Mathematical Induction

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Basics

- **The Well-Ordering Property** - Every nonempty set of nonnegative integers has a least element.
- Many theorems state that P(n) is true for all positive integers.
  - For example, P(n) could be the statement that the sum of the first n positive integers $1+2+3+ . . . + n = n(n+1)/2$
- **Mathematical Induction** is a technique for proving theorems of this kind.

Steps in an Induction Proof

1. Basis step : The proposition is shown to be true for n=1 (or, more generally, the first element in the set)
2. Inductive step: The implication P(n) →P(n+1) is shown to be true for every positive integer n (more generally, for every integer element above a lower bound, which could be negative).

For $n \in Z^+$

$$[P(1) \wedge \forall n(P(n) \rightarrow P(n+1))] \rightarrow \forall n P(n)$$

Example: If p(n) is the proposition that the sum of the first n positive integers is n(n+1)/2, prove p(n) for $n \in Z^+$.

Basis Step: We will show p(1) is true.

p(1) = 1(1+1)/2 = 2/2 = 1

Inductive Step:

We want to show that p(n) → p(n+1)

Assume 1+2+3+4+. . . + n = n(n+1)/2

Then 1+2+3+4+. . . + n + (n+1) = n(n+1)/2 + n+1 = n(n+1)/2 + (n+1)(2/2) =

[n(n+1) + 2(n+1)]/2 = [n^2 + 3n +2]/2 = [(n+1)(n+2)]/2

Since p(1) is true and p(n) → p(n+1), then p(n) is true for all positive integers n.

If p(n) is the proposition that the sum of the first n odd integers is n^2, prove p(n) for $n \in Z^+$

Induction Proof

Basis Step: We will show that p(1) is true.

$1 = 1^2$

Inductive Step

We want to show that $p(n) \rightarrow p(n+1)$

Assume $1 + 3 + 5 + 7 + \ldots + (2n-1) = n^2$

Then $1 + 3 + 5 + 7 + \ldots + (2n-1) + (2n + 1) = n^2 + 2n + 1 = (n+1)^2$

Since p(1) is true and $p(n) \rightarrow p(n+1)$, then p(n) is true for all positive integers n.

If p(n) is the proposition that $\sum_{j=0}^{n} 2^j = 2^{n+1} - 1$ prove p(n) when n is a non-negative integer.

Inductive Proof

Basis Step: We will show p(0) is true.

$2^0 = 1 = 2-1 = 2^{0+1} - 1$

Inductive step: We want to show that p(n) → p(n+1)

Assume $2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$, then

$2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^n + 2^{n+1} = 2^{n+1} - 1 + 2^{n+1}$

$= 2(2^{n+1}) - 1 = 2^{n+2} - 1$

Since p(0) is true and p(n) → p(n+1), then p(n) is true for all nonnegative integers n.

More General Example

Let p(n) be the statement that $n! > 2^n$. Prove p(n) for $n \geq 4$.

Inductive Proof:

Basis Step: We will show that p(4) is true.

$4! = 24 > 2^4 = 16$

Inductive Step: We want to show that $\forall k \geq 4$, $p(k) \rightarrow p(k+1)$. Assume $k! > 2^k$ for some arbitrary $k \geq 4$.

$n! > 2^n$ (cont.)

$(k+1)! \quad = (k+1)k!$

$\quad > (k+1)2^k$ (inductive hypothesis)

$\quad > 2 * 2^k$ (since $k \geq 4$)

$\quad = 2^{k+1}$

Since $p(4)$ is true and $p(n) \rightarrow p(n+1)$, then $p(n)$ is true for all integers $n \geq 4$.

Let p(n) be the statement that all numbers of the form 8^n-2^n for $n \in Z^+$ are divisible by 6 (i.e., can be written as 6k for some $k \in Z$). Prove p(n)

Inductive Proof

Basis Step: We will show that p(1) is true.

$8^1-2^1 = 6$ which is clearly divisible by 6.

Inductive Step: We must show that $[\forall k \in Z^+ (8^k-2^k)$ is divisible by $6 \rightarrow (8^{k+1}-2^{k+1})$ is divisible by 6].

Divisible by 6 Example (cont.)

$8^{k+1} - 2^{k+1} = 8(8^k) - 2^{k+1}$

$= 8(8^k) - 8(2^k) + 8*2^k - 2^{k+1}$

$= 8(8^k - 2^k) + 8*2^k - 2^{k+1}$

$= 8(8^k - 2^k) + 8*2^k - 2*2^k$

$= 8(8^k - 2^k) + 6*2^k$

By the inductive hypothesis $8(8^k-2^k)$ is divisible by 6 and clearly $6*2^k$ is divisible by 6. Thus $8^{k+1} - 2^{k+1}$ is divisible by 6. Since p(1) is true and p(n) → p(n+1), then p(n) is true for all positive integers n.

Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer

Basis Step: When n = 1, then $4^{n+1} + 5^{2n-1} = 4^{1+1} + 5^{2(1)-1} = 4^2+5 = 21$ which is clearly divisible by 21.

Inductive Step: Assume that $4^{n+1} + 5^{2n-1}$ is divisible by 21. We must show that $4^{n+1+1} + 5^{2(n+1)-1}$ is divisible by 21.

$$4^{n+1+1} + 5^{2(n+1)-1} = 4*4^{n+1} + 5^{2n+2-1}$$
$$= 4*4^{n+1} + 25*5^{2n-1}$$
$$= 4*4^{n+1} + (4+21)\ 5^{2n-1}$$
$$= 4(4^{n+1}+ 5^{2n-1}) +21*5^{2n-1}$$

The first term is divisible by 21 by the induction hypothesis and clearly the second term is divisible by 21. Therefore their sum is divisible by 21.

Second Principle of Mathematical Induction (Strong Induction)

1. <u>Basis Step:</u> The proposition p(1) is shown to be true.
2. <u>Inductive Step:</u> It is shown that $[p(1) \wedge p(2) \wedge \ldots \wedge p(n)] \rightarrow p(n+1)$ is true for every positive integer n.
3. Sometimes have multiple basis steps to prove.

Example of Strong Induction

Consider the sequence defined as follows:

$b_0 = 1$

$b_1 = 1$

$b_n = 2b_{n-1} + b_{n-2}$ for $n>1$

1,1, 3,7, 17,…

$b_{0,}\ b_{1,}\ b_{2,}\ b_{3,}\ b_{4,\ldots}$

Prove that b_n is odd

Inductive Proof Using Strong Induction

Basis Cases: (One for n=0 and one for n=1 since the general formula is not applicable until n>1, but it involves both b_0 and b_1.)

$b_0 = b_1 = 1$ so both b_0 and b_1 are odd.

Inductive Step:

Consider k>1 and assume that b_n is odd for all $0 \leq n \leq k$. We must show that b_{k+1} is odd.

Proof Example (cont.)

From the formula we know that

$b_{k+1} = 2b_k + b_{k-1}$. Clearly the first term is even. By the inductive hypothesis the second term is odd. Since the sum of an even integer and an odd integer is always odd (which we proved in number theory), then b_{k+1} is odd.

In this example we did not need all p(n), $0 \leq n \leq k$, but we did need p(k) and p(k-1). Note that a proof using weak induction would only be able to assume p(k).