Matrices

Introduction

A set of mn numbers, arranged in a rectangular formation (array or table) having m rows and n columns and enclosed by a square bracket [] is called mxn matrix (read "m by n matrix").

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The order or dimension of a matrix is the ordered pair having as first component the number of rows and as second component the number of columns in the matrix. If there are 3 rows and 2 columns in a matrix, then its order is written as (3, 2) or (3×2) read as three by two.

A matrix is denoted by a capital letter and the elements within the matrix are denoted by lower case letters

e.g. matrix [A] with elements a_{ii}

$$\mathbf{A}_{mxn} = \begin{bmatrix} a_{11} & a_{12} \dots & a_{ij} & a_{in} \\ a_{21} & a_{22} \dots & a_{ij} & a_{2n} \\ \mathbb{X} & \mathbb{X} & \mathbb{X} & \mathbb{X} \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

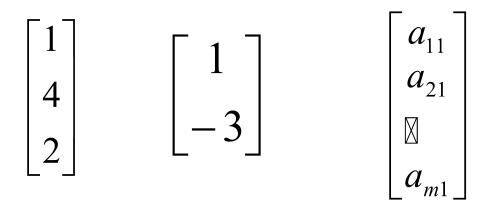
i goes from 1 to m

j goes from 1 to n

TYPES OF MATRICES

1. Column matrix or vector:

The number of rows may be any integer but the number of columns is always 1



TYPES OF MATRICES

2. Row matrix or vector

Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \boxtimes & a_{1n} \end{bmatrix}$$

TYPES OF MATRICES

3. Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns

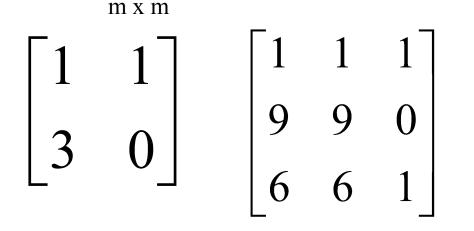
$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$
$$m \neq n$$

TYPES OF MATRICES

4. Square matrix

The number of rows is equal to the number of columns

(a square matrix **A** has an order of m)



The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which i=j

TYPES OF MATRICES

5. Diagonal matrix

A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$ $a_{ij} \neq 0$ for some or all i = j

TYPES OF MATRICES

6. Unit or Identity matrix - I

A diagonal matrix with ones on the main diagonal

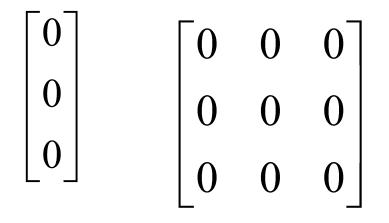
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$
 $a_{ij} = 1$ for some or all $i = j$

TYPES OF MATRICES

7. Null (zero) matrix - 0

All elements in the matrix are zero



 $a_{ij} = 0$ For all i,j

TYPES OF MATRICES

8. Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

TYPES OF MATRICES

8a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i > j$

TYPES OF MATRICES

8b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all i < j

Matrices – Introduction **TYPES OF MATRICES**

9. Scalar matrix

A diagonal matrix whose main diagonal elements are equal to the same scalar

A scalar is defined as a single number or constant

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$
 $a_{ij} = a$ for all $i = j$

Matrices

Matrix Operations

EQUALITY OF MATRICES

Two matrices are said to be equal only when all corresponding elements are equal

Therefore their size or dimensions are equal as well

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \mathbf{A} = \mathbf{B}$$

Some properties of equality: IIf A = B, then B = A for all A and B IIf A = B, and B = C, then A = C for all A, B and C

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

If $\mathbf{A} = \mathbf{B}$ then $a_{ij} = b_{ij}$

ADDITION AND SUBTRACTION OF MATRICES

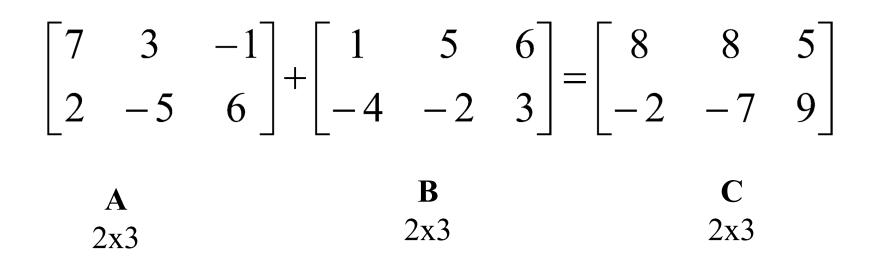
The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes cannot be added or subtracted

Commutative Law: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

Associative Law: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + \mathbf{B} + \mathbf{C}$



 $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$

 $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$ (where $-\mathbf{A}$ is the matrix composed of $-\mathbf{a}_{ij}$ as elements)

$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

SCALAR MULTIPLICATION OF MATRICES

Matrices can be multiplied by a scalar (constant or single element)

 $\mathbf{k}\mathbf{A} = \mathbf{A}\mathbf{k}$

Let k be a scalar quantity; then

Ex. If k=4 and $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$

Matrices - Operations $4 \times \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} \times 4 = \begin{bmatrix} 12 & -4 \\ 8 & 4 \\ 8 & -12 \\ 16 & 4 \end{bmatrix}$

Properties:

- $k (\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$
- $(\mathbf{k} + \mathbf{g})\mathbf{A} = \mathbf{k}\mathbf{A} + \mathbf{g}\mathbf{A}$
- k(AB) = (kA)B = A(k)B
- k(gA) = (kg)A

MULTIPLICATION OF MATRICES

The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication to be possible

i.e. the number of columns of \mathbf{A} must equal the number of rows of \mathbf{B}

Example.

A x **B** = **C** (1x3) (3x1) (1x1)

 $\mathbf{B} \times \mathbf{A} = \text{Not possible!}$ (2x1) (4x2)

 $\mathbf{A} \times \mathbf{B} = \text{Not possible!}$ (6x2) (6x3)

Example

A x **B** = **C** (2x3) (3x2) (2x2)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$

$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$

$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$

$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row *i* of **A** with column *j* of **B** – row by column multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1 \times 4) + (2 \times 6) + (3 \times 5) & (1 \times 8) + (2 \times 2) + (3 \times 3) \\ (4 \times 4) + (2 \times 6) + (7 \times 5) & (4 \times 8) + (2 \times 2) + (7 \times 3) \end{bmatrix}$$
$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Remember also:

 $\mathbf{IA} = \mathbf{A}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix} = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

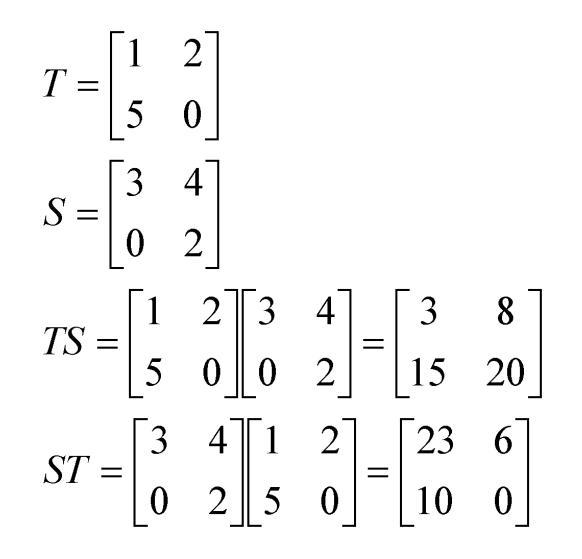
Assuming that matrices **A**, **B** and **C** are conformable for the operations indicated, the following are true:

- $1. \quad \mathbf{AI} = \mathbf{IA} = \mathbf{A}$
- 2. A(BC) = (AB)C = ABC (associative law)
- 3. A(B+C) = AB + AC (first distributive law)
- 4. (A+B)C = AC + BC (second distributive law)

Caution!

- 1. AB not generally equal to BA, BA may not be conformable
- 2. If AB = 0, neither A nor B necessarily = 0
- 3. If AB = AC, B not necessarily = C

AB not generally equal to BA, BA may not be conformable



If AB = 0, neither A nor B necessarily = 0

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

TRANSPOSE OF A MATRIX

If:

$$A = A^{3} = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

 $2x^{3} = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$

Then transpose of A, denoted A^{T} is:

$$A^{T} = {}_{2}A^{3^{T}} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$
$$a_{ij} = a_{ji}^{T} \quad \text{For all } i \text{ and } j$$

To transpose:

Interchange rows and columns

The dimensions of \mathbf{A}^{T} are the reverse of the dimensions of \mathbf{A}

$$A = {}_{2}A^{3} = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix} \qquad 2 \ge 3$$
$$A^{T} = {}_{3}A^{T^{2}} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix} \qquad 3 \ge 2$$

Properties of transposed matrices:

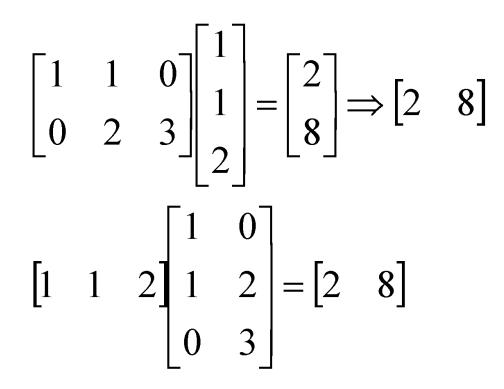
- 1. $(\mathbf{A} + \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$
- 2. $(\mathbf{A}\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}} \mathbf{A}^{\mathrm{T}}$
- 3. $(\mathbf{k}\mathbf{A})^{\mathrm{T}} = \mathbf{k}\mathbf{A}^{\mathrm{T}}$
- $4. \quad (\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$

1.
$$(\mathbf{A} + \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 5 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

 $(\mathbf{A}\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}} \mathbf{A}^{\mathrm{T}}$



SYMMETRIC MATRICES

A Square matrix is symmetric if it is equal to its transpose:

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$
$$A^{T} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

When the original matrix is square, transposition does not affect the elements of the main diagonal

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$A^{T} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

The identity matrix, **I**, a diagonal matrix **D**, and a scalar matrix, **K**, are equal to their transpose since the diagonal is unaffected.

INVERSE OF A MATRIX

Consider a scalar k. The inverse is the reciprocal or division of 1 by the scalar.

Example:

k=7 the inverse of k or $k^{-1} = 1/k = 1/7$

Division of matrices is not defined since there may be AB = ACwhile $B \neq C$

Instead matrix inversion is used.

The inverse of a square matrix, A, if it exists, is the unique matrix A^{-1} where:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$$

Example:

$$A = {}_{2}A^{2} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$
$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Because:

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Properties of the inverse:

$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(A^{-1})^{-1} = A$$
$$(A^{-1})^{-1} = (A^{-1})^{T}$$
$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

A square matrix that has an inverse is called a nonsingular matrix A matrix that does not have an inverse is called a singular matrix Square matrices have inverses except when the determinant is zero When the determinant of a matrix is zero the matrix is singular

DETERMINANT OF A MATRIX

To compute the inverse of a matrix, the determinant is required Each square matrix **A** has a unit scalar value called the

determinant of A, denoted by det A or |A|

If
$$A = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$$

then $|A| = \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix}$

If A = [A] is a single element (1x1), then the determinant is defined as the value of the element

Then $|\mathbf{A}| = \det \mathbf{A} = \mathbf{a}_{11}$

If A is $(n \times n)$, its determinant may be defined in terms of order (n-1) or less.

MINORS

If A is an n x n matrix and one row and one column are deleted, the resulting matrix is an $(n-1) \times (n-1)$ submatrix of A.

The determinant of such a submatrix is called a minor of **A** and is designated by m_{ij} , where *i* and *j* correspond to the deleted row and column, respectively.

 m_{ij} is the minor of the element a_{ij} in A.

eg.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in A has a minor

Delete first row and column from A.

The determinant of the remaining 2 x 2 submatrix is the minor of a₁₁

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Therefore the minor of a_{12} is:

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for a_{13} is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

COFACTORS

The cofactor C_{ii} of an element a_{ii} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number *i* and column *j* is even, $c_{ij} = m_{ij}$ and when *i*+*j* is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1, j=1) = (-1)^{1+1}m_{11} = +m_{11}$$

$$c_{12}(i=1, j=2) = (-1)^{1+2}m_{12} = -m_{12}$$

$$c_{13}(i=1, j=3) = (-1)^{1+3}m_{13} = +m_{13}$$

DETERMINANTS CONTINUED

The determinant of an n x n matrix \mathbf{A} can now be defined as

$$|A| = \det A = a_{11}c_{11} + a_{12}c_{12} + \mathbb{A} + a_{1n}c_{1n}$$

The determinant of A is therefore the sum of the products of the elements of the first row of A and their corresponding cofactors.

(It is possible to define $|\mathbf{A}|$ in terms of any other row or column but for simplicity, the first row only is used)

Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Has cofactors :

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

And:

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of A is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$
$$|A| = (3)(2) - (1)(1) = 5$$

For a 3 x 3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$
$$c_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$
$$c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

The determinant of a matrix A is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Which by substituting for the cofactors in this case is:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$A = (1)(2-0) - (0)(0+3) + (1)(0+2) = 4$$

ADJOINT MATRICES

A cofactor matrix C of a matrix A is the square matrix of the same order as A in which each element a_{ij} is replaced by its cofactor c_{ij} .

Example:

If
$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

The cofactor C of A is $C = \begin{vmatrix} 4 & 3 \\ -2 & 1 \end{vmatrix}$

The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adjA = C^{T}$$

It can be shown that:

$$\mathbf{A}(\mathrm{adj}\;\mathbf{A}) = (\mathrm{adj}\mathbf{A})\;\mathbf{A} = |\mathbf{A}|\;\mathbf{I}$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$
$$|A| = (1)(4) - (2)(-3) = 10$$
$$adjA = C^{T} = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

$$A(adjA) = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$
$$(adjA)A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

USING THE ADJOINT MATRIX IN MATRIX INVERSION

Since

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$$

and

$$\mathbf{A}(\mathrm{adj}\;\mathbf{A}) = (\mathrm{adj}\mathbf{A})\;\mathbf{A} = |\mathbf{A}|\;\mathbf{I}$$

then

$$A^{-1} = \frac{adjA}{|A|}$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$
$$A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$$
To check
$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$
$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$
$$A^{-1}A = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Example 2

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

The determinant of A is

 $|\mathbf{A}| = (3)(-1-0)-(-1)(-2-0)+(1)(4-1) = -2$

The elements of the cofactor matrix are

$$\begin{split} c_{11} &= +(-1), & c_{12} &= -(-2), & c_{13} &= +(3), \\ c_{21} &= -(-1), & c_{22} &= +(-4), & c_{23} &= -(7), \\ c_{31} &= +(-1), & c_{32} &= -(-2), & c_{33} &= +(5), \end{split}$$

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

so
$$adjA = C^{T} = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-2} \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

The result can be checked using

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$$

The determinant of a matrix must not be zero for the inverse to exist as there will not be a solution

Nonsingular matrices have non-zero determinants

Singular matrices have zero determinants