Sets

What is a set?

- A set is a group of “objects”
  - People in a class: { Alice, Bob, Chris }
  - Classes offered by a department: { CS 101, CS 202, … }
  - Colors of a rainbow: { red, orange, yellow, green, blue, purple }
  - States of matter { solid, liquid, gas, plasma }
  - States in the US: { Alabama, Alaska, Virginia, … }
  - Sets can contain non-related elements: { 3, a, red, Virginia }

- Although a set can contain (almost) anything, we will most often use sets of numbers
  - All positive numbers less than or equal to 5: {1, 2, 3, 4, 5}
  - A few selected real numbers: { 2.1, π, 0, -6.32, e }

Set properties 1

- Order does not matter
  - We often write them in order because it is easier for humans to understand it that way
  - {1, 2, 3, 4, 5} is equivalent to {3, 5, 2, 4, 1}

- Sets are notated with curly brackets

Set properties 2

- Sets do not have duplicate elements
  - Consider the set of vowels in the alphabet.
    - It makes no sense to list them as {a, a, a, e, i, o, o, o, o, o, u}
    - What we really want is just {a, e, i, o, u}
  - Consider the list of students in this class
    - Again, it does not make sense to list somebody twice
- Note that a list is like a set, but order does matter and duplicate elements are allowed
  - We won't be studying lists much in this class

Specifying a set 1

- Sets are usually represented by a capital letter (A, B, S, etc.)

- Elements are usually represented by an italic lower-case letter (*a*, *x*, *y*, etc.)

- Easiest way to specify a set is to list all the elements: A = {1, 2, 3, 4, 5}
  - Not always feasible for large or infinite sets

Specifying a set 2

- Can use an ellipsis (…): B = {0, 1, 2, 3, …}
  - Can cause confusion. Consider the set C = {3, 5, 7, …}. What comes next?
  - If the set is all odd integers greater than 2, it is 9
  - If the set is all prime numbers greater than 2, it is 11
- Can use set-builder notation
  - D = {x | x is prime and x > 2}
  - E = {x | x is odd and x > 2}
  - The vertical bar means "such that"
  - Thus, set D is read (in English) as: "all elements x such that x is prime and x is greater than 2"

Specifying a set 3

- A set is said to "contain" the various "members" or "elements" that make up the set
  - If an element *a* is a member of (or an element of) a set S, we use then notation $a \in S$
    - $4 \in \{1, 2, 3, 4\}$
  - If an element is not a member of (or an element of) a set S, we use the notation $a \notin S$
    - $7 \notin \{1, 2, 3, 4\}$
    - Virginia $\notin \{1, 2, 3, 4\}$

Often used sets

- **N** = {0, 1, 2, 3, …} is the set of natural numbers
- **Z** = {…, -2, -1, 0, 1, 2, …} is the set of integers
- $\mathbf{Z}^+$ = {1, 2, 3, …} is the set of positive integers (a.k.a whole numbers)
  - Note that people disagree on the exact definitions of whole numbers and natural numbers
- **Q** = $\{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$ is the set of rational numbers
  - Any number that can be expressed as a fraction of two integers (where the bottom one is not zero)
- **R** is the set of real numbers

The universal set 1

- U is the universal set – the set of all of elements (or the “universe”) from which given any set is drawn
  - For the set {-2, 0.4, 2}, U would be the real numbers
  - For the set {0, 1, 2}, U could be the natural numbers (zero and up), the integers, the rational numbers, or the real numbers, depending on the context

The universal set 2

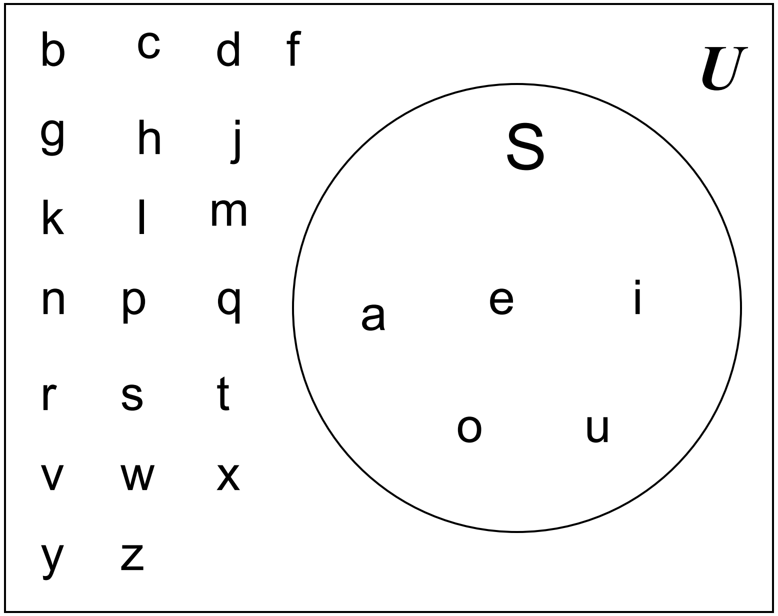
– For the set of the students in this class, $\boldsymbol{U}$ would be all the students in the University (or perhaps all the people in the world)

– For the set of the vowels of the alphabet, $\boldsymbol{U}$ would be all the letters of the alphabet

– To differentiate $\boldsymbol{U}$ from U (which is a set operation), the universal set is written in a different font (and in bold and italics)

Venn diagrams

- Represents sets graphically
  - The box represents the universal set
  - Circles represent the set(s)
- Consider set S, which is the set of all vowels in the alphabet
- The individual elements are usually not written in a Venn diagram



Sets of sets

- Sets can contain other sets
  - S = { {1}, {2}, {3} }
  - T = { {1}, {{2}}, {{{3}}} }
  - V = { {{1}, {{2}}}, {{{3}}}, { {1}, {{2}}, {{{3}}} } }
    - V has only 3 elements!
- Note that 1 ≠ {1} ≠ {{1}} ≠ {{{1}}}
  - They are all different

The empty set 1

- If a set has zero elements, it is called the empty (or null) set
  - Written using the symbol ∅
  - Thus, ∅ = { } VERY IMPORTANT
  - If you get confused about the empty set in a problem, try replacing ∅ by { }
- As the empty set is a set, it can be a element of other sets
  - { ∅, 1, 2, 3, x } is a valid set

The empty set 1

- Note that $\varnothing \neq \{ \varnothing \}$
  - The first is a set of zero elements
  - The second is a set of 1 element (that one element being the empty set)

- Replace $\varnothing$ by { }, and you get: $\{ \} \neq \{ \{ \} \}$
  - It's easier to see that they are not equal that way

Set equality

- Two sets are equal if they have the same elements
  - $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\}$
    - Remember that order does not matter!
  - $\{1, 2, 3, 2, 4, 3, 2, 1\} = \{4, 3, 2, 1\}$
    - Remember that duplicate elements do not matter!
- Two sets are not equal if they do not have the same elements
  - $\{1, 2, 3, 4, 5\} \neq \{1, 2, 3, 4\}$

Subsets 1

- If all the elements of a set S are also elements of a set T, then S is a subset of T
  - For example, if S = {2, 4, 6} and T = {1, 2, 3, 4, 5, 6, 7}, then S is a subset of T
  - This is specified by $S \subseteq T$
    - Or by $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$
- If S is not a subset of T, it is written as such: $S \nsubseteq T$
  - For example, $\{1, 2, 8\} \nsubseteq \{1, 2, 3, 4, 5, 6, 7\}$

Subsets 2

- Note that any set is a subset of itself!
  - Given set S = {2, 4, 6}, since all the elements of S are elements of S, S is a subset of itself
  - This is kind of like saying 5 is less than or equal to 5
  - Thus, for any set S, $S \subseteq S$

Subsets 3

- The empty set is a subset of *all* sets (including itself!)
  - Recall that all sets are subsets of themselves
- *All* sets are subsets of the universal set
- A horrible way to define a subset:
  - $\forall x\ (x \in A \rightarrow x \in B)$
  - English translation: for all possible values of x, (meaning for all possible elements of a set), if x is an element of A, then x is an element of B
  - This type of notation will be gone over later

Proper Subsets 1

- If S is a subset of T, and S is not equal to T, then S is a proper subset of T
  - Let T = {0, 1, 2, 3, 4, 5}
  - If S = {1, 2, 3}, S is not equal to T, and S is a subset of T
  - A proper subset is written as $S \subset T$
  - Let R = {0, 1, 2, 3, 4, 5}. R is equal to T, and thus is a subset (but not a proper subset) or T
    - Can be written as: $R \subseteq T$ and $R \not\subset T$ (or just R = T)
  - Let Q = {4, 5, 6}. Q is neither a subset or T nor a proper subset of T

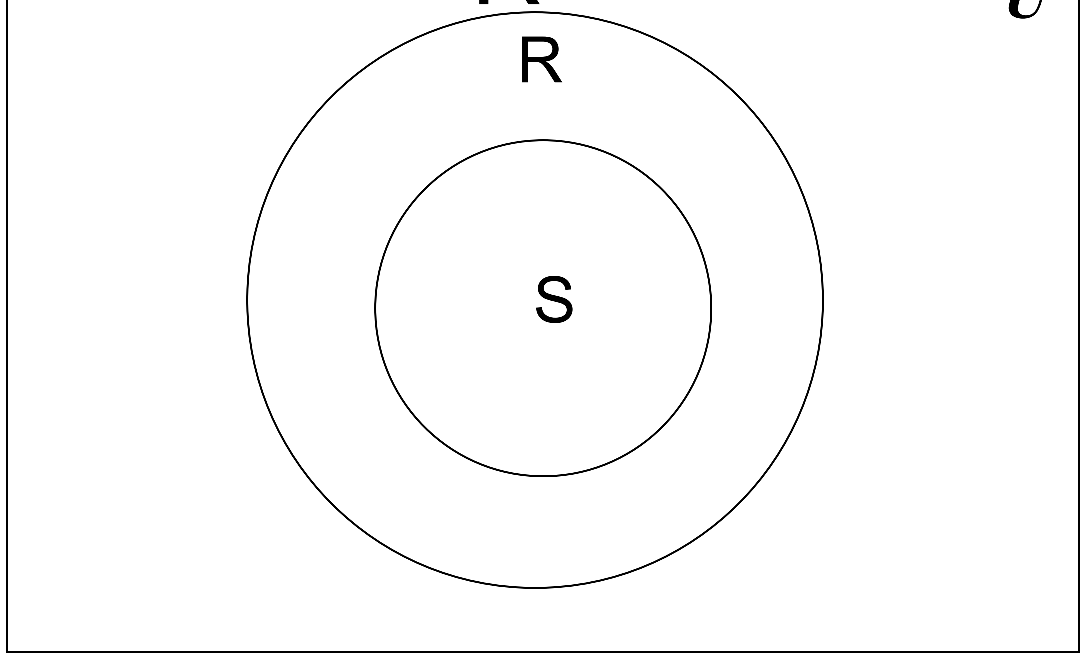
Proper Subsets 2

- The difference between “subset” and “proper subset” is like the difference between “less than or equal to” and “less than” for numbers

- The empty set is a proper subset of all sets other than the empty set (as it is equal to the empty set)

Proper subsets: Venn diagram

$S \subset R$



Set cardinality

- The cardinality of a set is the number of elements in a set
  - Written as |A|
- Examples
  - Let R = {1, 2, 3, 4, 5}. Then |R| = 5
  - $|\varnothing| = 0$
  - Let S = {∅, {a}, {b}, {a, b}}. Then |S| = 4
- This is the same notation used for vector length in geometry
- A set with one element is sometimes called a singleton set

Power sets 1

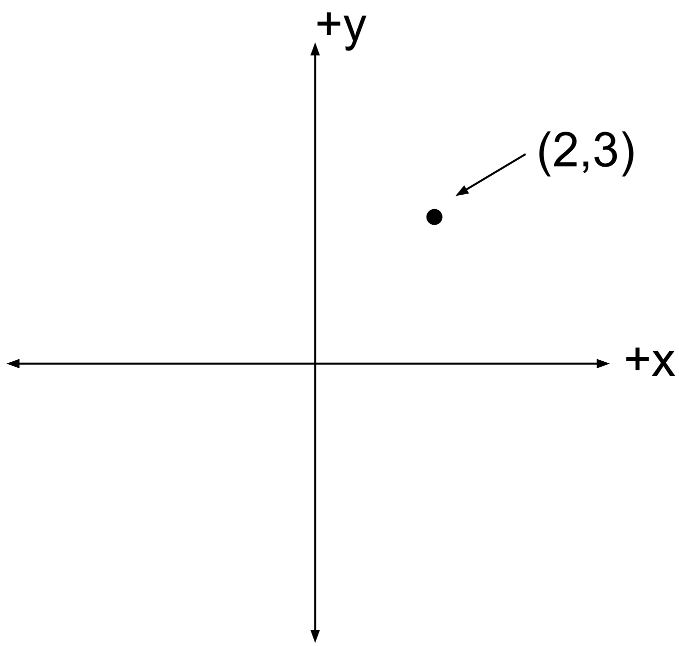
- Given the set S = {0, 1}. What are all the possible subsets of S?
  - They are: ∅ (as it is a subset of all sets), {0}, {1}, and {0, 1}
  - The power set of S (written as P(S)) is the set of all the subsets of S
  - P(S) = { ∅, {0}, {1}, {0,1} }
    - Note that |S| = 2 and |P(S)| = 4

Power sets 2

- Let T = {0, 1, 2}. The P(T) = { ∅, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2} }
  - Note that |T| = 3 and |P(T)| = 8
- P(∅) = { ∅ }
  - Note that |∅| = 0 and |P(∅)| = 1
- If a set has *n* elements, then the power set will have 2^n elements

Tuples

- In 2-dimensional space, it is a (*x*, *y*) pair of numbers to specify a location
- In 3-dimensional (1,2,3) is not the same as (3,2,1) – space, it is a (*x*, *y*, *z*) triple of numbers
- In *n*-dimensional space, it is a *n*-tuple of numbers
  - Two-dimensional space uses pairs, or 2-tuples
  - Three-dimensional space uses triples, or 3-tuples
- Note that these tuples are **ordered**, unlike sets
  - the *x* value has to come first



Cartesian products 1

- A Cartesian product is a set of all ordered 2-tuples where each "part" is from a given set
  - Denoted by A x B, and uses parenthesis (not curly brackets)
  - For example, 2-D Cartesian coordinates are the set of all ordered pairs **Z** x **Z**
    - Recall **Z** is the set of all integers
    - This is all the possible coordinates in 2-D space
  - Example: Given A = { a, b } and B = { 0, 1 }, what is their Cartiesian product?
    - C = A x B = { (a,0), (a,1), (b,0), (b,1) }

Cartesian products 2

- Note that Cartesian products have only 2 parts in these examples (later examples have more parts)
- Formal definition of a Cartesian product:
  - A x B = { (a,b) | $a \in$ A and $b \in$ B }

Cartesian products 3

- All the possible grades in this class will be a Cartesian product of the set S of all the students in this class and the set G of all possible grades
  - Let S = { Alice, Bob, Chris } and G = { A, B, C }
  - D = { (Alice, A), (Alice, B), (Alice, C), (Bob, A), (Bob, B), (Bob, C), (Chris, A), (Chris, B), (Chris, C) }
  - The final grades will be a subset of this: { (Alice, C), (Bob, B), (Chris, A) }
    - Such a subset of a Cartesian product is called a relation (more on this later in the course)

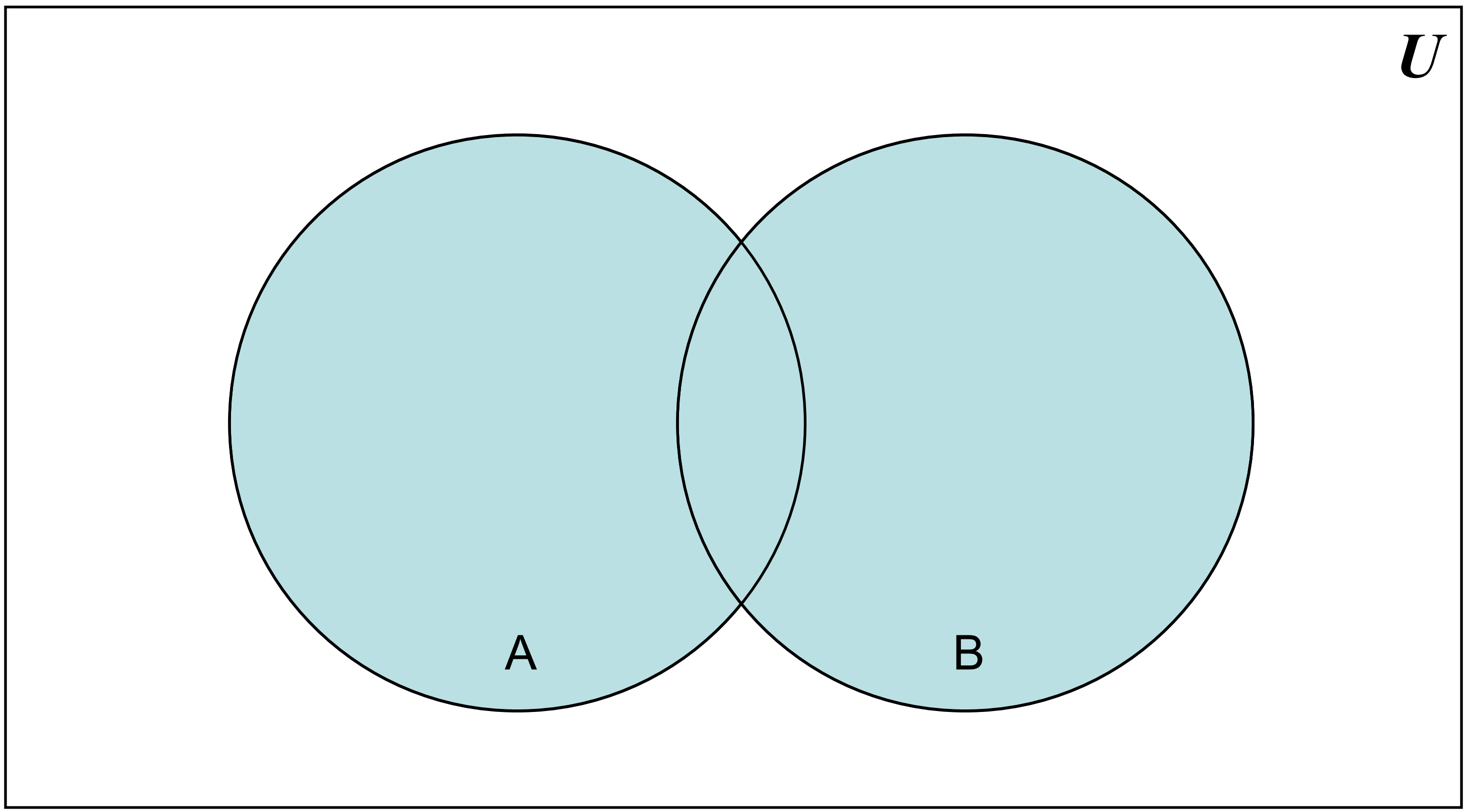
Cartesian products 4

- There can be Cartesian products on more than two sets
- A 3-D coordinate is an element from the Cartesian product of **Z** x **Z** x **Z**

Set Operations

Set operations: Union

A U B



Set operations: Union

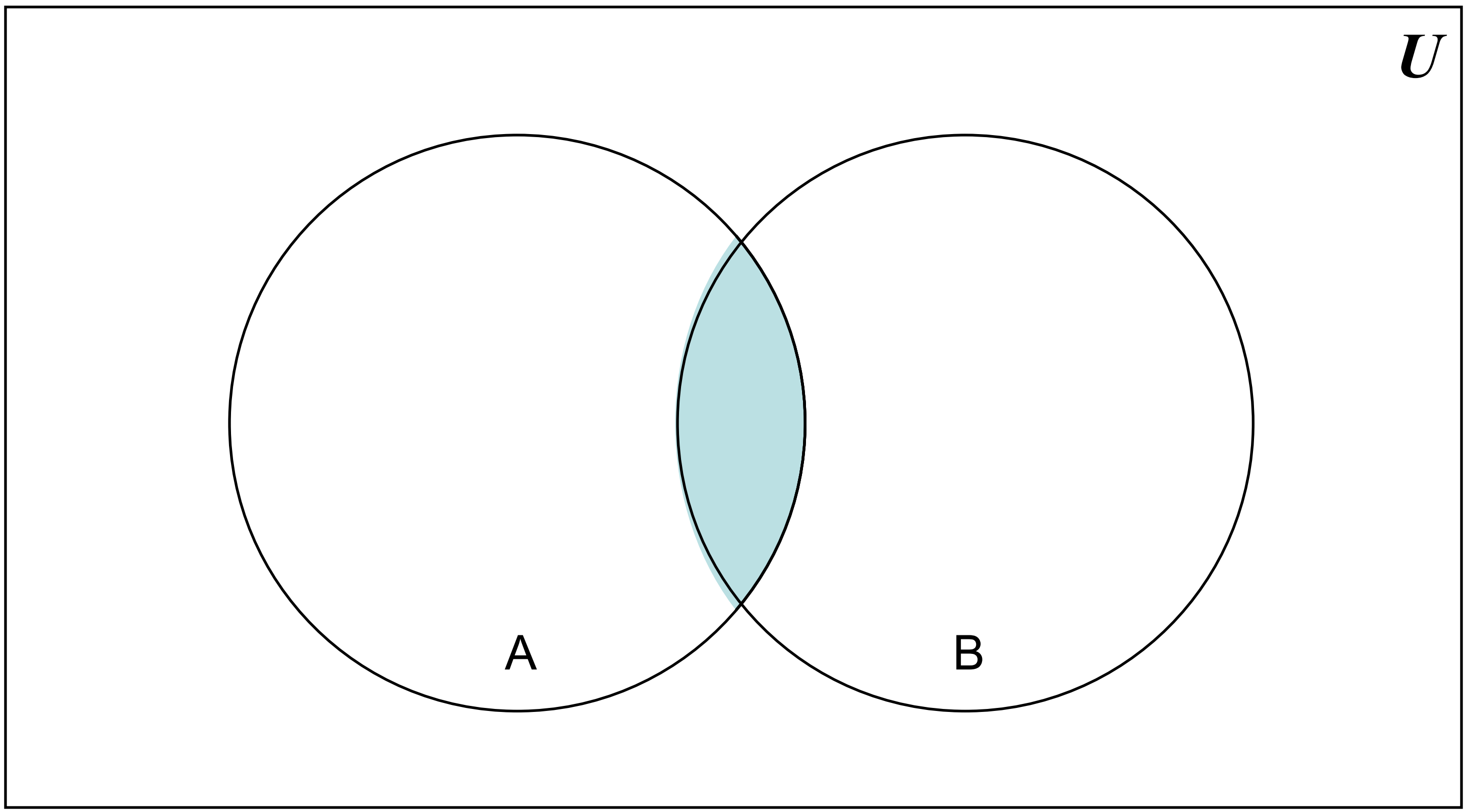
- Formal definition for the union of two sets: $A \cup B = \{ x \mid x \in A \textbf{ or } x \in B \}$
- Further examples
  - {1, 2, 3} U {3, 4, 5} = {1, 2, 3, 4, 5}
  - {New York, Washington} U {3, 4} = {New York, Washington, 3, 4}
  - {1, 2} U ∅ = {1, 2}

Set operations: Union

- Properties of the union operation
  - A U ∅ = A Identity law
  - A U ***U*** = ***U*** Domination law
  - A U A = A Idempotent law
  - A U B = B U A Commutative law
  - A U (B U C) = (A U B) U C Associative law

Set operations: Intersection

A ∩ B



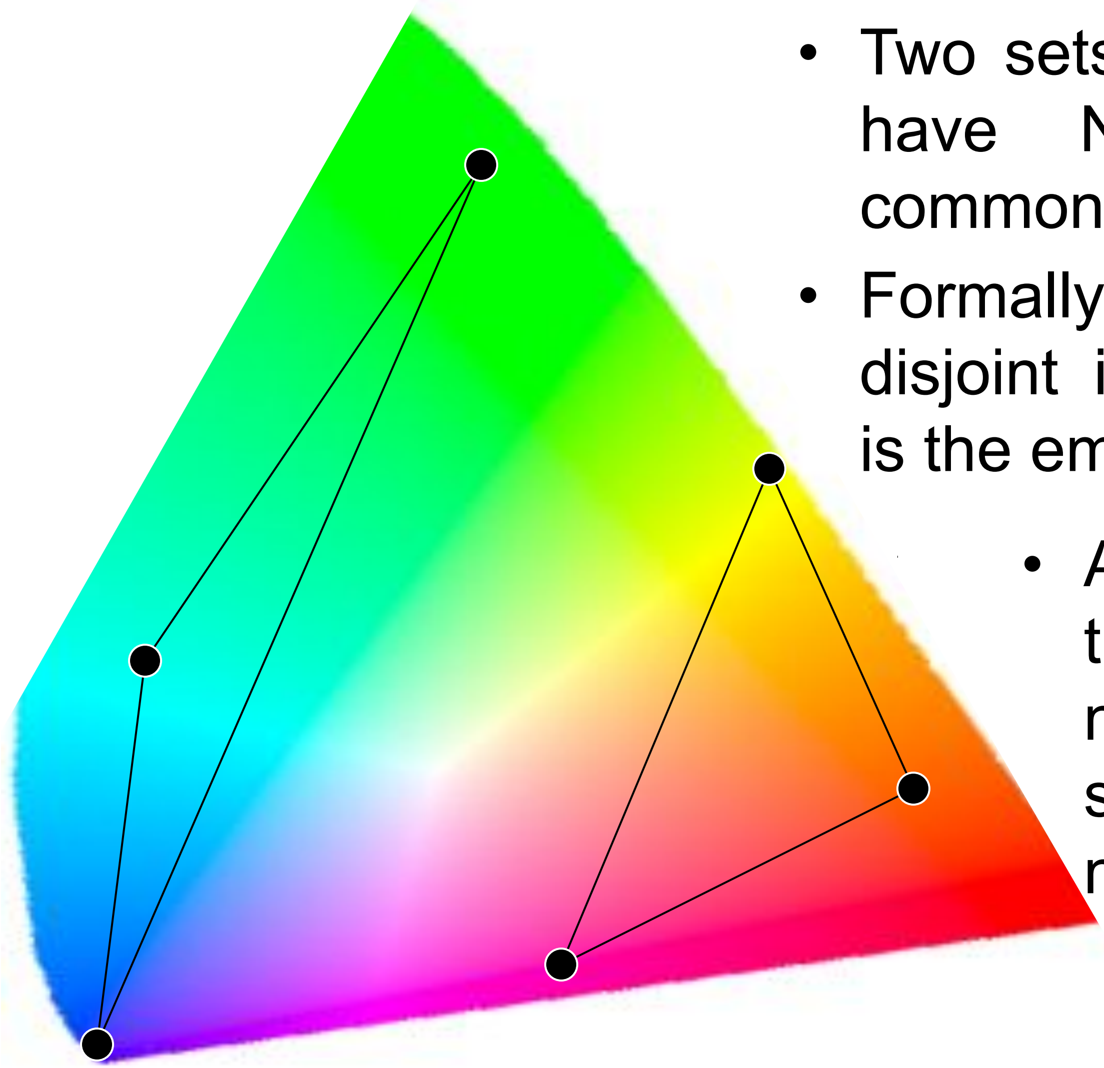
Set operations: Intersection

- Formal definition for the intersection of two sets: $A \cap B = \{ x \mid x \in A$ **and** $x \in B \}$
- Further examples
  - $\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$
  - {New York, Washington} $\cap$ {3, 4} = $\varnothing$
    - No elements in common
  - $\{1, 2\} \cap \varnothing = \varnothing$
    - Any set intersection with the empty set yields the empty set

Set operations: Intersection

- Properties of the intersection operation
  - $A \cap \boldsymbol{U} = A$ Identity law
  - $A \cap \varnothing = \varnothing$ Domination law
  - $A \cap A = A$ Idempotent law
  - $A \cap B = B \cap A$ Commutative law
  - $A \cap (B \cap C) = (A \cap B) \cap C$ Associative law

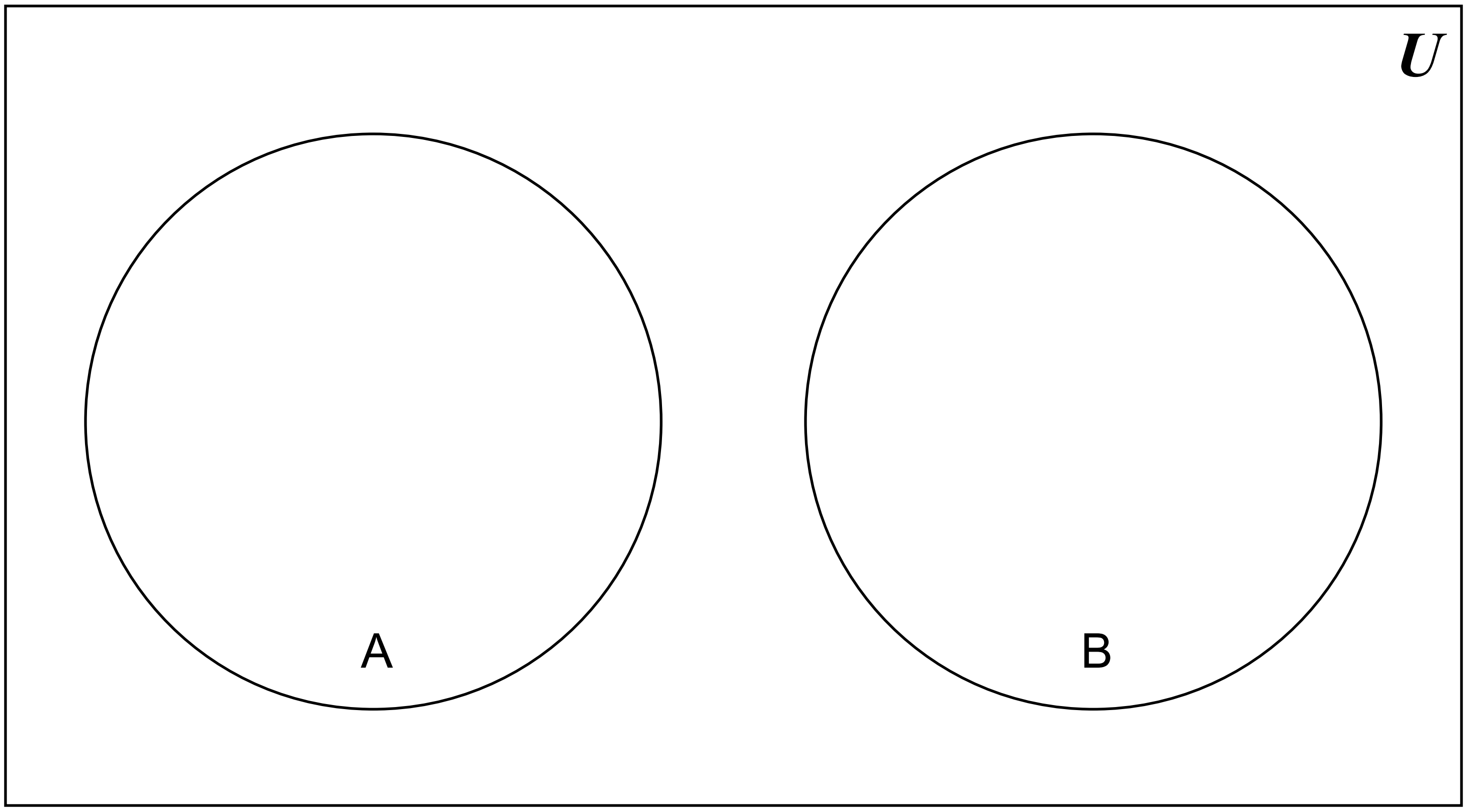
Disjoint sets 1



- Two sets are disjoint if the have NO elements in common
- Formally, two sets are disjoint if their intersection is the empty set

- Another example: the set of the even numbers and the set of the odd numbers

Disjoint sets 2

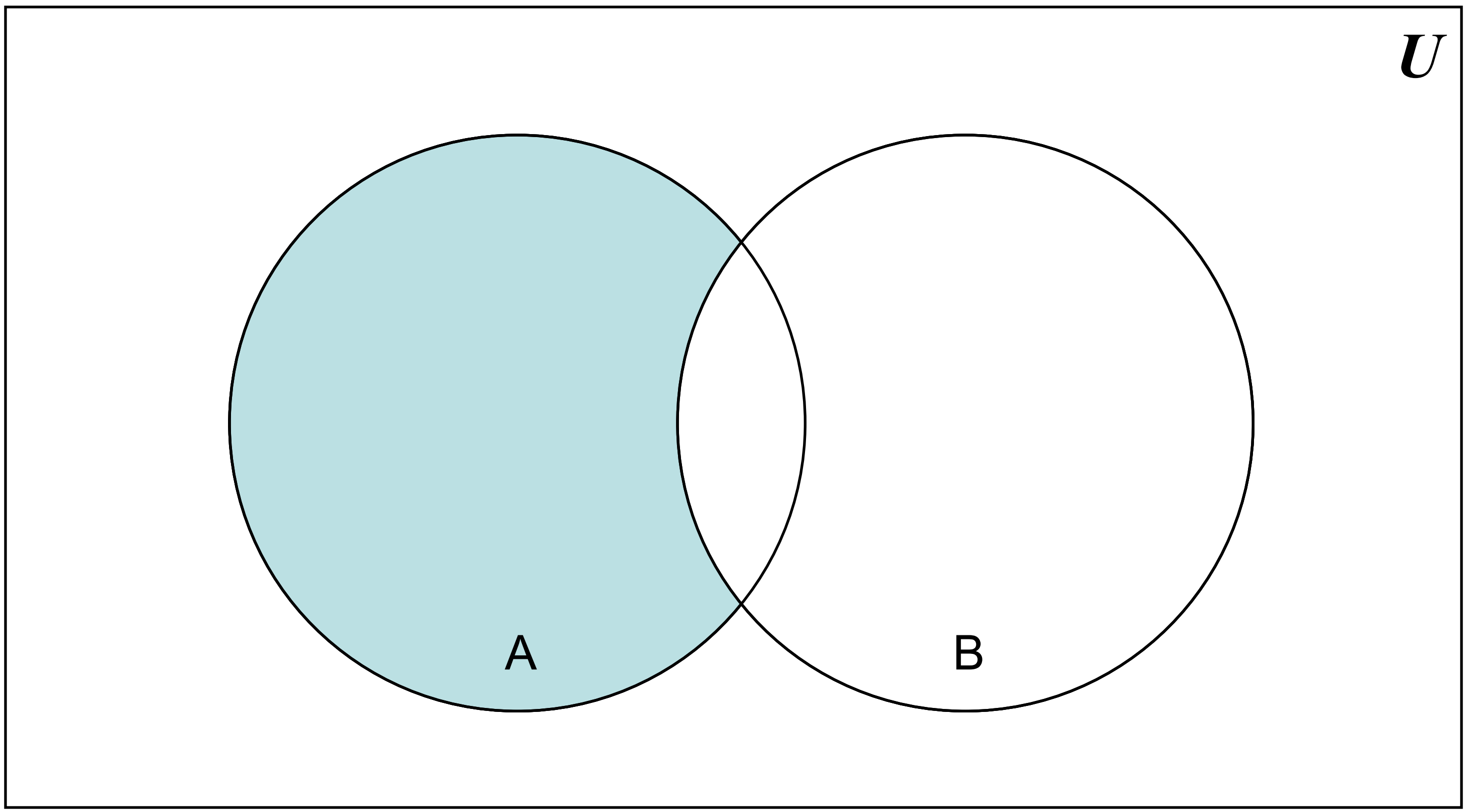


Disjoint sets 3

- Formal definition for disjoint sets: two sets are disjoint if their intersection is the empty set
- Further examples
  - {1, 2, 3} and {3, 4, 5} are not disjoint
  - {New York, Washington} and {3, 4} are disjoint
  - {1, 2} and $\varnothing$ are disjoint
    - Their intersection is the empty set
  - $\varnothing$ and $\varnothing$ are disjoint!
    - Their intersection is the empty set

Set operations: Difference

A - B



Set operations: Difference

- Formal definition for the difference of two sets:
  $A - B = \{ x \mid x \in A \textbf{ and } x \notin B \}$
  $A - B = A \cap \overline{B}$ □ Important!
- Further examples
  - {1, 2, 3} - {3, 4, 5} = {1, 2}
  - {New York, Washington} - {3, 4} = {New York, Washington}
  - {1, 2} - ∅ = {1, 2}
    - The difference of any set S with the empty set will be the set S

Set operations: Symmetric Difference

- A symmetric difference of the sets contains all the elements in either set but NOT both
- Formal definition for the symmetric difference of two sets:

  $A \oplus B = \{ x \mid (x \in A \text{ or } x \in B) \text{ and } x \notin A \cap B\}$

  $A \oplus B = (A \cup B) - (A \cap B)$ □ Important!
- Further examples
  - $\{1, 2, 3\} \oplus \{3, 4, 5\} = \{1, 2, 4, 5\}$
  - {New York, Washington} $\oplus$ {3, 4} = {New York, Washington, 3, 4}
  - $\{1, 2\} \oplus \varnothing = \{1, 2\}$
    - The symmetric difference of any set S with the empty set will be the set S

Complement sets

- A complement of a set is all the elements that are NOT in the set
- Formal definition for the complement of a set: $\overline{A} = \{ x \mid x \notin A \}$
- Further examples (assuming $\boldsymbol{U}$ = **Z**)
  - $\overline{\{1, 2, 3\}} = \{ \ldots, -2, -1, 0, 4, 5, 6, \ldots \}$

Complement sets

- Properties of complement sets
  - $\bar{\bar{A}} = A$ Complementation law
  - $A \cup \bar{A} = \boldsymbol{U}$ Complement law
  - $A \cap \bar{A} = \varnothing$ Complement law