
Calculus++ Light
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Sudoku no more!
Playtime's Over

Question 4. Find the limit of the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

Solution. Our sequence can be written down as follows

$$2^{\frac{1}{2}}, 2^{\frac{1}{2}+(\frac{1}{2})^2}, 2^{\frac{1}{2}+(\frac{1}{2})^2+(\frac{1}{2})^3}, 2^{\frac{1}{2}+(\frac{1}{2})^2+(\frac{1}{2})^3+(\frac{1}{2})^4}, \ldots$$

Therefore the n^{th} term of the sequence is given by

$$a_n = 2^{\frac{1}{2}+(\frac{1}{2})^2+(\frac{1}{2})^3+(\frac{1}{2})^4+\cdots+(\frac{1}{2})^n}.$$

Using the formula for the sum of a geometric series we obtain

$$\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 1-\left(\frac{1}{2}\right)^n$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{1-(\frac{1}{2})^n} = 2^{\lim\limits_{n\to\infty} 1-(\frac{1}{2})^n} = 2.$$

Stolz-Cesaro Theorem

Let a_n and b_n be two sequences of real numbers.

Assume that:

I. $b_n \to \infty$ as $n \to \infty$,

II. b_n is increasing for sufficiently large n,

III. $\lim\limits_{n\to\infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} = L.$

Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L.$

Question 1. $a_n = \ln n, \quad b_n = n, \quad \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = ?$

Solution. The conditions I and II of the Stolz-Cesaro theorem are satisfied.

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\lim_{n\to\infty}\frac{\ln(n+1)-\ln n}{n+1-n}$$

$$=\lim_{n\to\infty}\ln\frac{n+1}{n}=\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)$$

$$=\ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)=\ln 1=0.$$

Thus, the Stolz-Cesaro Theorem tells us that

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\ln n}{n}=0.$$

To find the limit $\lim_{n\to\infty}\frac{n^2}{2^n}$

either apply the Stolz-Cesaro Theorem twice, or write it down as a product

$$\lim_{n\to\infty} \frac{n}{\sqrt{2}^n} \frac{n}{\sqrt{2}^n},$$

apply the Stolz-Cesaro Theorem to $\lim_{n\to\infty} \frac{n}{\sqrt{2}^n}$

and then use the product rule

$$\lim_{n\to\infty} x_n y_n = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n.$$

Answers to Questions from Light #1: Sequences and Limits

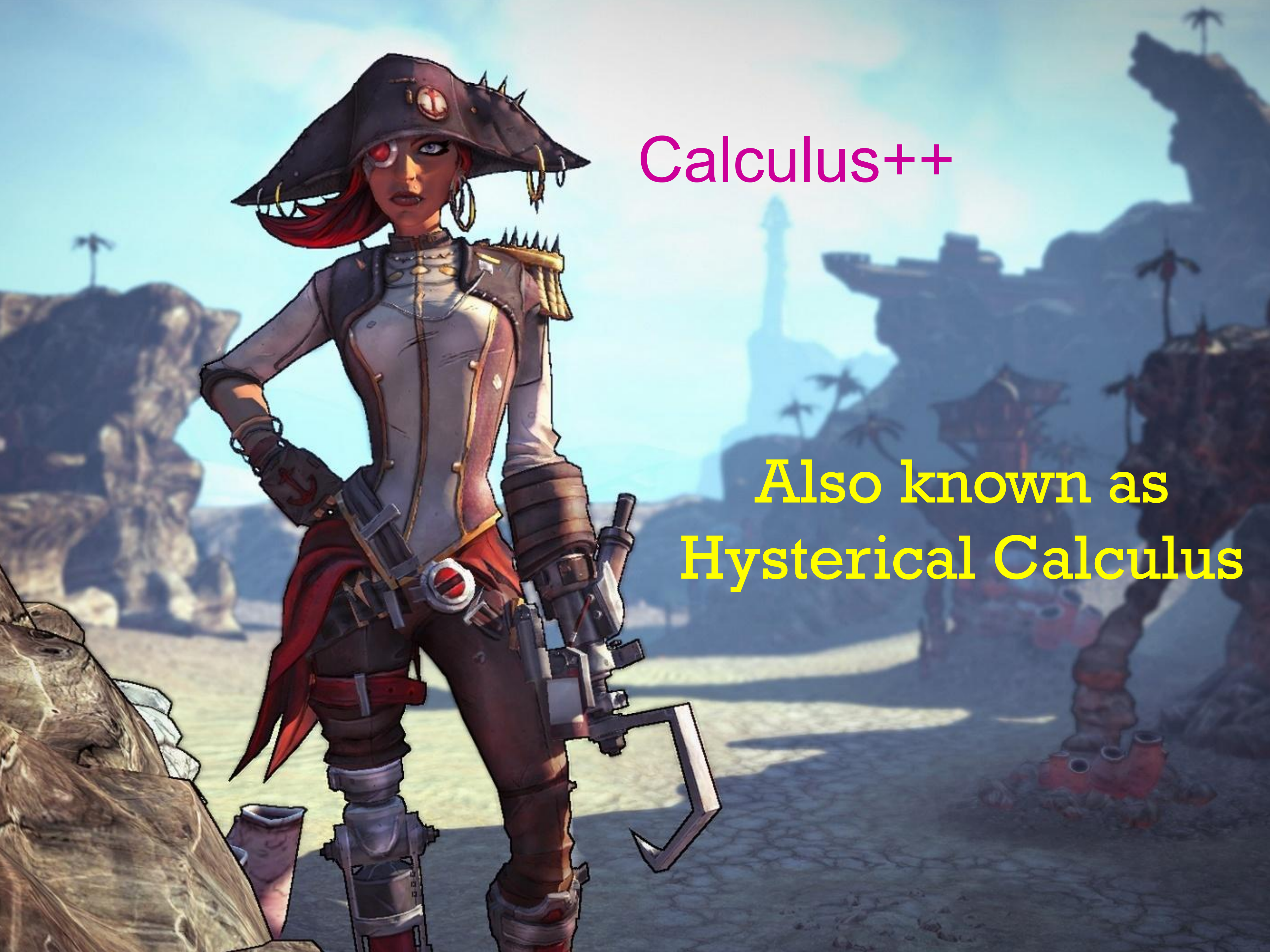
Question 0: $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

Question 1: $\frac{17}{20} = a_{650}$ Question 2: $n_{\min} = 11$

Question 4: 10

Question 5: $x = \log_2\left(\frac{3}{1+2^{2018}}\right)$

$A = 3, \quad B = 2, \quad C = 2018, \quad D = 2$


Calculus++
Also known as Hysterical Calculus

Question 1a. Find the following limit

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}}{\sqrt{n}}.$$

Solution. Use the Stolz-Cesaro theorem.

In this case $b_n = \sqrt{n}.$

The sequence b_n is infinitely large and increasing. Hence, the conditions I and II of the Stolz-Cesaro theorem are satisfied.

$$a_n = \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}$$

$$a_{n+1} = \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}$$

$$\Rightarrow a_{n+1}-a_n = \frac{1}{\sqrt{n+1}}$$

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\frac{1}{\sqrt{n+1}}\frac{1}{\sqrt{n+1}-\sqrt{n}}$$

$$=\frac{1}{\sqrt{n+1}}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$=\frac{1}{\sqrt{n+1}}\frac{\sqrt{n+1}+\sqrt{n}}{1}$$

Hence $\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\lim\limits_{n\to\infty}\left(1+\sqrt{\frac{n}{n+1}}\right)=2.$

The Stolz-Cesaro Theorem tells us that

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}}{\sqrt{n}}=2.$$

Cauchy Criterion

A sequence x_n, $n = 1,2,3,\ldots$ is called a fundamental sequence (or Cauchy sequence) if for any $\varepsilon > 0$ we can find a number N such that, for any $n > N$ and any $m > 0$:

$$\left| x_{n+m} - x_n \right| < \varepsilon.$$

Theorem (Cauchy Criterion). A sequence x_n, $n = 1,2,3,\ldots$, converges if and only if it is a Cauchy sequence.

Definition (of non-fundamental sequences).

A sequence x_n, $n = 1,2,3,\ldots$ is not a Cauchy sequence if we can find $\varepsilon > 0$ such that, for any number N, we can find $n > N$ and $m > 0$, such that $\left|x_{n+m} - x_n\right| > \varepsilon$.

Question 3. The sequence $-1, +1, -1, +1, \ldots$ is not a Cauchy sequence (and, hence, it diverges).

Solution. Let $\varepsilon = 1$, and let N be any natural number. Take $n = 2N + 1,\ m = 1.$

Since n is odd and $n + m$ is even, we have $x_n = -1$ and $x_{n+m} = +1.$

Hence $|x_{n+m} - x_n| = |1 - (-1)| = 2 > \varepsilon.$

Therefore, the sequence

$$\{x_n\} = -1, +1, -1, +1, \ldots$$

is not a Cauchy sequence.

Question 4. Use the Cauchy criterion to show that the sequence

$$x_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n = 1,2,3,\ldots,$$

diverges.

Solution: According to the Cauchy criterion it is sufficient to show that $\{x_n\}$ is not a fundamental sequence:

$$\exists \varepsilon > 0, \forall N, \exists n > N, \exists m > 0 : \left| x_{n+m} - x_n \right| > \varepsilon.$$

We have

$$\left| x_{n+m} - x_n \right| = \left| \sum_{k=n+1}^{n+m} \frac{1}{k} \right| \geq \left| \sum_{k=n+1}^{n+m} \frac{1}{n+m} \right| = \frac{m}{n+m}.$$

Choosing $m = n$ we obtain

$$\left|x_{n+m} - x_n\right| \geq \frac{m}{n+m} = \frac{n}{n+n} = \frac{1}{2}.$$

Thus, $\exists \varepsilon > 0$ (for instance, $\varepsilon = \frac{1}{4}$), $\forall N$
$\exists n > N$ (for instance, $n = N + 1$),
$\exists m > 0$ (we set $m = n$): $\underbrace{\left|x_{n+m} - x_n\right|}_{\geq \frac{1}{2}} > \varepsilon = \frac{1}{4}.$

Therefore, our sequence $\{x_n\}$ is not fundamental, and the Cauchy criterion tells us that $\{x_n\}$ diverges.

Question 5. Use the Cauchy criterion to show that the sequence $x_n = \sum_{k=1}^{n} \frac{1}{k^2}, \quad n = 1, 2, \dots$, converges.

Solution: It is sufficient to show that the sequence x_n is fundamental:

$$\forall \varepsilon > 0, \exists N, \forall n > N, \forall m > 0 : |x_{n+m} - x_n| < \varepsilon.$$

We have $x_{n+m} - x_n = \sum_{k=n+1}^{n+m} \frac{1}{k^2} \leq \sum_{k=n+1}^{n+m} \frac{1}{(k-1)k}$

$$= \sum_{k=n+1}^{n+m} \left(\frac{1}{k-1} - \frac{1}{k} \right) = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} + \dots + \frac{1}{n+m-1} - \frac{1}{n+m}$$

$$= \frac{1}{n} - \frac{1}{n+m} = \frac{m}{n(n+m)} < \frac{1}{n}.$$

Thus $\left| x_{n+m} - x_n \right| < \frac{1}{n}.$

Therefore $\forall \varepsilon > 0, \exists N,$ we set $N = \left\lceil \frac{1}{\varepsilon} \right\rceil,$

$$\forall n > N, \forall m > 0:$$

$$\left| x_{n+m} - x_n \right| < \frac{1}{n} < \frac{1}{N} \le \frac{1}{\varepsilon^{-1}} = \varepsilon.$$

Thus, the sequence x_n is fundamental, and therefore it converges to some limit L.

In fact, $L = \frac{\pi^2}{6}.$

Picture of the Week

Question 8. Draw the curve defined by the equation $\lim_{n\to\infty} \frac{1}{\sqrt[n]{|x|^{-n}+|y|^{-n}}} = \frac{x^2+y^2}{2}$.

Solution. We already know that

$$\lim_{n\to\infty} \sqrt[n]{|x|^n+|y|^n} = \max(|x|,|y|).$$

Therefore $\lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{|x|^n+|y|^n}{|x|^n|y|^n}}} =$

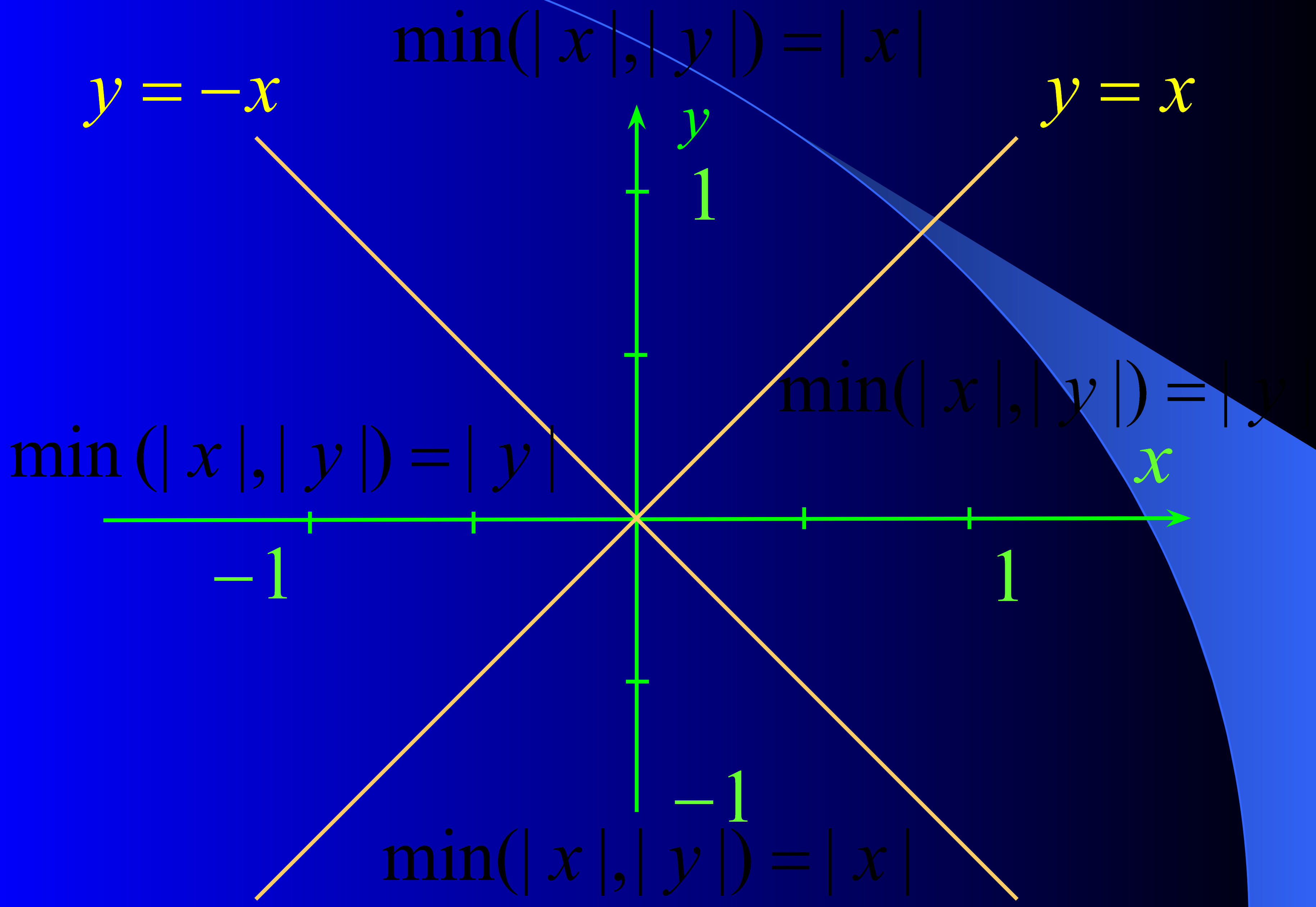
$$\lim_{n\to\infty} \frac{|x||y|}{\sqrt[n]{|x|^n+|y|^n}} = \frac{|x||y|}{\max(|x|,|y|)} =$$

$$= \min(|x|,|y|).$$

Thus, we have to draw the curve defined by the equation

$$\min(|x|,|y|) = \frac{x^2+y^2}{2}.$$

Let us look at the $xy-$plane:



The curve defined by the equation $y = \frac{x^2 + y^2}{2}$

is the circle with the radius 1, centred at the point (0,1).

Indeed, $x^2 + y^2 - 2y = 0$

$$\Leftrightarrow \; x^2 + y^2 - 2y + 1 = 1$$

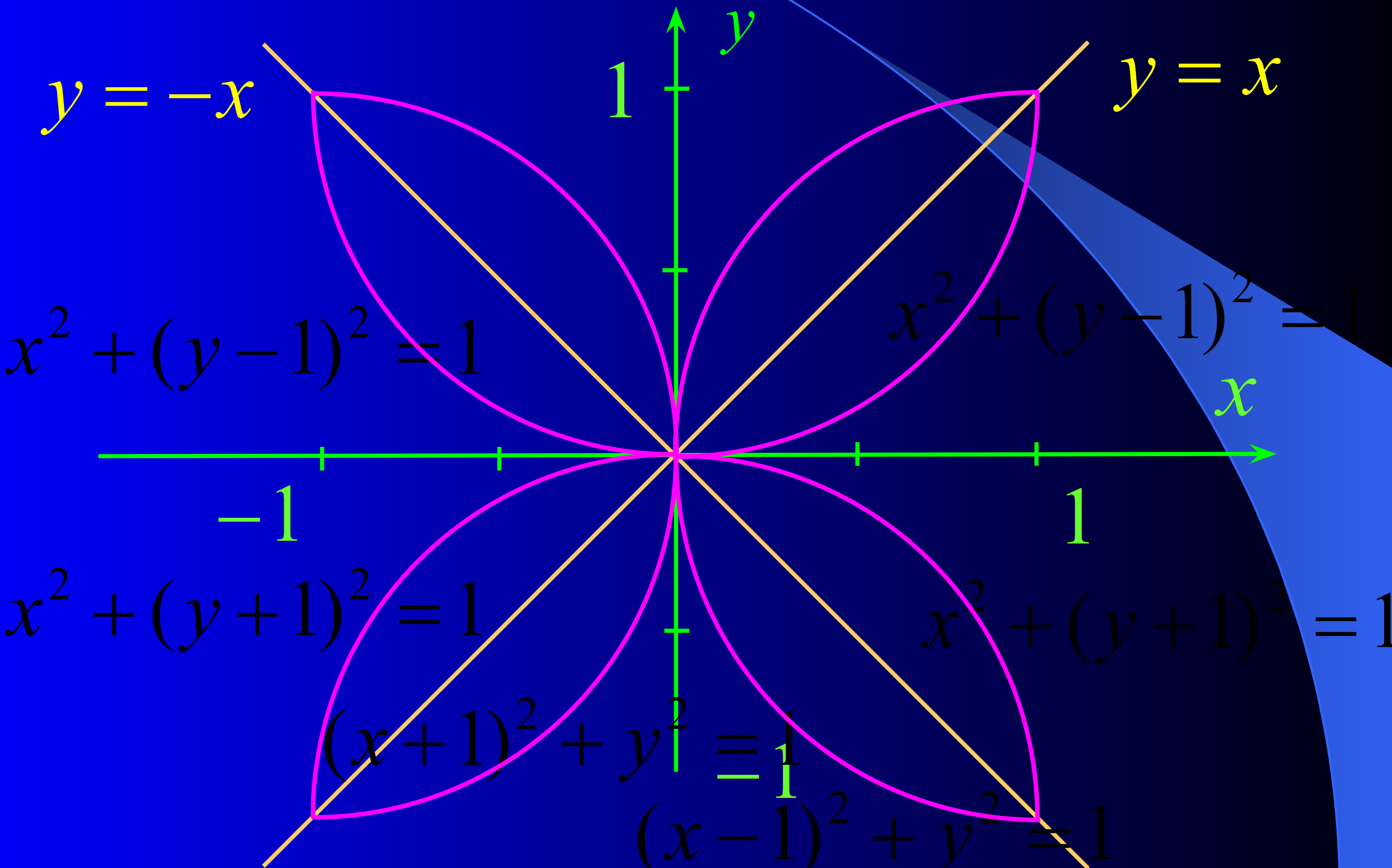
$$\Leftrightarrow \; x^2 + (y-1)^2 = 1.$$

The curve defined by the equation $x = \frac{x^2 + y^2}{2}$

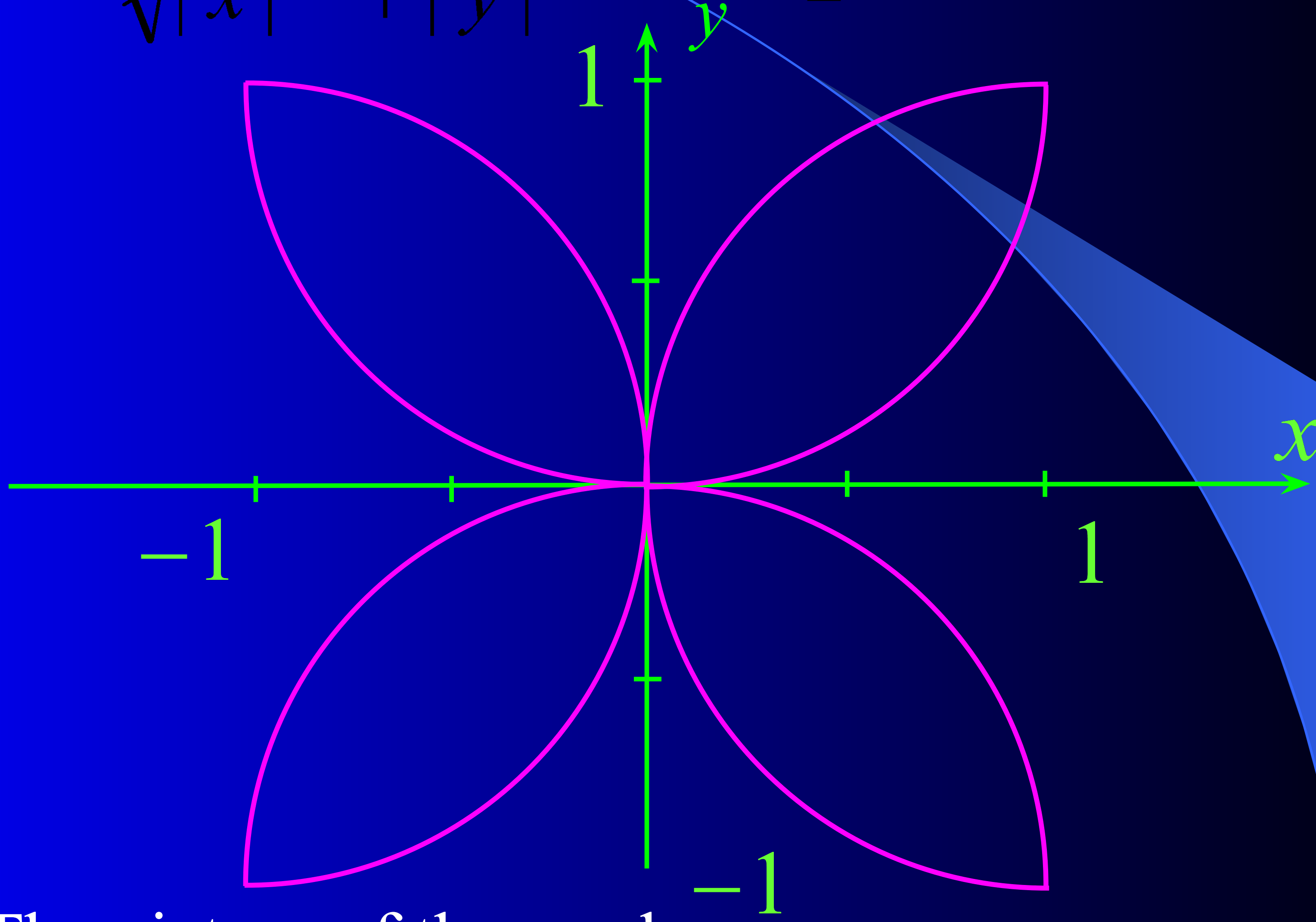
is the circle with the radius 1, centred at the point (1,0): $(x-1)^2 + y^2 = 1.$

The equations of our curve.

$$(x+1)^2 + y^2 = 1 \quad (x-1)^2 + y^2 = 1$$



$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{|x|^{-n}+|y|^{-n}}}=\frac{x^2+y^2}{2}.$$



The picture of the week.

Answers to Questions from Seminar 1.

Question 2: $\gamma = \frac{1}{3}$.

Question 3: $a_n = \frac{5}{6} \cdot 2^n - \frac{1}{3} \cdot (-1)^n$

$a_{100} + a_{99} = 5 \cdot 2^{98}$

Question 8b: $\lim\limits_{n\to\infty} \sin^2\left(\pi\sqrt{n^2+n}\right) = 1.$

Question 8c: $\lim\limits_{n\to\infty}\left(\sin\frac{2\pi n}{3n+1}\right)^n = 0.$

Question 9: $y = \begin{cases} 1, & \text{for } 0 \le x \le 1; \\ x, & \text{for } 1 < x < 2; \\ \frac{1}{2}x^2, & \text{for } 2 \le x. \end{cases}$