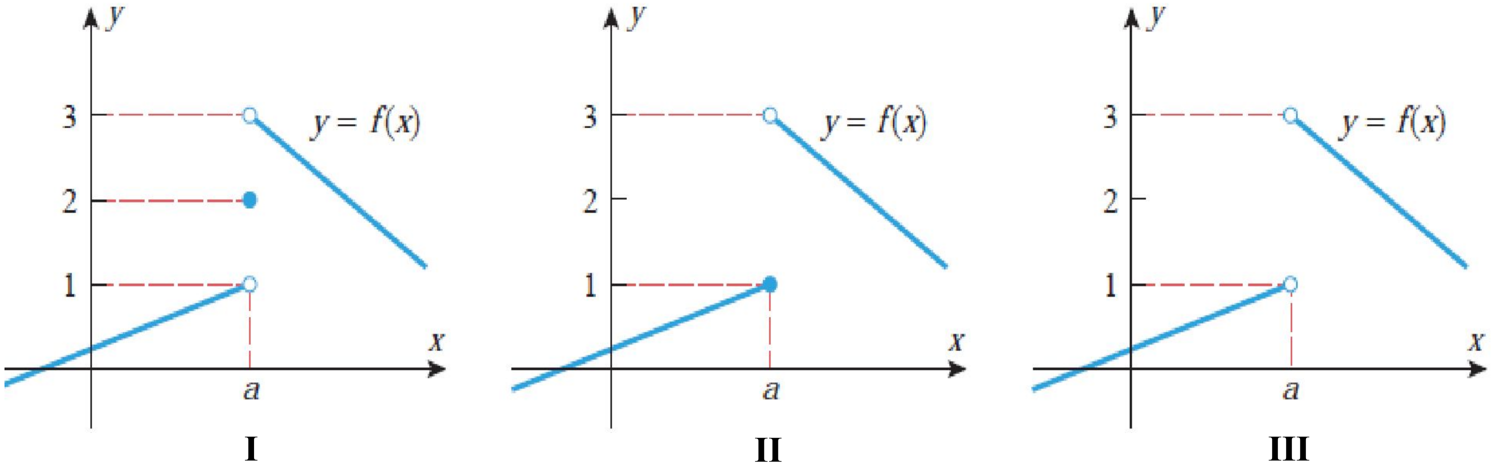
$$\lim_{x \to a} f(x) = L \;\; iif \;\; \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

What does the iif mean?

What can you conclude about the limit at $x = 0$

Example 4: For the functions presented find the one-sided and the two-sided limits at $x = a$, if they exist, as well as the value of the function at a.





In all three cases we have

$$\lim_{x\to a^-} f(x) = 1,\ \lim_{x\to a^+} f(x) = 3,\ \lim_{x\to a} f(x) \text{ DNE}$$

The values of the functions at a are:

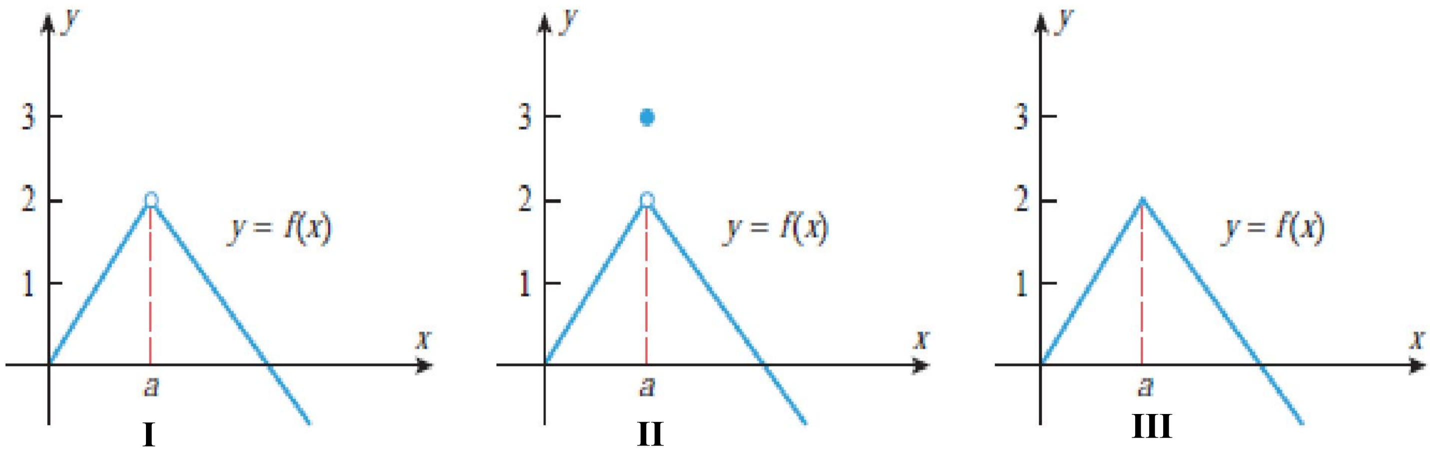
I. $f(a) = 2$

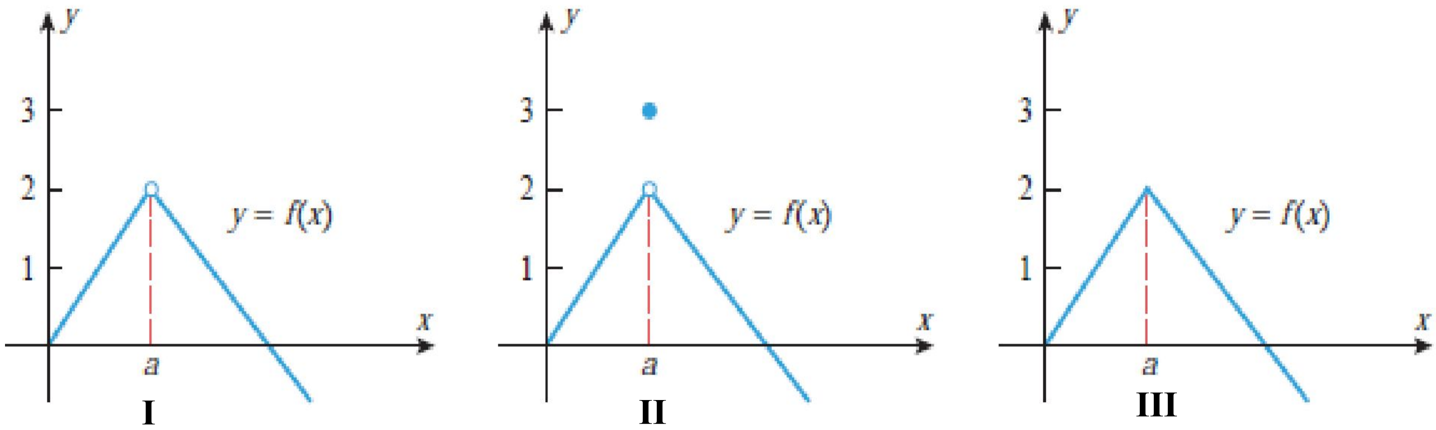
II. $f(a) = 1$

III. $f(a)$ is not defined.

Your turn!

- For the functions presented find the one-sided and the two-sided limits at $x = a$, if they exist, as well as the value of the function at a.





I. $\lim_{x \to a^-} f(x) = 2, \lim_{x \to a^+} f(x) = 2, \lim_{x \to a} f(x) = 2$

f(a) is not defined

II. $\lim_{x \to a^-} f(x) = 2, \lim_{x \to a^+} f(x) = 2, \lim_{x \to a} f(x) = 2$

$f(a) = 3$

III. $\lim_{x \to a^-} f(x) = 2, \lim_{x \to a^+} f(x) = 2, \lim_{x \to a} f(x) = 2$,

$f(a) = 2$

Example 5: Let

$$f(x) = \begin{cases} 1/(x+2) & x < -2 \\ x^2 - 5, & -2 < x \leq 3 \\ \sqrt{x+13} & x > 3 \end{cases}$$

Find

a) $\lim_{x \to -2} f(x)$

b) $\lim_{x \to 0} f(x)$

Solution:

$$f(x) = \begin{cases} 1/(x+2) & x < -2 \\ x^2 - 5, & -2 < x \le 3 \\ \sqrt{x+13} & x > 3 \end{cases}$$

a) $\lim_{x \to -2} f(x)$

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{1}{x+2} = \frac{1}{0^-} = -\infty$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x^2 - 5) = -1$$

- $$f(x) = \begin{cases} 1/(x+2), & x < -2 \\ x^2 - 5, & -2 < x \leq 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$$

b) $\lim_{x\to 0} f(x) = \lim_{x\to 0}(x^2 - 5) = -5$

c) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(x^2 - 5) = 4$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}\sqrt{x+13} = 4$

Therefore, $\lim_{x\to 3} f(x) = 4$

Indeterminations

Remember from the previous lecture that situations such as

$$\frac{0}{0}, \qquad 0^0, \qquad \infty \cdot 0, \qquad \infty^0$$

$$+\infty - \infty, \qquad \frac{\infty}{\infty}, \qquad 1^\infty$$

are **indeterminations**. We can work our way around them by

- factorising and cancelling out the like terms;
- rewriting the expression; or
- conjugate multiplication.

Example 6:

In the previous lecture we conjectured that

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}\ -1}=2$$

Now let's compute the limit analytically to show it is true.

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}\ -1}\qquad \left(=\frac{0}{0}\quad Indetermination!!!\ \right)$$

Using conjugate multiplication, we have

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}\ -1}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}\ -1)(\sqrt{x}+1)}$$

$$=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-1}=\lim_{x\to 1}(\sqrt{x}+1)=2$$

Example 7: Compute, if possible, the limit

$$\lim_{n\to+\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$$

Solution:

$$\lim_{n\to+\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=+\infty-\infty$$

Multiplying and dividing by $\sqrt{n+1}+\sqrt{n}$ leads to

$$\lim_{n\to+\infty}\left(\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}\right)$$

$$=\lim_{n\to+\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to+\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=0$$

Example 8:

Compute the following limits

1. $lim_{x\to 2}\frac{x^4-16}{x-2}$
2. $lim_{x\to 1}(x^7-2x^5+1)^{35}$
3. $lim_{x\to 4}\frac{x-2}{(x-4)(x+2)}$
4. $lim_{x\to +\infty}\frac{6x+4}{2x-7}$ and $lim_{x\to -\infty}\frac{6x+4}{2x-7}$
5. $lim_{x\to +\infty}\frac{4x^2-x}{2x^3-5}$ and $lim_{x\to -\infty}\frac{4x^2-x}{2x^3-5}$

Solution:

1. $\lim_{x\to 2}\frac{x^4-16}{x-2} = \frac{2^4-16}{2-2}\ \left(= \frac{0}{0}\ Indetermination!!!\right)$

$x^4 - 16 = (x+2)(x-2)(x^2+4)$

$\lim_{x\to 2}\frac{x^4-16}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)(x^2+4)}{x-2}$

$= \lim_{x\to 2}(x+2)(x^2+4) = 32$

2. $\lim_{x\to 1}(x^7 - 2x^5 + 1)^{35} = (1 - 2 + 1)^{35} = 0$

3. $lim_{x \to 4} \frac{x-2}{(x-4)(x+2)}$

The first thing to notice is that the sign of $(x - 2)$ and $(x + 2)$ does not change but the sign of $(x - 4)$ in the denominator will change depending on how $x \to 4$:

- When $x \to 4^+$, $x - 4 > 0$
- When $x \to 4^-$, $x - 4 < 0$

So, we can suspect that the left and right limits will be different.

- $$lim_{x\to 4^+}\frac{x-2}{(x-4)(x+2)}=\frac{2}{0^+\cdot 6}=+\infty$$
$$lim_{x\to 4^-}\frac{x-2}{(x-4)(x+2)}=\frac{2}{0^-\cdot 6}=-\infty$$

Therefore, $lim_{x\to 4}\frac{x-2}{(x-4)(x+2)}$ does not exist.

4. $lim_{x\to +\infty}\frac{6x+4}{2x-7}$ $\left(=\frac{+\infty}{+\infty}\ Indetermination\right)$

$$lim_{x\to +\infty}\frac{\frac{6x+4}{x}}{\frac{2x-7}{x}}=lim_{x\to +\infty}\frac{6+\frac{4}{x}}{2-\frac{7}{x}}=\frac{6}{2}=3$$

$6+\frac{4}{}$ (

- 5. $lim_{x\to+\infty} \frac{4x^2-x}{2x^3-5} \quad \left(= \frac{+\infty}{+\infty} \; indetermination\right)$

$$lim_{x\to+\infty} \frac{\frac{4x^2-x}{x^2}}{\frac{2x^3-5}{x^2}} = lim_{x\to+\infty} \frac{4-\frac{1}{x}}{2x-\frac{5}{x^2}} = \frac{4-0}{+\infty-0} = 0$$

$$lim_{x\to-\infty} \frac{4-\frac{1}{x}}{2x-\frac{5}{x^2}} = \frac{4-0}{-\infty-0} = 0$$

Example 9:

Find the limit $lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}$ and $lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}$

Solution:

$$lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6} \qquad \left(=\frac{+\infty}{+\infty} \quad Indetermination!!!\right)$$

Note the x^2 under the square root, and recall that $\sqrt{x^2}=|x|$.

Dividing the numerator and denominator by $|x|$ we have

$$lim_{x\to+\infty}\frac{\frac{\sqrt{x^2+2}}{|x|}}{\frac{3x-6}{|x|}}=lim_{x\to+\infty}\frac{\frac{\sqrt{x^2+2}}{\sqrt{x^2}}}{\frac{3x-6}{x}}=lim_{x\to+\infty}\frac{\sqrt{1+\frac{2}{x^2}}}{3-\frac{6}{x}}=\frac{1}{3}$$

Following the same reasoning,

$$lim_{x\to-\infty}\frac{\frac{\sqrt{x^2+2}}{|x|}}{\frac{3x-6}{|x|}}=lim_{x\to-\infty}\frac{\frac{\sqrt{x^2+2}}{\sqrt{x^2}}}{\frac{3x-6}{(-x)}}=lim_{x\to-\infty}\frac{\sqrt{1+\frac{2}{x^2}}}{-3+\frac{6}{x}}=-\frac{1}{3}$$

Your turn!

- Compute $\lim_{x\to+\infty}\left(\sqrt{x^2+3x+2}-\sqrt{x^2+1}\right).$

Your turn!

- Compute $\lim_{x\to+\infty}\left(\sqrt{x^2+3x+2}-\sqrt{x^2+1}\right)$

$(= -\infty + \infty \ \ \mathit{Indetermination}!!)$

$$= \lim_{x\to+\infty}\frac{\left(\sqrt{x^2+3x+2}-\sqrt{x^2+1}\right)\left(\sqrt{x^2+3x+2}+\sqrt{x^2+1}\right.}{\sqrt{x^2+3x+2}+\sqrt{x^2+1}}$$

$$= \lim_{x\to+\infty}\frac{x^2+3x+2-x^2-1}{\sqrt{x^2+3x+2}+\sqrt{x^2+1}}$$

$$= \lim_{x\to+\infty}\frac{3x+1}{\sqrt{x^2+3x+2}+\sqrt{x^2+1}}$$

- $$= \lim_{x\to+\infty} \frac{3x+1}{\sqrt{x^2\left(1+\frac{3}{x}+\frac{2}{x^2}\right)}+\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}$$

$$= \lim_{x\to+\infty} \frac{3x+1}{\sqrt{x^2\left(1+\frac{3}{x}+\frac{2}{x^2}\right)}+\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}$$

$$= \lim_{x\to+\infty} \frac{3x+1}{x\sqrt{1+\frac{3}{x}+\frac{2}{x^2}}+x\sqrt{1+\frac{1}{x^2}}}$$

1

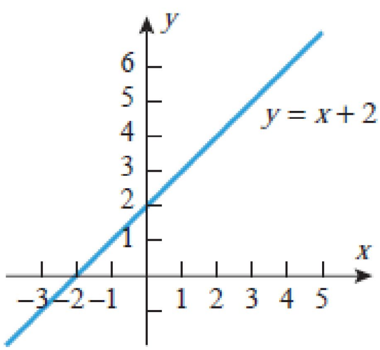
Limits at infinity

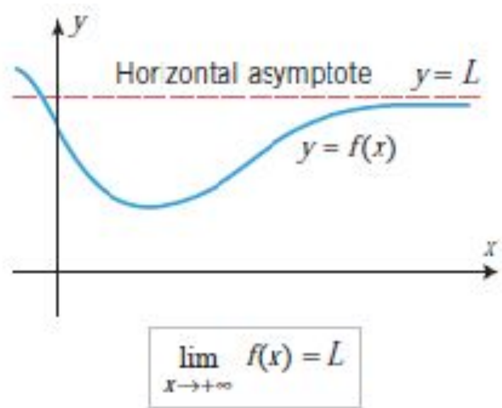
- Just like we studied what happens to sequences as n became larger and larger, we can study what happens to a function $f(x)$ as x becomes larger and larger (positive or negative), i.e. as $x \to +\infty$ or $x \to -\infty$.

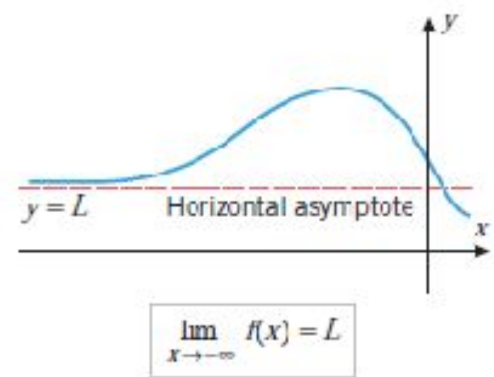
Since these values of x are “at the end” of the line of real numbers, what happens to the function there is often called the **end behaviour of the function**.

Most often, one of two cases will happen:

1. The function keeps increasing (or decreasing) without bound.
2. The function will approach a finite value. We say the function has a **horizontal asymptote**.







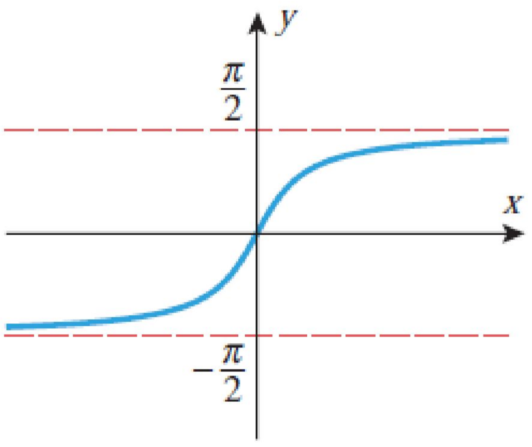
an a graph cross a horizontal asymptote?

End behaviour of a function

The **end behaviour of the function** $f(x)$ is determined by computing the limits:

$$lim_{x\to+\infty}\ f(x) \quad \textbf{and} \quad lim_{x\to-\infty}\ f(x)$$

Note that a function can have two horizontal asymptotes if the limits at limits at positive / negative infinity have different values.



$y = \tan^{-1} x$

Example 10:

Determine the end behaviour of the functions:

1. $f(x) = 3x^2 - 77$
2. $g(x) = \sqrt{x + 26}$
3. $h(x) = \arctan x + 4$

Solution:

1. $f(x) = 3x^2 - 77$

$lim_{x \to +\infty} f(x) = +\infty$ and $lim_{x \to -\infty} f(x) = +\infty$

2. $g(x) = \sqrt{x + 26}$

$lim_{x \to +\infty} g(x) = +\infty$

$g(x)$ is not defined for $x < -26$, so we cannot compute $lim_{x \to -\infty} g(x)$.

3. $h(x) = \arctan x + 4$

Your turn!

Determine the end behaviour of the functions:

1. $f(x) = 3x^4 + 5x$
2. $g(x) = 2x^3 + 7$
3. $h(x) = 3x^5 + 5x^2$
4. $s(x) = \frac{26}{\sqrt{x+26}}$

How can we relate the leading term of a

Your turn!

- Determine the end behaviour of the functions:

1. $f(x) = 3x^4 + 5x$

$lim_{x \to +\infty} f(x) = +\infty$ and $lim_{x \to -\infty} f(x) = +\infty$

2. $g(x) = 2x^3 + 7$

$lim_{x \to +\infty} g(x) = +\infty$ and $lim_{x \to -\infty} g(x) = -\infty$

3. $h(x) = 3x^5 + 5x^2$

$lim_{x \to +\infty} h(x) = +\infty$ and $lim_{x \to -\infty} h(x) = -\infty$

Infinite limits

Sometimes a function is not defined for some values of x. It can happen that around those points it increases or decreases without bound.

Suppose the function $f(x)$ is not defined when $x = a$. If

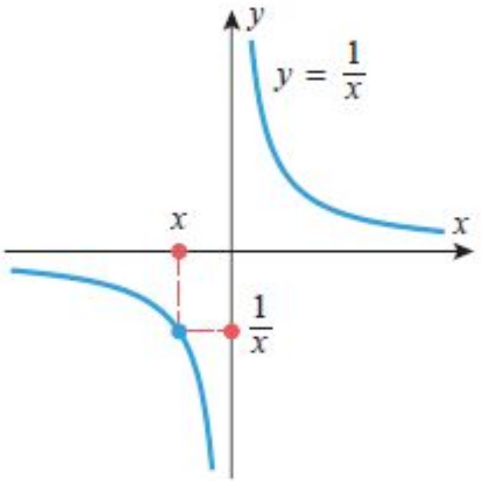
$lim_{x \to a^-} f(x) = +\infty$ or $lim_{x \to a^+} f(x) = +\infty$

or

$lim_{x \to a^-} f(x) = -\infty$ or $lim_{x \to a^+} f(x) = -\infty$

Example 11:

Show that the line $x = 0$ is a vertical asymptote of the function $y = \frac{1}{x}$



Solution: All we need to do is to show that one of the limits from the previous slide is indeed infinite. For practice, we will compute both of them.

$lim \frac{1}{x} = -\infty$ or $lim \frac{1}{x} = +\infty$

Example 12:

- Compute, if possible, the following limits:

1. $\lim_{x\to 3} \frac{-3^x}{3-x}$
2. $\lim_{x\to 2} \frac{1-2x+x^2}{x^2-4}$
3. $\lim_{x\to 1} \frac{\cos x}{x-1}$

Solution:

1. $\lim_{x \to 3} \frac{-3^x}{3-x}$

Note how the sign of the denominator changes depending on how we approach 3:

$$\lim_{x \to 3^+} \frac{-3^x}{3-x} = \frac{-27}{0^-} = +\infty$$

$$\lim_{x \to 3^-} \frac{-3^x}{3-x} = \frac{-27}{0^+} = -\infty$$

2. $\lim_{x\to 2} \frac{1-2x+x^2}{x^2-4}$

Note how the sign of the denominator changes depending on how we approach 2:

$$\lim_{x\to 2^+} \frac{1-2x+x^2}{x^2-4} = \frac{1}{0^+} = +\infty$$

$$\lim_{x\to 2^-} \frac{1-2x+x^2}{x^2-4} = \frac{1}{0^-} = -\infty$$

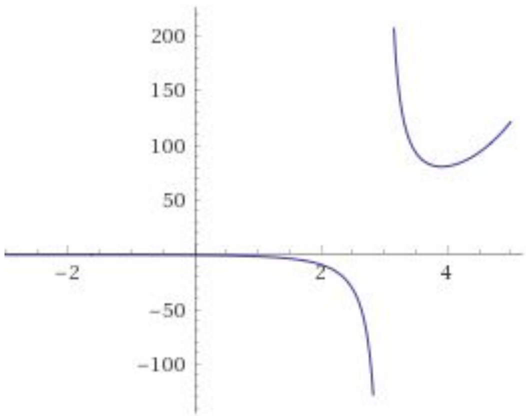
3. $$\lim_{x\to 1} \frac{\cos x}{x-1}$$

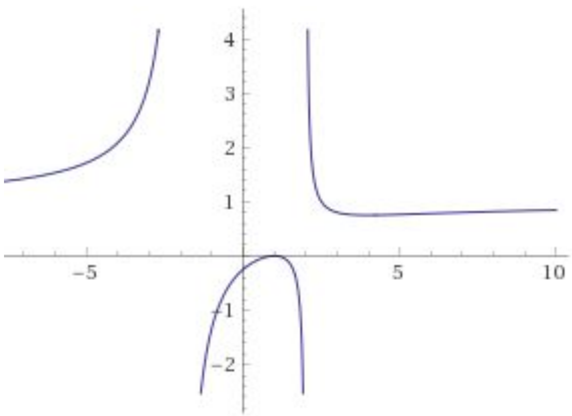
Note how the sign of the denominator changes depending on how we approach 1:

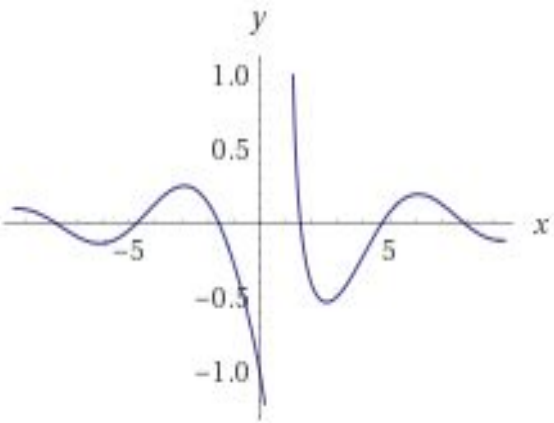
$$\lim_{x\to 1^+} \frac{\cos x}{x-1} = \frac{\cos 1}{0^+} = +\infty$$

$$\lim_{x\to 1^-} \frac{\cos x}{x-1} = \frac{\cos 1}{0^-} = -\infty$$

All the functions in the previous example have vertical asymptotes at the values of x for which we computed the limit:







$$y = \frac{-3^x}{3-x}$$

$$y = \frac{1-2x+x^2}{x^2-4}$$

$$y = \frac{\cos x}{x-1}$$

Solution:

1. $\lim\limits_{x\to+\infty}(-3^x+2^x) \quad (= -\infty + \infty \quad \mathit{Indetermination!!})$

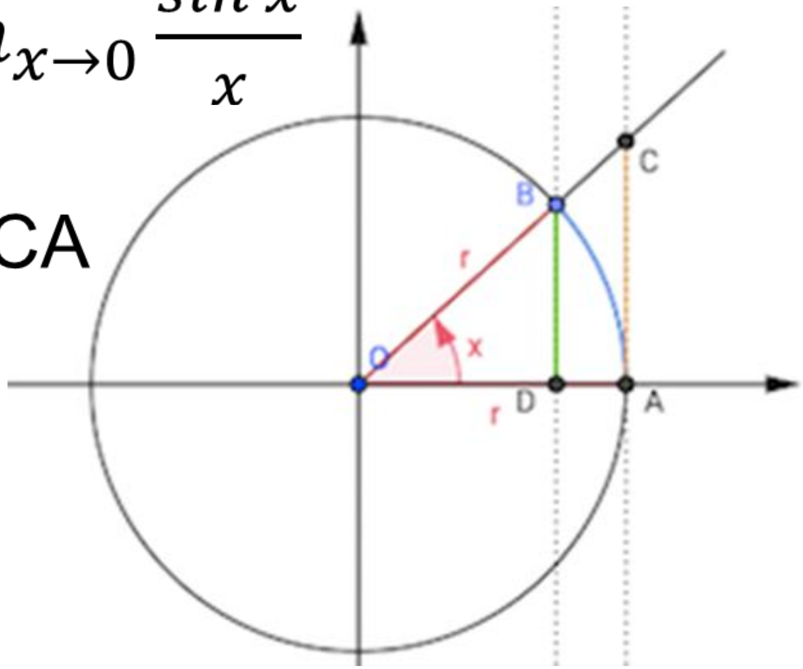
$$\lim_{x\to+\infty} -3^x\left(1-\frac{2^x}{3^x}\right) = \lim_{x\to+\infty} -3^x\left(1-\left(\frac{2}{3}\right)^x\right)$$

$$= \lim_{x\to+\infty} -3^x \cdot \lim_{x\to+\infty}\left(1-\left(\frac{2}{3}\right)^x\right) = (-3^\infty \cdot 1) = -\infty$$

2. $\lim\limits_{x\to+\infty}\frac{1-2x+x^2}{1-x^4} \qquad \left(=\frac{+\infty}{-\infty} \quad \mathit{Indetermination!!}\right)$

$$= \lim_{x\to+\infty}\frac{\frac{1}{x^4}-\frac{2}{x^3}+\frac{1}{x^2}}{\frac{1}{x^4}-1} = \frac{0}{1} = 0$$

Example 13: Compute $\lim_{x\to 0} \frac{\sin x}{x}$



- Since the triangles OBD and OCA are similar, we have

$$\frac{BD}{OD} = \frac{AC}{OA} \Rightarrow \frac{r \sin x}{r \cos x} = \frac{AC}{r} \Rightarrow AC = r \tan x$$

We also see that

$$BD < arc\ AB < AC \Rightarrow r \sin x < rx < r \tan x$$
$$\Rightarrow \sin x < x < \tan x$$

- Dividing all by $\sin x$:

$$\frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\tan x}{\sin x} \Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

Taking the limit when $x \to 0$:

$$\Rightarrow lim_{x\to 0} 1 \leq lim_{x\to 0} \frac{x}{\sin x} \leq lim_{x\to 0} \frac{1}{\cos x}$$

$$\Rightarrow 1 \leq lim_{x\to 0} \frac{x}{\sin x} \leq 1$$

Therefore,

$$lim_{x\to 0} \frac{x}{\sin x} = 1$$

- From

$$lim_{x \to 0} \frac{x}{\sin x} = 1$$

We can conclude that

$$lim_{x \to 0} \frac{\sin x}{x} = 1$$

Which is the same as

$$lim_{h \to 0} \frac{\sin h}{h} = 1$$

Example 14: Compute $lim_{x\to 0}\frac{\cos x-1}{x}$

- We can apply the previous exercise to compute this limit:

$$lim_{x\to 0}\frac{\cos x-1}{x}=lim_{x\to 0}\frac{\cos x-\cos 0}{x}$$

$$=lim_{x\to 0}\frac{-2\sin\left(\frac{x+0}{2}\right)\sin\left(\frac{x-0}{2}\right)}{x}$$

$$=-lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.lim_{x\to 0}\sin\left(\frac{x}{2}\right)=-1\times 0=0$$

Your turn!

Compute $lim_{h\to 0}\frac{e^h-1}{h}$. You may want to use $u = e^h - 1$

Your turn!

Compute $lim_{h\to 0}\frac{e^h-1}{h}$. You may want to use $u = e^h - 1$

- Let us focus on the numerator and introduce a new variable:

$$u = e^h - 1$$

Which is equivalent to:

$u = e^h - 1 \Leftrightarrow e^h = u + 1 \Leftrightarrow \ln\left(e^h\right) = \ln(u+1)$

$\Leftrightarrow h = \ln(u+1)$

Note that as $h \to 0$, $u = e^h - 1$ also approaches 0.

- Replacing on the expression of the limit (and changing *h* for *u* in the limit):

$$lim_{h\to 0}\frac{e^h-1}{h}=lim_{u\to 0}\frac{u}{\ln(u+1)}$$

$$=lim_{u\to 0}\frac{\frac{u}{u}}{\frac{\ln(u+1)}{u}}=lim_{u\to 0}\frac{1}{\frac{1}{u}\ln(u+1)}$$

- $$= lim_{u\to 0}\frac{1}{\ln(u+1)^{\frac{1}{u}}} = lim_{u\to 0}\frac{1}{\ln\left(1+\frac{1}{\frac{1}{u}}\right)^{\frac{1}{u}}}$$

$$= \frac{1}{\ln lim_{u\to 0}\left(1+\frac{1}{\frac{1}{u}}\right)^{\frac{1}{u}}} = \frac{1}{\ln e} = \frac{1}{1} = 1$$

Learning outcomes

After this lecture, you should be able to

4.3.1 Compute limits graphically and analitically;

4.3.2 Compute one-sided limits and determine if the limit of a function at a point exists;

4.3.3 Identify horizontal and vertical asymptotes.

Preview activity: Binomial expansions

Find the pattern of the numbers and guess the numbers in the next two rows.

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

???