Chapter 4
Probability

Terminology
Example: Rolling a dice

- **Event**
  any collection of outcomes of a procedure
  - EX) {1}, {2}, {1,3}, {2,4,5},….
- **Simple Event**
  an outcome or an event that cannot be further broken down into simpler components
  - EX) {1}, {2},…,{6}
- **Sample Space**
  collection of all possible simple events
  - EX) {1,2,3,4,5,6}

  Event is a subset of sample space
- **What is 'Probability'?** 'weight' of each event

Notation for ‘Probability’

P **- denotes a probability.**

A**,** ***B*****, and** ***C*** **- denote events.**

P**(*****A*****) - denotes the probability of event** ***A*** **occurring.**

Probability is a set function that maps a set (event) into a real value between 0 and 1

Example

- **Suppose we role two dice simultaneously**
- **What are simple events?**
  - **We have 36 simple events for this experiment**
  - **(1,1), (1,2),(1,3),…,(6,6)**
- **Sample space: collection of all the possible simple events**
  - **{(1,1), (1,2),(1,3),…,(6,6)}**

Basic Rules for Computing Probability

Rule 1: Classical Approach to Probability

(Requires Equally Likely Outcomes)

Assume that a given procedure has *n* different simple events and that each of those simple events has an equal chance of occurring. If event *A* can occur in *s* of these *n* ways, then

$$P(A) = \frac{s}{n} = \frac{\text{number of ways } A \text{ can occur}}{\text{number of different simple events}}$$

Basic Rules for Computing Probability

Rule 2: Relative Frequency Approximation of Probability

Conduct (or observe) a procedure, and count the number of times event *A* actually occurs. Based on these actual results, $P(A)$ is approximated as follows:

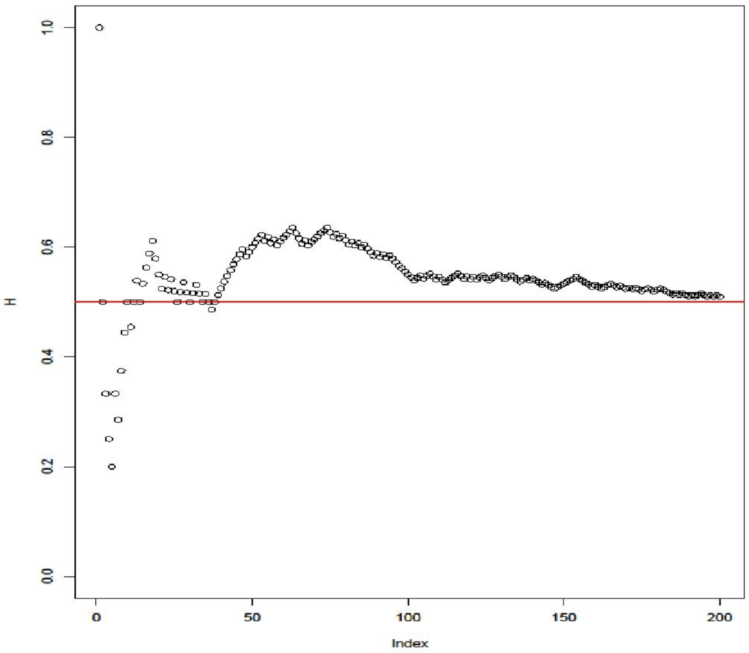
$$P(A) = \frac{\text{\# of times } A \text{ occurred}}{\text{\# of times procedure was repeated}}$$

Law of Large Numbers

As a procedure is repeated again and again, the relative frequency probability of an event tends to approach the actual probability.

Try this R code

```
R=runif(200);C=round(R)
C
H=c()
for (i in 1:length(C)){
  H[i]=sum(C[1:i])/i
}
plot(H, ylim=c(0,1))
abline(h=0.5, col="red")
```



Basic conditions of Probability

- The probability of an impossible event is 0.
- The probability of an event that is certain to occur is 1.
- For any event *A*, the probability of *A* is between 0 and 1 inclusive.
  That is, $0 \leq P(A) \leq 1$.

Compound Event (OR)

Any event combining 2 or more simple events

Notation

$$P(A \text{ or } B) = P(A \cup B)$$
$$= P(A, B, \text{or } Both \text{ occur in a single trial})$$

Example

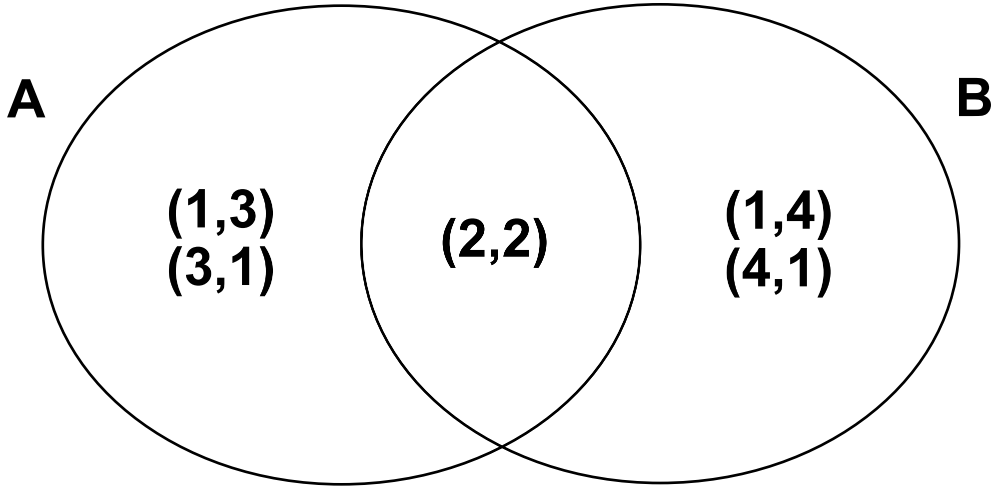
- Consider the previous example: Rolling two dice
  - event A: **sum** of two outcome values is 4
  - event B: **product** of two outcome value is 4
- Event A occurs if we have an outcome from {(1,3),(2,2),(3,1)}
- Event B occurs if we have an outcome from {(1,4),(2,2),(4,1)}
- $P(A \text{ or } B) = P$ ({(1,3),(2,2),(3,1),(1,4),(4,1)})

Compound Event

Formal Addition Rule

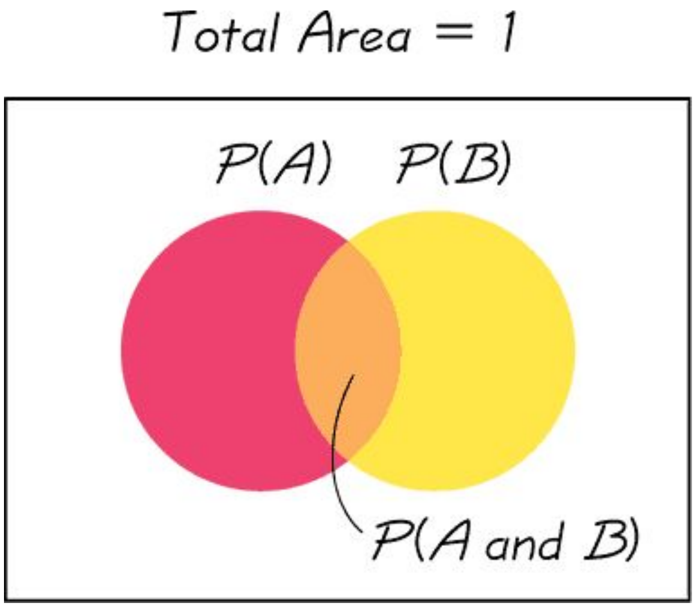
$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

where $P(A \text{ and } B)$ denotes the probability that A and B both occur at the same time

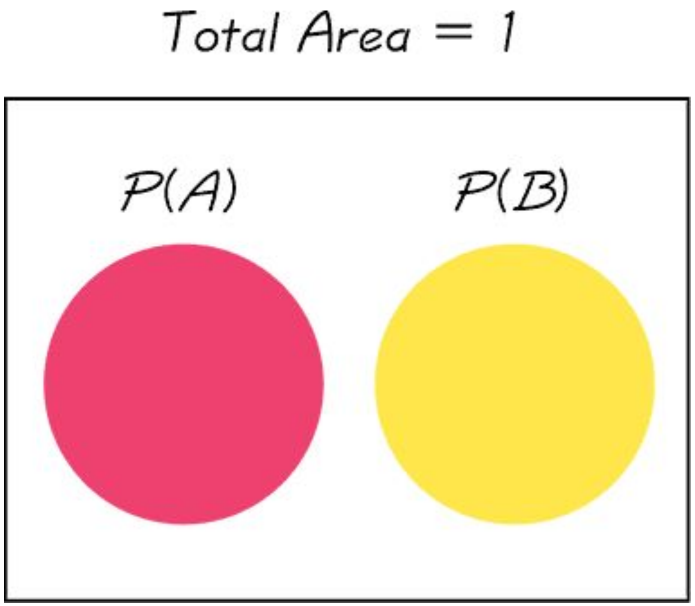


Disjoint or Mutually Exclusive

Events *A* and *B* are disjoint (or mutually exclusive) if they cannot occur at the same time. (That is, disjoint events do not overlap.)

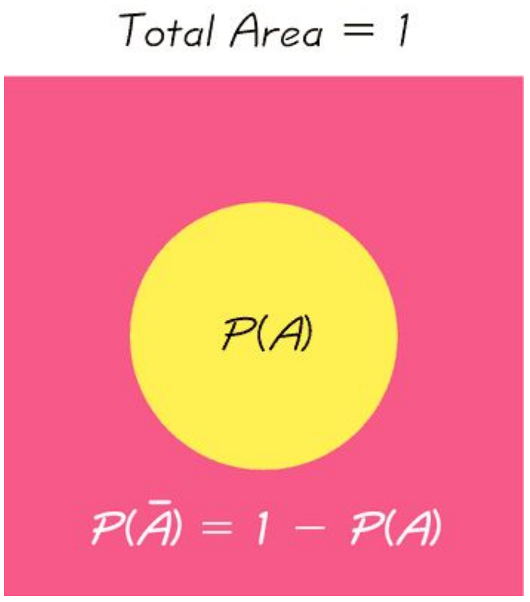


Venn Diagram for Events That Are Not Disjoint



Venn Diagram for Disjoint Events

Complementary Events



$P(A)$ and $P(\bar{A})$ are disjoint

Rule of Complementary Event

$$P(A) + P(\bar{A}) = 1$$

Chapter 4
Probability

4-1 Review and Preview

4-2 Basic Concepts of Probability

4-3 Addition Rule

4-4 Multiplication Rule: Basics

4-5 Multiplication Rule: Complements and Conditional Probability

4-6 Counting

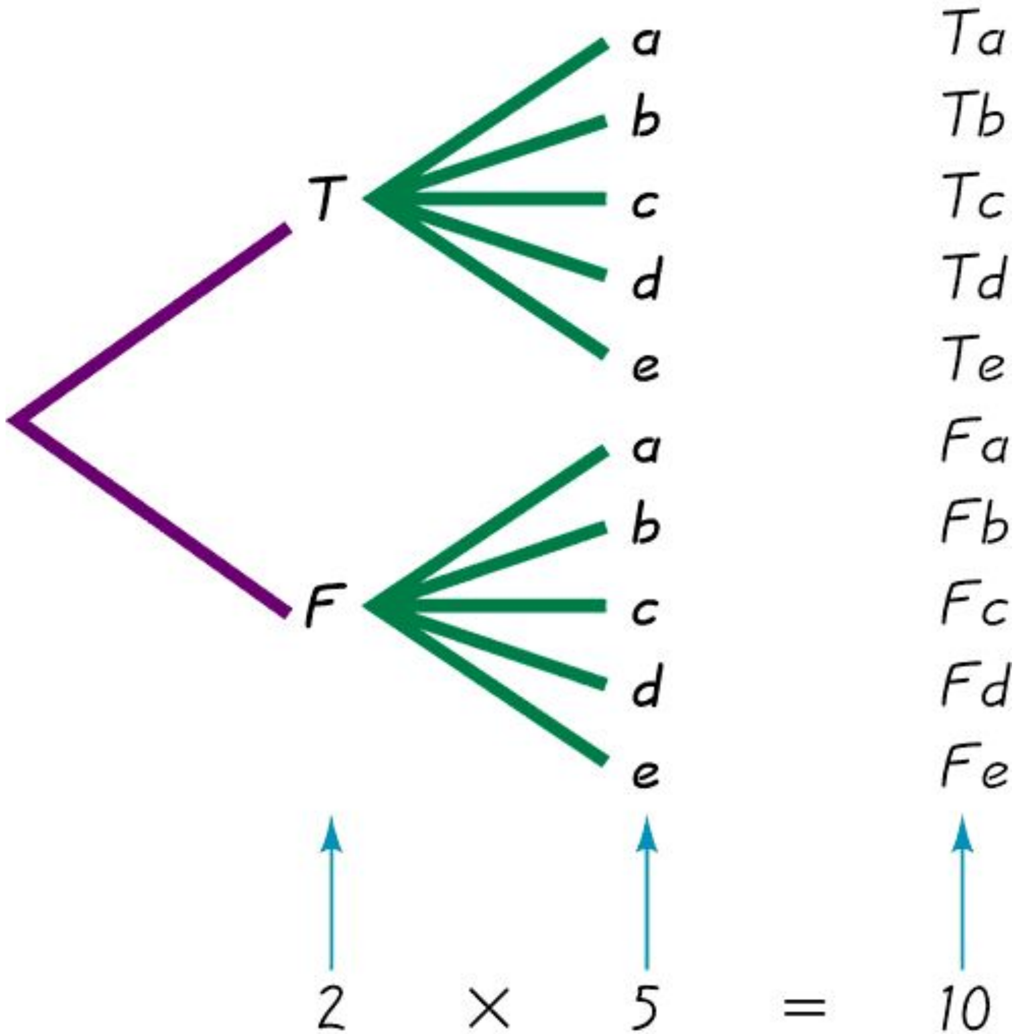
Notation

$$P(A \text{ and } B) = P(A \cap B)$$
$$= P(\text{Both A and B occur in a single trial})$$

Tree Diagrams
:Sequential Trial

This figure summarizes the possible outcomes for a true/false question followed by a multiple choice question.

Note that there are 10 possible combinations.



Multiplication Rule for Several Events

In general, the probability of any **sequence of independent events** is simply the product of their corresponding probabilities.

Conditional Probability -Example

Suppose we have one fair coin and one biased coin. We want to compute the probability of '**Head**' given that we chose a '**Biased coin**'

- Use a tree diagram

Conditional Probability

***P*(*B*|*A*)** represents the probability of event *B* occurring after it is assumed that event *A* has already occurred (read *B*|*A* as "*B* given *A*.")

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

- Note that if A and B are independent events, P(B|A) is the same as P(B)

Dependent and Independent

Two events *A* and *B* are independent if the occurrence of one does not affect the *probability* of the occurrence of the other. Otherwise, they are said to be dependent.

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Key Concepts

Probability of "at least one":

Find the probability that among several trials, we get at least one of some specified event.

Conditional probability:

Find the probability of an event when we have additional information that some other event has already occurred.

Complements: The Probability of "At Least One"

- **'At least one'** is equivalent to **'one or more'.**
- **The complement of getting 'at least one' item is that you get no items**
- **What is the complement of 'at most *k*' ?**

Finding the Probability of "At Least One"

To find the probability of at least one of something, calculate the probability of 'none' first, then subtract that result from 1.

P(at least one) = 1 – P(none).

Use a similar rule for 'At most k' probability

Example

- A student wants to ensure that she is not late for an early class because of a mal-functioning alarm clock. Instead of using one alarm clock, she decides to use **three**. If each alarm clock has an 90% chance of working correctly, what is the probability that at least one of her alarm clocks works correctly?

Bayes Rule

- In some cases, P(B|A) is easier to compute than P(A|B). So we use the formula called ***Bayes Rule***

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

$$\text{where } P(B) = P(B \cap A) + P(B \cap \overline{A})$$
$$= P(B \mid A) \cdot P(A) + P(B \mid \overline{A}) \cdot P(\overline{A})$$

Example – Bayes Rule

- A dealer has three coins, one fair coin and two biased coins with the probability of Head, 1/2, 1/3, and 1/4, respectively. Suppose a gambler observed a Tail, find the probability that it came from the fair coin. That is P(Fair|Tail).

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Notation

The **factorial symbol !** denotes the product of decreasing positive whole numbers.

For example,

$$4! = 4 \bullet 3 \bullet 2 \bullet 1 = 24.$$

By special definition, $0! = 1$.

Factorial Rule

n **different items can be arranged in order** ***n*!** **different ways:**

- **This factorial rule reflects the fact that the first item may be selected in *n* different ways, the second item may be selected in *n* – 1 ways, and so on**

Factorial Rule

(when some items are identical to others)

There are n items available, and some items are identical to others. If there are n_1 alike, n_2 alike, . . . n_k alike, the number of permutations (or sequences) of all items selected without replacement is

$$\frac{n!}{n_1! \cdot n_2! \cdots\cdots\cdots n_k!}$$

- There are eight balls number as 1,1,1,2,2,3,4,5. What is the number of possible sequences of these balls?

$$\frac{8!}{3!\,2!}$$

Permutations Rule

Requirements:

1. **There are *n* different items available.**
2. **We select *r* of the *n* items (without replacement).**
3. **We consider rearrangements of the *r* items to be different sequences. (The permutation of ABC is different from CBA and is counted separately.)**

If the preceding requirements are satisfied, the number of permutations (or sequences) of *r* items selected from *n* available items (without replacement) is

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

How do you interpret this?

Example - Permutation

- There are 10 members on the board of directors for a certain non-profit institution. If they must select a chairperson, vice chairperson, and secretary, how many different cases are possible?

$$\frac{10!}{(10-3)!} = 10 \cdot 9 \cdot 8$$

Combinations Rule

Requirements:

1. **There are *n* different items available.**
2. **We select *r* of the *n* items (without replacement).**
3. **We consider rearrangements of the same items to be the same (The combination of ABC is the same as CBA)**

If the preceding requirements are satisfied, the number of combinations of *r* items selected from *n* different items is

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r} $$

How do you interpret this?

Example – Permutation and Combination

- There are 10 members on the board of directors for a certain non-profit institution. If they must select a chairperson, vice chairperson, secretary **as well as three additional ethics subcommittee members**, how many different cases are possible?

$$\binom{10}{3} * 3! * \binom{7}{3}$$