Relations

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- Relations, properties of relations
- Equivalence relations
- Partial orderings
- Hasse diagrams
- The topological sorting algorithm

Relations

- The **Cartesian product** of A and B, denoted by

$$A \times B,$$

is the set of all **ordered pairs** (a, b), where

$a \in A$ and $b \in B$.

Hence, $A \times B = \{(a, b) | a \in A \wedge b \in B\}$.

Relations

Definition 1

A **binary relation** between two sets A and B is defined to be a subset R of the Cartesian product $A \times B$.

We use the notation $\boldsymbol{aRb}$ to denote that $(a, b) \in R$. Moreover, when (a, b) belongs to R, a is said to be related to b by R.

In the special case when $A = B$, we simply refer to R as a relation on A.

Relations

<u>Example 1</u>

Write down all ordered pairs belonging to the following binary relations between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$:

a) $U = \{(x, y) | \, x + y = 9\}$;

b) $V = \{(x, y) | \, \mathrm{x} < \mathrm{y}\}$.

<u>Solution</u>

a) $U = \{(3, 6), (5, 4), (7, 2)\}$;

b) $V = \{(1, 2), (1, 4), (1, 6), (3, 4), (3, 6), (5, 6)\}$. ■

Relations

<u>Example 2</u>

The following defines a relation on $A = \{1, 2, 3, 4, 5, 6\}$:

$$R = \{(x, \text{y}) \mid x \text{ is a divisor of } y\}.$$

Write down the ordered pairs belonging to R.

<u>Solution</u>

$R = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\}$ ■

Relations

We can use the concept of a **directed graph** to describe the ordered pairs belonging to a given binary relation.

Relations

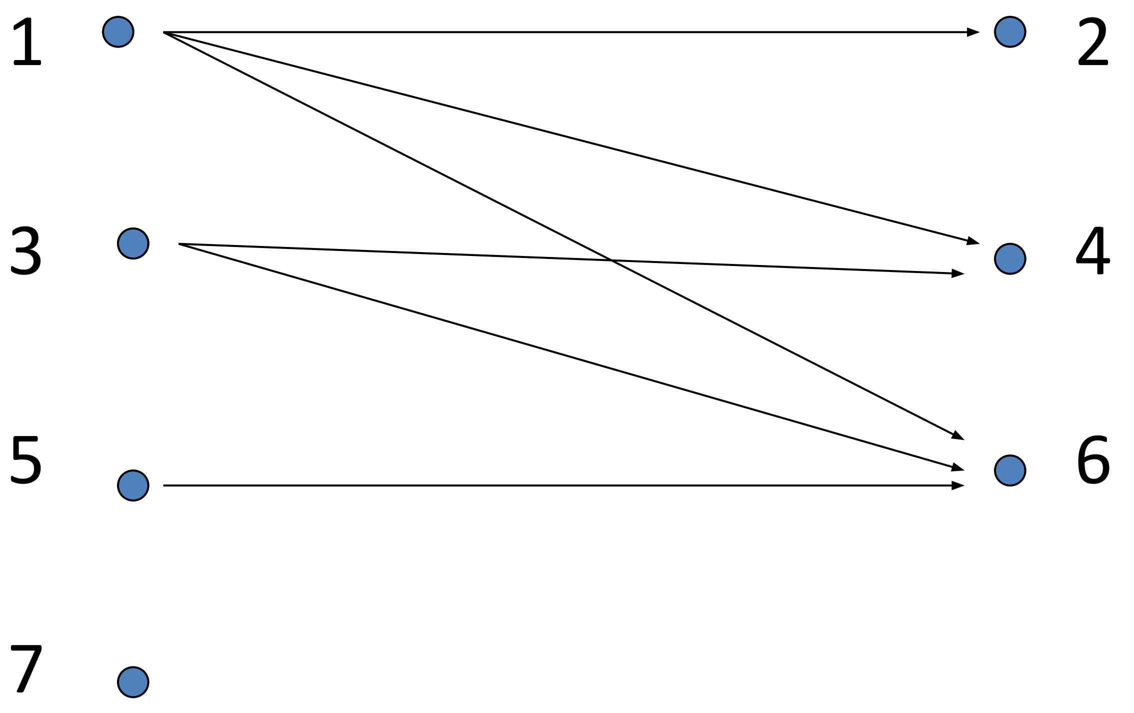
- Let A and B be two finite sets and let R be a binary relation between these two sets.

We represent the elements of these two sets as the vertices of a graph.

For each of the ordered pairs in a relation R, draw an arrow linking the related elements.

This is called a **directed graph** or **digraph**.

<u>Example 3</u> Consider the relation V between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$ given in example 1, b): $V = \{(1, 2), (1, 4), (1, 6), (3, 4), (3, 6), (5, 6)\}$. The corresponding directed graph is given in the figure below.



Relations

- For a relation on a single set A we use a directed graph in which a single set of vertices represents the elements of A and arrows link the related elements.

Properties of relations

- We now restrict our attention to relations defined on a single set A and define a number of properties which a given relation on A may or may not possess.

Properties of relations

Definition 2

A relation R on a set A is **reflexive** if xRx for all x in A.

In terms of ordered pairs a given relation is reflexive if (x, x) belongs to R for all possible values of x.

Properties of relations

- In terms of directed graph representation R is reflexive if there is always an arrow from every vertex to itself.

Properties of relations

Definition 3

A relation R on a set A is **symmetric** when xRy implies yRx for all x and y in A.

In terms of ordered pairs a given relation is symmetric if when (x, y) belongs to R then (y, x) belongs to R for all possible values of x and y.

Properties of relations

- In terms of directed graph representation R is symmetric if whenever there is an arc from x to y then there is also an arc from y to x.

Properties of relations

Definition 4

A relation R on a set A is antisymmetric when xRy and yRx implies $x = y$ for all x and y in A.

In terms of ordered pairs a given relation is antisymmetric if when (x, y) belongs to R and (y, x) belongs to R then $x = y$ for all possible values of x and y.

Properties of relations

- In terms of directed graph representation R is antisymmetric if whenever there is an arc from x to y and x is not equal to y then there is no arc from y to x.

Properties of relations

Definition 5

A relation R on a set A is **transitive** when xRy and yRz implies xRz for all x, y and z in A.

In terms of ordered pairs a given relation is transitive if when (x, y) belongs to R and (y, z) belongs to R then (x, z) belongs to R for all possible values of x, y and z.

Properties of relations

- In terms of directed graph representation R is transitive if whenever there is an arc from x to y and there is an arc from y to z then there is no arc from x to z.

Properties of relations

<u>Example 4</u>

Which of the following define a relation that is reflexive, symmetric, antisymmetric or transitive?

a) «x divides y» on the set of natural numbers;

b) «$x \neq y$» on the set of integers;

c) «x has the same age as y» on the set of all people?

Equivalence relation

Definition 1

A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Equivalence relations are important throughout mathematics and computer science.

One reason for this is that in an equivalence relation, when two elements are related it makes sense to say they are equivalent.

Equivalence relation

Definition 2

Two elements a and b that are related by an equivalence relation are called **equivalent**.

The notation $\boldsymbol{a \sim b}$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Equivalence relation

<u>Example 1</u>

Let R be the relation on the set of integers such that

$aRb \overset{def}{\iff} a = b$ or $a = -b$.

R is reflexive, symmetric, and transitive. It follows that R is an equivalence relation.

Equivalence relation

Example 2

Let R be the relation on the set of real numbers such that

$aRb \overset{def}{\Longleftrightarrow} a - b$ is an integer.

R is reflexive, symmetric, and transitive. It follows that R is an equivalence relation.

Equivalence relation

Example 3 (**Congruence modulo m**)

Let m be an integer with $m > 1$. Let $a, b \in \boldsymbol{Z}$.

$a \equiv b(mod\ m) \overset{def}{\Longleftrightarrow} m \mid (a - b)$

$R = \{(a, b) | a \equiv b(mod\ m)\}$ is an equivalence relation on the set of integers.

1) R is reflexive : $m \mid (a - a), \forall a \in \boldsymbol{Z}$;
2) R is symmetric : $m \mid (a - b) \Rightarrow m \mid (b - a), \forall a, b \in \boldsymbol{Z}$;
3) R is transitive : $\big(m \mid (a - b)\big) \wedge \big(m \mid (b - c)\big) \Rightarrow \big(m \mid (a - c)\big), \forall a, b, c \in \boldsymbol{Z}$.

Equivalence relation

Example 4

Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x:

$$aRb \overset{def}{\Longleftrightarrow} l(a) = l(b).$$

R is reflexive, symmetric, and transitive. It follows that R is an equivalence relation.

Equivalence relation

<u>Example 5</u>

Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that **sR_nt if and only if $s = t$, or both s and t have at least n characters and the first n characters of s and t are the same**.

That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t begins with the n characters at the start of s.

R_n is reflexive, symmetric, and transitive. It follows that R is an equivalence relation.

Equivalence relation

Example 6

The "divides" relation $|$ in the set of positive integers is not an equivalence relation.

This relation is not symmetric (for instance, $2|4$ but $4 \nmid 2$).

Equivalence relation

Example 7

Let R be the relation on the set of real numbers such that

$$xRy \overset{def}{\Longleftrightarrow} |x - y| < 1$$

R is not an equivalence relation because it is not transitive:

$$2{,}8 \; R \; 1{,}9 \;\; \Leftrightarrow \;\; |2{,}8 - 1{,}9| < 1$$
$$1{,}9 \; R \; 1{,}1 \;\; \Leftrightarrow \;\; |1{,}9 - 1{,}1| < 1$$
$$2{,}8 \; \not R \; 1{,}1 \;\; \Leftrightarrow \;\; |2{,}8 - 1{,}1| \geq 1$$

Equivalence relation

Definition 3

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.

The equivalence class of a with respect to R is denoted by $[a]_R$:

$$[a]_R = \{s|(a,s) \in R\}.$$

When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

Equivalence relation

Definition 3

If $b \in [a]_R$, then b is called a **representative** of this equivalence class.

Any element of a class can be used as a representative of this class.

That is, there is nothing special about the particular element chosen as the representative of the class.

Equivalence relation

Example 8

Let R be the relation on the set of real numbers such that

$$aRb \overset{def}{\iff} a = b \text{ or } a = -b.$$

Find $[a]_R = [a]$.

Solution

$[a] = \{-a, a\}$

For example,

$[5] = \{-5,5\}$, $[7] = \{-7,7\}$, $[0] = \{0\}$.

Equivalence relation

Example 9

What are the equivalence classes of 0 and 1 for congruence modulo 4?

Solution

$[0]_4 = \{a | a \equiv 0 (mod\ 4)\} = \{\dots, -8, -4, 0, 4, 8, \dots\}$

$[1]_4 = \{a | a \equiv 1 (mod\ 4)\} = \{\dots, -7, -3, 1, 5, 9, \dots\}$

Equivalence relation

Example 10

What is the equivalence class of the string 0111 with respect to the equivalence relation R_3 on the set of all bit strings?

Solution

$$[0111]_{R_3} = \\ = \{011, 0110, 0111, 01100, 01101, 01110, 01111, \ldots\}$$

Equivalence relation

Theorem 1

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

1) aRb,
2) $[a] = [b]$,
3) $[a] \cap [b] \neq \emptyset$.

Определение 1
Бинарным отношением между множествами A и B называется подмножество R декартова произведения $A \times B$.
Если упорядоченная пара (a, b) принадлежит отношению R, то пишут aRb и говорят, что **a находится в отношении R с b.**
В том случае, когда $A = B$, мы говорим просто об **отношении** R на A.

Proof

Assume that $c \in [a] \Rightarrow aRc$

aRb, R is symmetric $\Rightarrow bRa$

bRa, aRc, R is transitive $\Rightarrow bRc$

$bRc \Rightarrow c \in [b]$

This shows that $[a] \subseteq [b]$.

The proof that $[b] \subseteq [a]$ is similar.

? $[a] = [b] \Rightarrow [a] \cap [b] \neq \emptyset$

<u>Proof</u>

Let $[a] = [b] \Rightarrow$

$[a] \cap [b] =$

$[a] \cap [a] =$

$[a] \neq \emptyset,$

(because $a \in [a]$ because R is reflexive).

Пример 1

Выписать упорядоченные пары, принадлежащие следующим бинарным отношениям между множествами $A = \{1, 3, 5, 7\}$ и $B = \{2, 4, 6\}$:

a) $U = \{(x, y) | x + y = 9\}$;

b) $V = \{(x, y) | x < y\}$.

Решение

a) $U = \{(3, 6), (5, 4), (7, 2)\}$;

b) $V = \{(1, 2), (1, 4), (1, 6), (3, 4), (3, 6), (5, 6)\}$. ▬

Proof

Assume that $[a] \cap [b] \neq \emptyset \Rightarrow$

Then there is an element c with:

$$c \in [a] \wedge c \in [b].$$

$$\Downarrow$$

$$aRc \wedge bRc$$

$$\Downarrow$$

$$aRc \wedge cRb$$

$$\Downarrow$$

$$aRb \ \blacksquare$$

Equivalence relation

- It follows that these equivalence classes are either equal or disjoint and that the equivalence classes form a partition of A.

Equivalence relation

Definition 4

A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union.

Equivalence relation

<u>Example 11</u>

What are the sets in the partition of the integers arising from congruence modulo 4?

<u>Solution</u>

There are four congruence classes: $[0]_4$, $[1]_4$, $[2]_4$, $[3]_4$. These congruence classes are disjoint, and every integer is in exactly one of them.

In other words these congruence classes form a partition.

Equivalence relation

Example 12

What are the sets in the partition of the set of all bit strings arising from the relation R_3 on the set of all bit strings?

Solution

Note that every bit string of length less than three is equivalent only to itself. Every bit string of length three or more is equivalent to one of the eight bit strings 000,001,010,011,100,101,110 and 111. We have

$[000]_{R_3}$, $[001]_{R_3}$, $[010]_{R_3}$, $[011]_{R_3}$,
$[100]_{R_3}$, $[101]_{R_3}$, $[110]_{R_3}$, $[111]_{R_3}$.

Example 12 What are the sets in the partition of the set of all bit strings arising from the relation R_3 on the set of all bit strings?

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Note that every bit string of length less than three is equivalent only to itself. Every bit string of length three or more is equivalent to one of the eight bit strings $000, 001, 010, 011, 100, 101, 110$ and 111. We have

$[000]_{R_3}$, $[001]_{R_3}$, $[010]_{R_3}$, $[011]_{R_3}$,
$[100]_{R_3}$, $[101]_{R_3}$, $[110]_{R_3}$, $[111]_{R_3}$.

These 15 equivalence classes are disjoint and every bit string is in exactly one of them. These equivalence classes partition the set of all bit strings.

Partial Orderings

Definition 1

A relation R on a set S is called a **partial ordering** or partial order if it is reflexive, antisymmetric, and transitive.

A set S together with a partial ordering R is called a **partially ordered set**, or **poset**, and is denoted by (S, R). Members of S are called elements of the poset.

Partial Orderings

- <u>Example 1</u>

The “greater than or equal” relation ($\geq$) is a partial ordering on the set of integers.

Partial Orderings

Example 2

The divisibility relation | is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive.

Partial Orderings

<u>Example 3</u>

The inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Partial Orderings

Example 4

Let R be the relation on the set of people such that xRy if x and y are people and x is older than y.

R is not reflexive, because no person is older than himself or herself. It follows that R is not a partial ordering.

Partial Orderings

Definition 2

The elements a and b of a poset $(S, \preccurlyeq)$ are called **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$.

When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called **incomparable**.

Partial Orderings

Definition 3

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a **totally ordered** or **linearly ordered set**, and $\preccurlyeq$ is called a **total order** or a **linear order**.

A totally ordered set is also called a **chain**.

Partial Orderings

Example 5

The poset $(\mathbb{Z}, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.

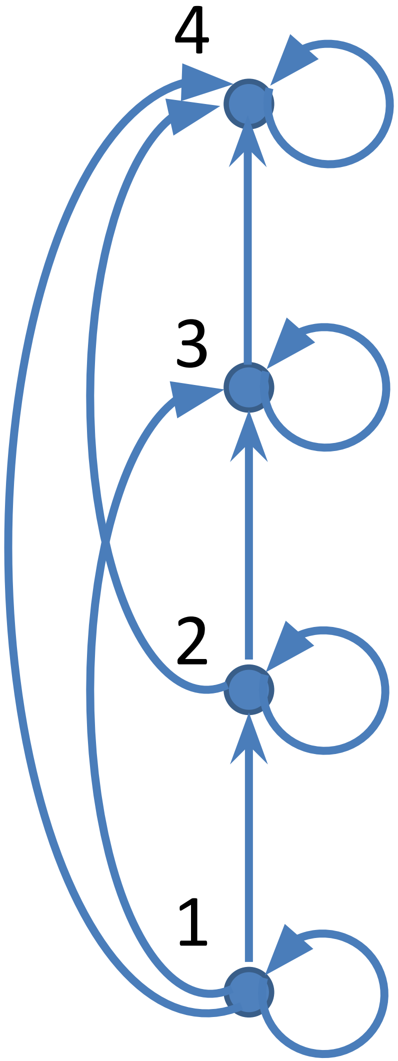
Partial Orderings

Example 6

The poset $(\mathbb{Z}^{+}, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7.

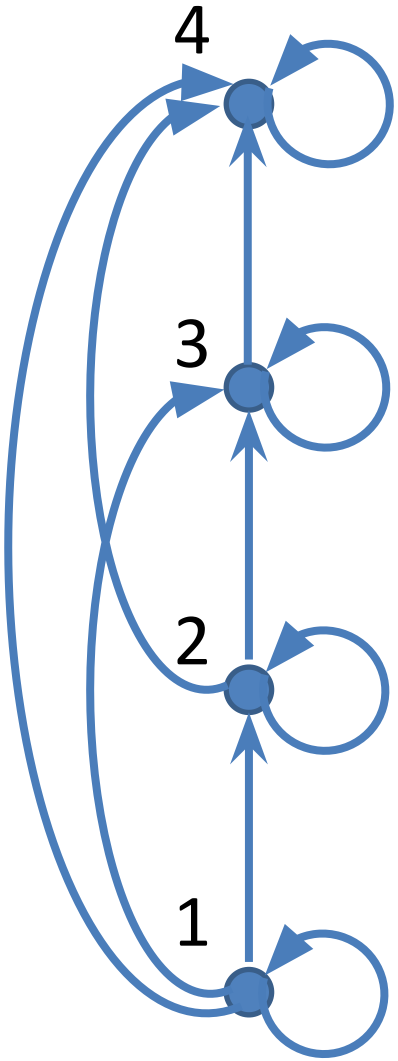
Constructing the Hasse Diagram for $(S = \{1, 2, 3, 4\}, \leq)$

Start with the directed graph for this relation.



Constructing the Hasse Diagram for
$(S = \{1, 2, 3, 4\}, \leq)$

Remove all loops.



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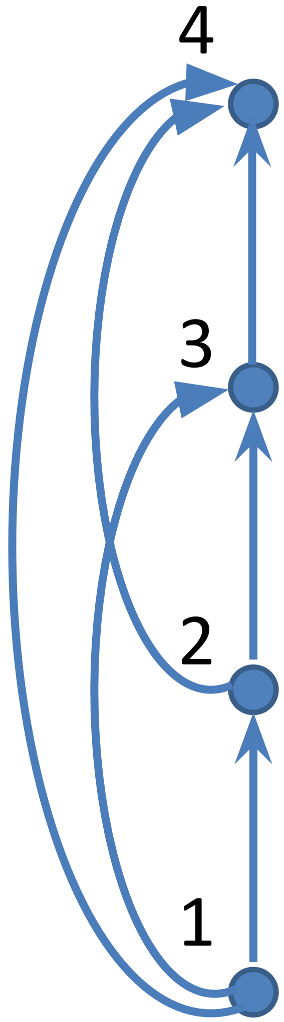
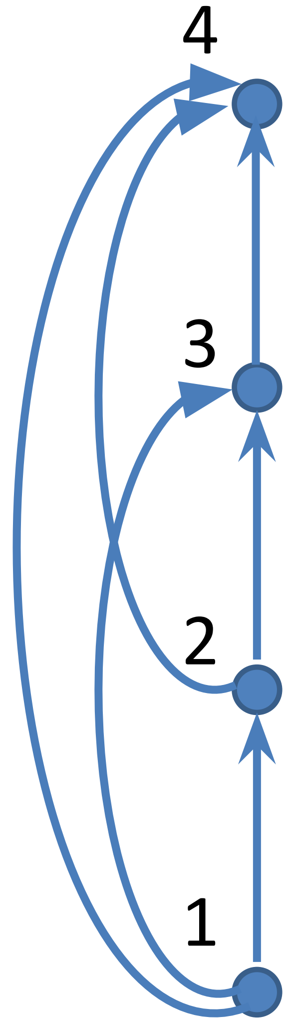


Constructing the Hasse Diagram for $(S = \{1, 2, 3, 4\}, \leq)$

Remove all edges (x, y) for which there is an element $z \in S$ such that $x \leq z$ and $z \leq y$.



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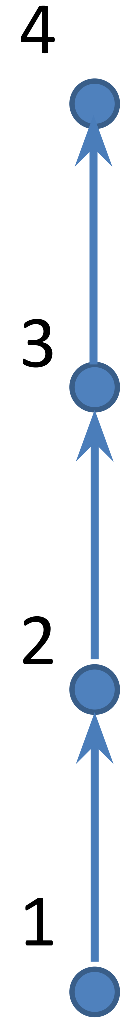
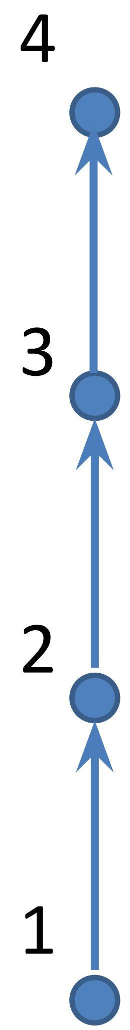


Constructing the Hasse Diagram for $(S = \{1, 2, 3, 4\}, \leq)$

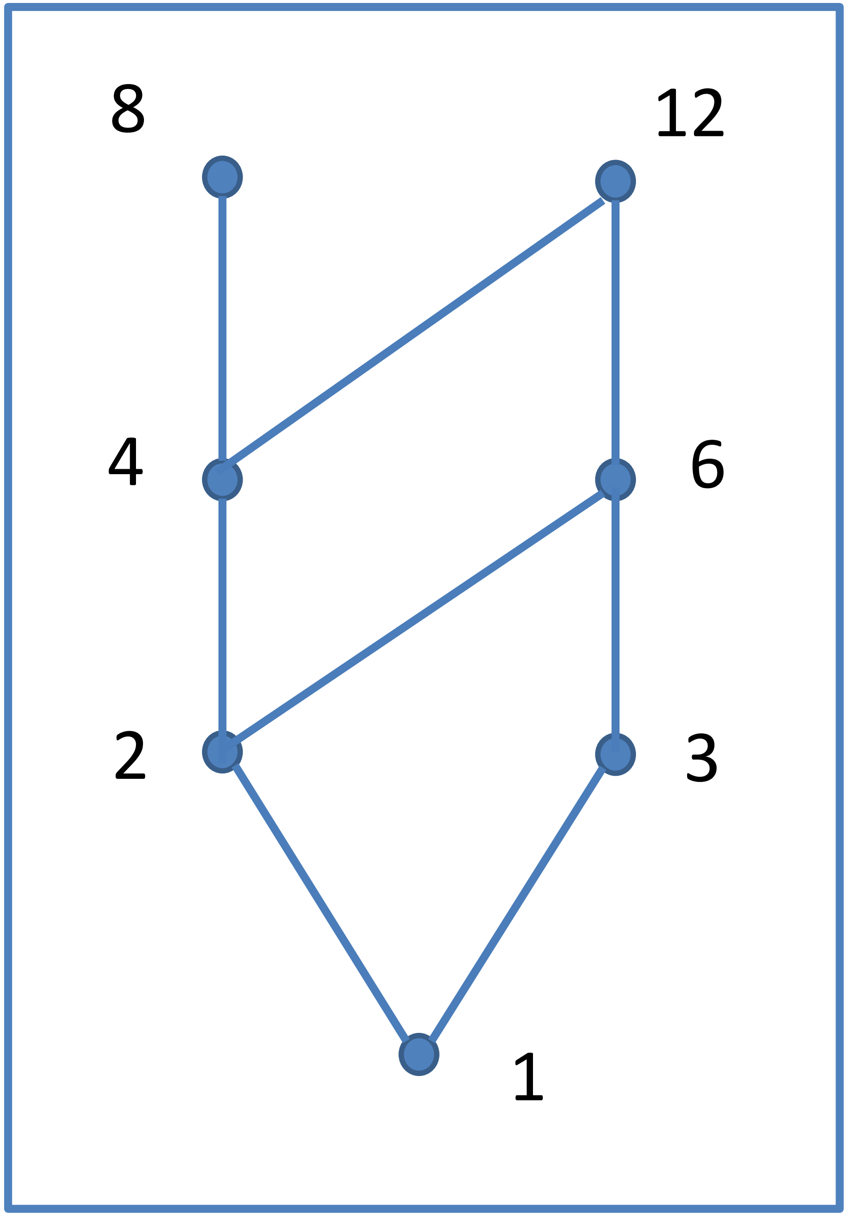
Remove all the arrows on the directed edges, because all edges point "upward" toward their terminal vertex.



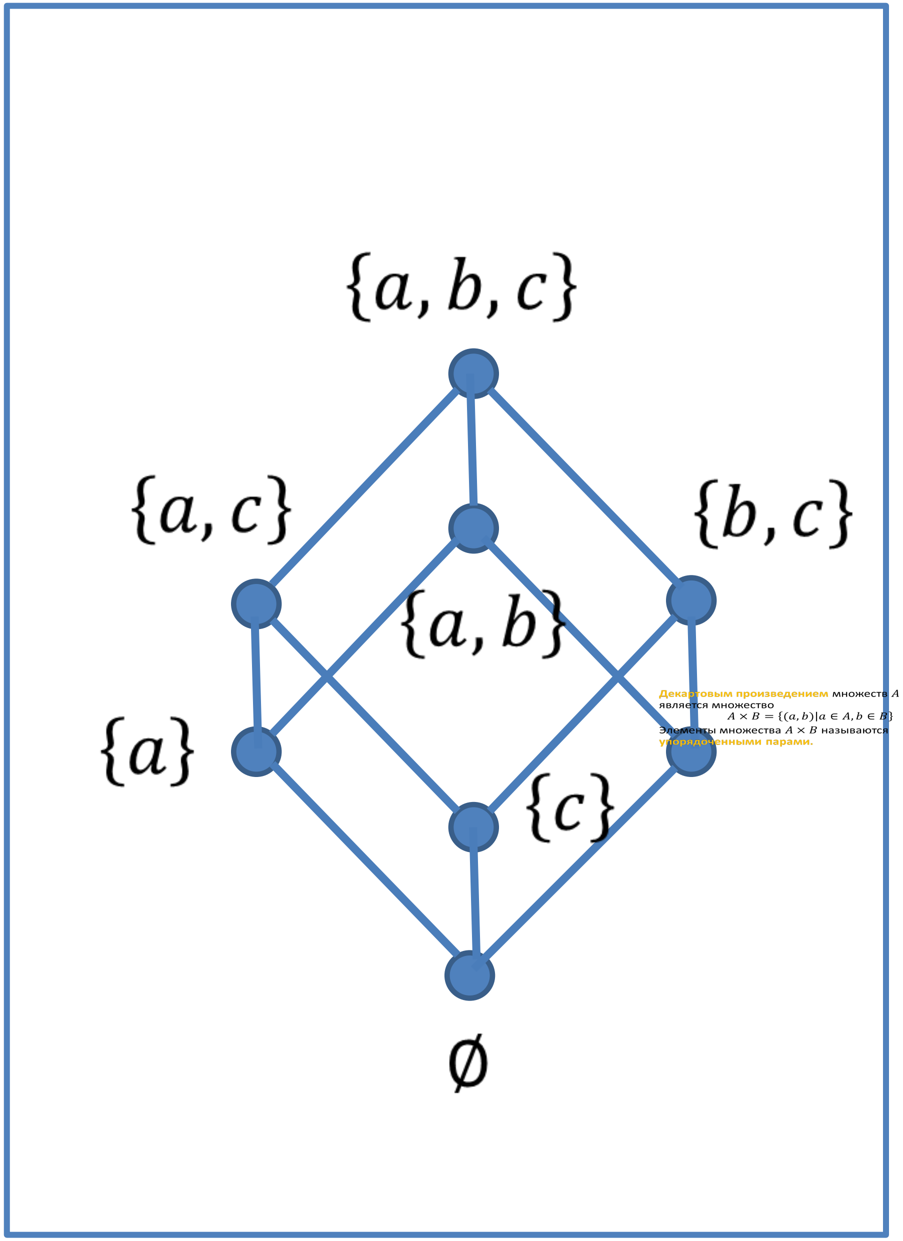
Constructing the Hasse Diagram for $(S = \{1, 2, 3, 4\}, \leq)$

Remove all the arrows on the directed edges, because all edges point “upward” toward their terminal vertex.

Draw the Hasse diagram representing the partial ordering $\{(a, b) | b \vdots a\}$, on $\{1, 2, 3, 4, 6, 8, 12\}$.



Draw the Hasse diagram representing the partial ordering on $(P(\{a, b, c\}), \subseteq)$.



Декартовым произведением множеств A и B является множество

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Элементы множества $A \times B$ называются **упорядоченными парами.**

Maximal and minimal elements

Definition 4

An element of a poset is called **maximal** if it is not less than any element of the poset. That is, a is maximal in the poset $(S, \prec)$ if there is no $b \in S$ such that $a \prec b$.

Similarly, an element of a poset is called **minimal** if it is not greater than any element of the poset. That is, a is minimal if there is no element $b \in S$ such that $b \prec a$.

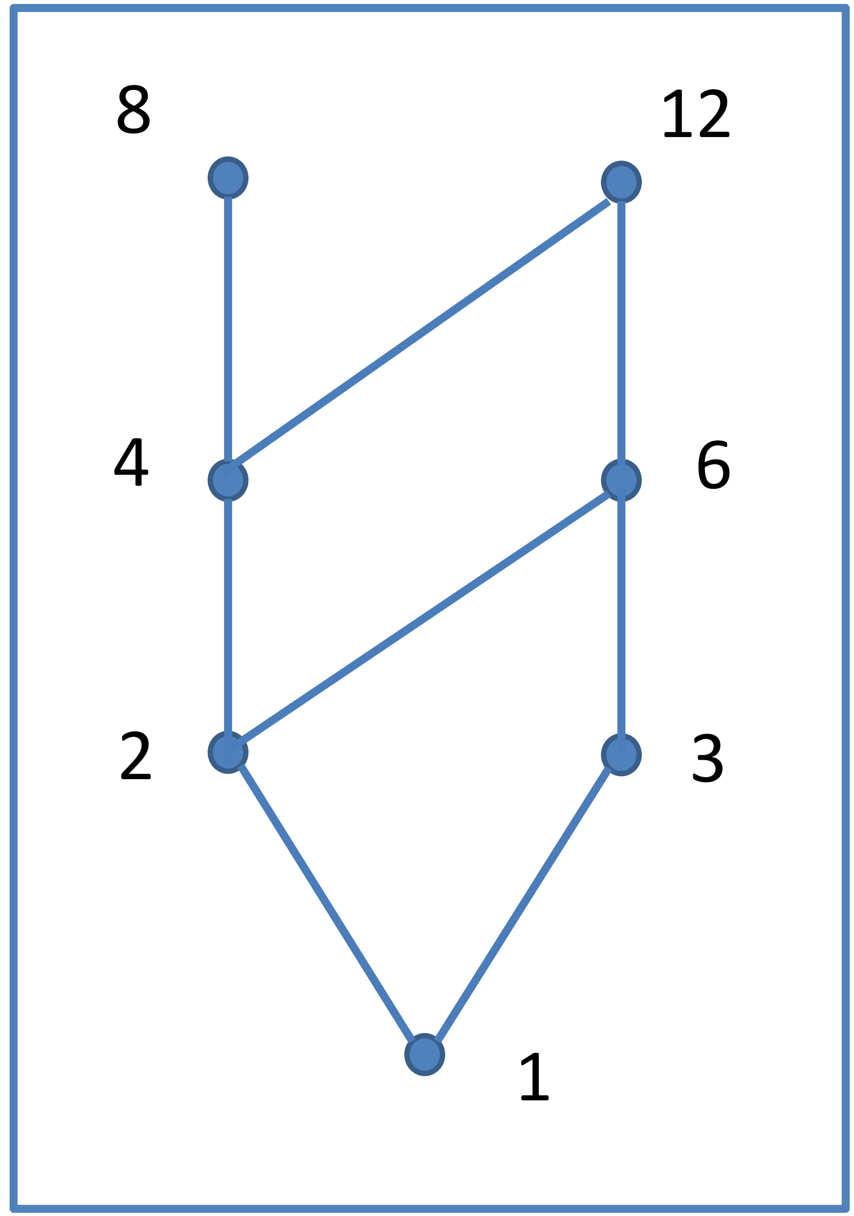
Maximal and minimal elements

Maximal and minimal elements are easy to spot using a Hasse diagram.

They are the “top” and “bottom” elements in the diagram.

The Hasse diagram representing the partial ordering $\{(a, b) | b : a\}$, on $\{1, 2, 3, 4, 6, 8, 12\}$.

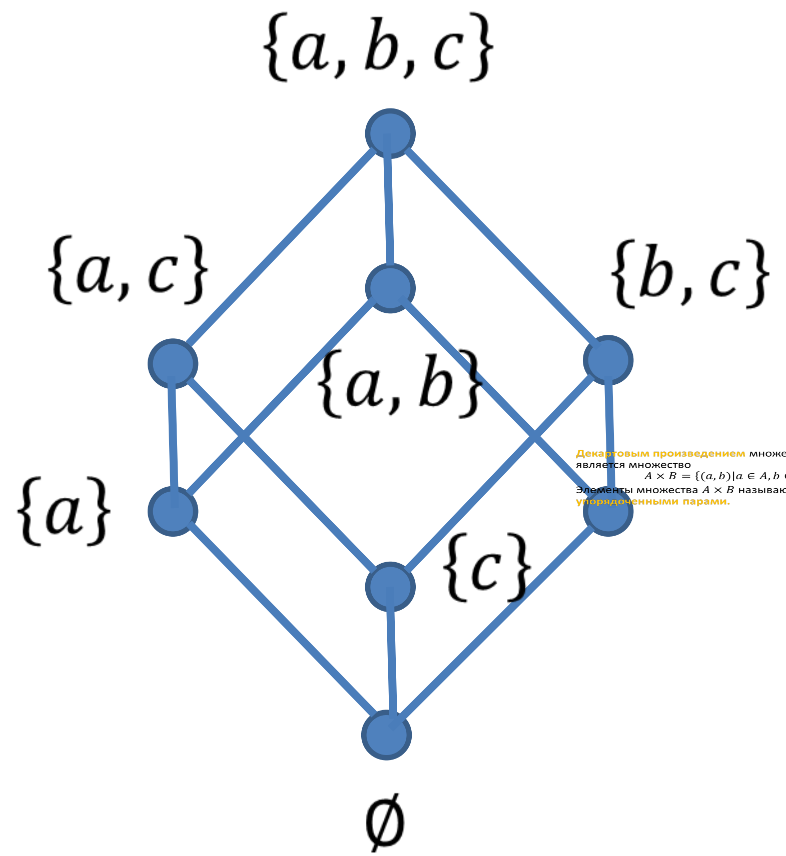
The maximal elements are 8, 12, and the minimal element is 1.



The Hasse diagram representing the partial ordering on $(\boldsymbol{P}(\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}), \subseteq)$.

The maximal element is $\{a, b, c\}$

and the minimal element is $\emptyset$.



Декартовым произведением множеств A и B является множество

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Элементы множества $A \times B$ называются **упорядоченными парами.**

Topological sorting

- Suppose that a project is made up of 20 different tasks. Some tasks can be completed only after others have been finished.

How can an order be found for these tasks?

To model this problem we set up a partial order on the set of tasks so that $a \prec b$ if and only if a and b are tasks where b cannot be started until a has been completed.

To produce a schedule for the project, we need to produce an order for all 20 tasks that is compatible with this partial order.

We will show how this can be done.

Topological sorting

Definition 5

A total ordering $\prec$ is said to be compatible with the partial ordering R if $a \prec b$ whenever aRb.

Constructing a compatible total ordering from a partial ordering is called **topological sorting**.

Topological sorting

Lemma

Every finite nonempty poset $(S, \prec)$ has at least one minimal element.

Proof

Choose an element a_0 of S.

If a_0 is not minimal, then there is an element a_1 in S with $a_1 \prec a_0$.

If a_1 is not minimal, then there is an element a_2 in S with $a_2 \prec a_1$.

Continue this process.

Because there are only a finite number of elements in the poset S, this process must end with a minimal element a_n.

The topological sorting algorithm

- Let $(A, \prec)$ be finite poset.

First choose a minimal element a_1 in A. Such an element exists by lemma.

$(A - \{a_1\}, \prec)$ is also a poset.

If $A - \{a_1\} \neq \emptyset$ choose a minimal element a_2 of this poset. Such an element exists by lemma.

If $A - \{a_1, a_2\} \neq \emptyset$ choose a minimal element a_3 of this poset.

Continue this process.

Because A is a finite set, this process must terminate.

The topological sorting algorithm

- The desired total ordering $\prec_l$ is defined by:

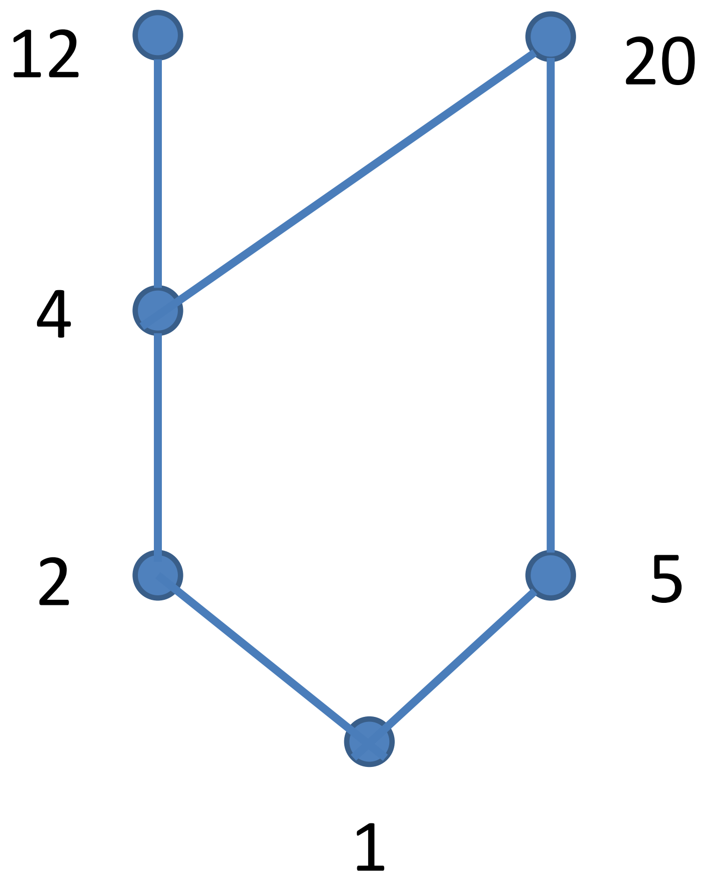
$$a_1 \prec_l a_2 \prec_l \ldots \prec_l a_n$$

This total ordering is compatible with the original partial ordering. ■

The topological sorting algorithm

Example 7

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.



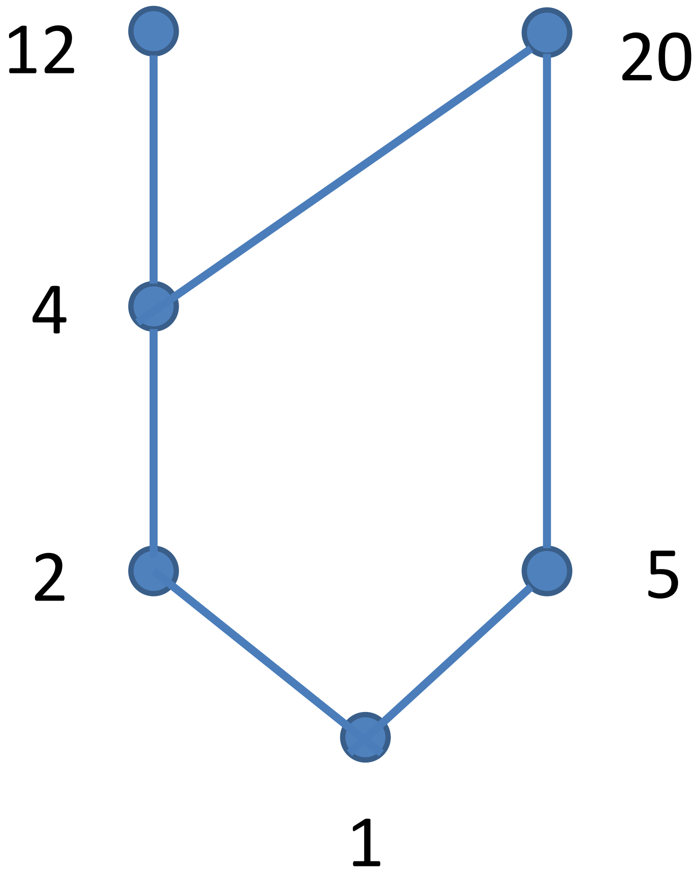
The topological sorting algorithm

Example 7

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

Solution

1



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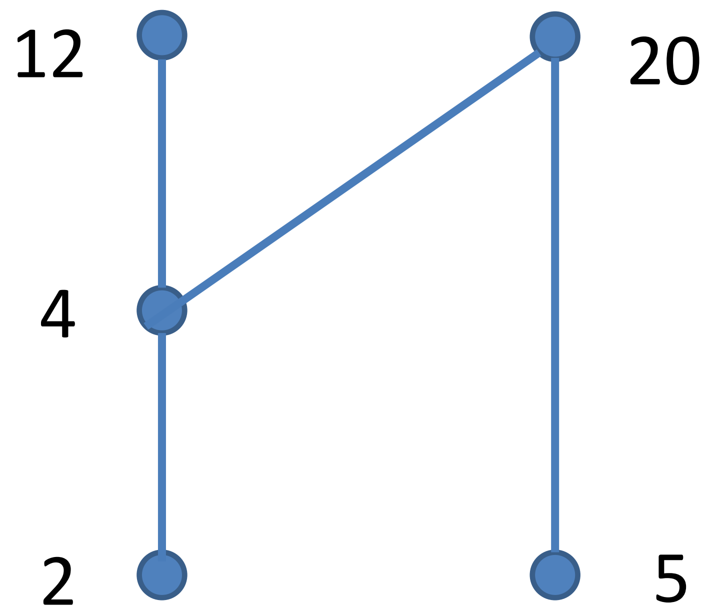

1

The topological sorting algorithm

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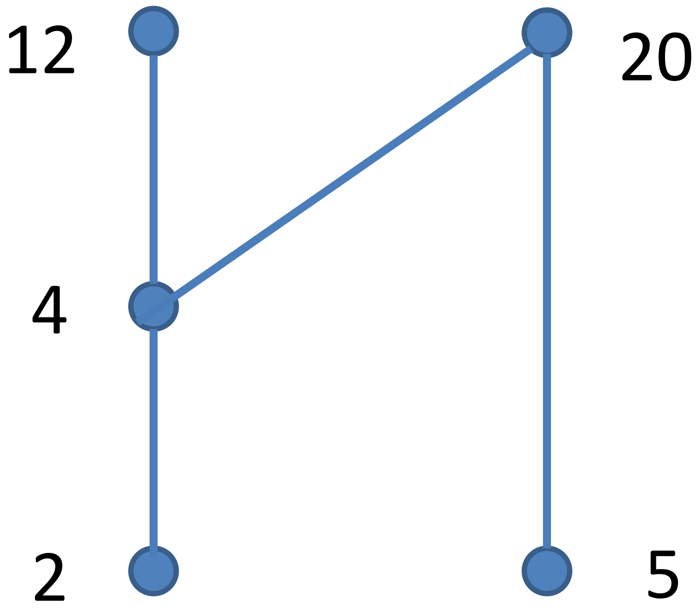
$1 \prec 5$



The topological sorting algorithm

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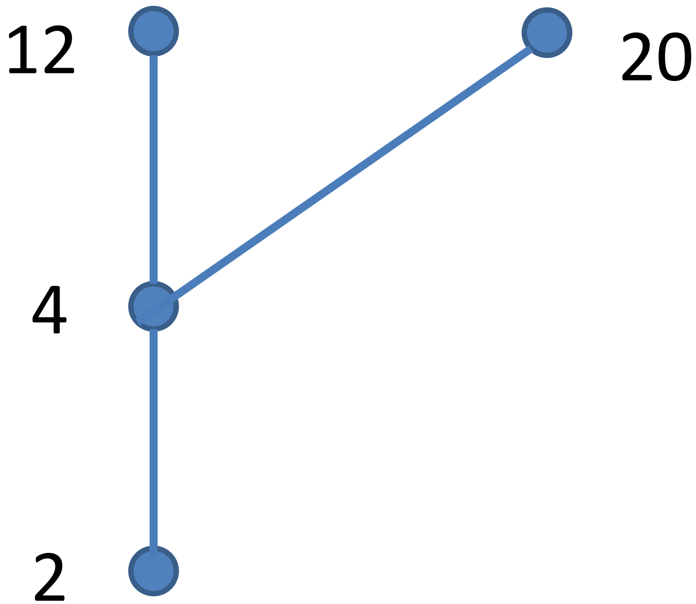


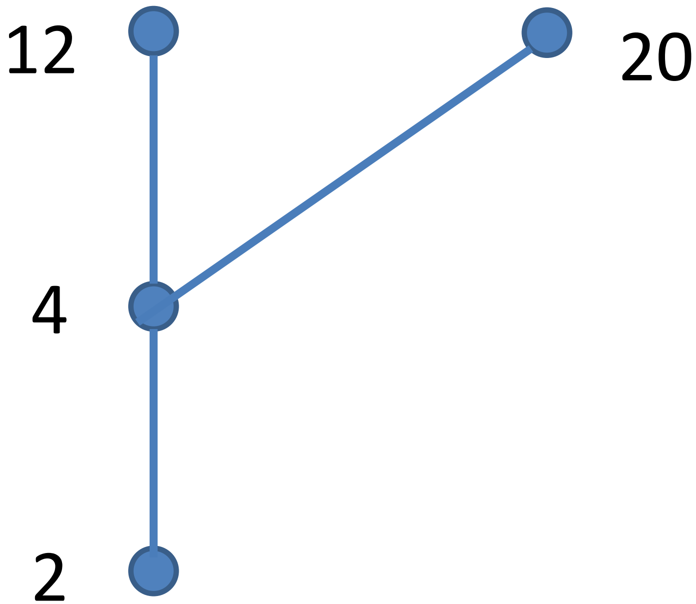
The topological sorting algorithm

Example 7

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

Solution

$1 \prec 5 \prec 2$



The topological sorting algorithm

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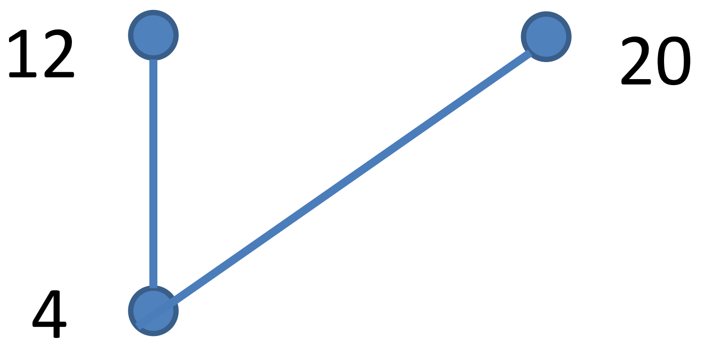
Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

Solution



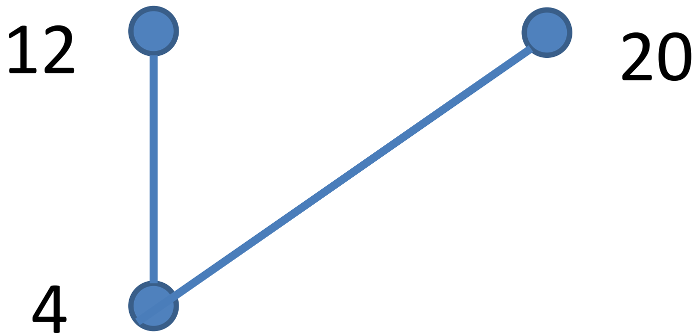
$1 \prec 5 \prec 2$

The topological sorting algorithm

<u>Example 7</u>

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

<u>Solution</u>



$1 \prec 5 \prec 2 \prec 4$

The topological sorting algorithm

<u>Example 7</u>

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

<u>Solution</u>

$$1 \prec 5 \prec 2 \prec 4$$

The topological sorting algorithm

<u>Example 7</u>

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, \mid)$.

<u>Solution</u>

$$1 \prec 5 \prec 2 \prec 4 \prec 20$$

The topological sorting algorithm

<u>Example 7</u>

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, \mid)$.

<u>Solution</u>

$$1 \prec 5 \prec 2 \prec 4 \prec 20$$

The topological sorting algorithm

Example 7

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

Solution

$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$

$\blacksquare$

The topological sorting algorithm

<u>Example 8</u>

A development project at a computer company requires the completion of seven tasks.

Some of these tasks can be started only after other tasks are finished.

A partial ordering on tasks is set up by considering $taskX \prec taskY$ if task Y cannot be started until task X has been completed.

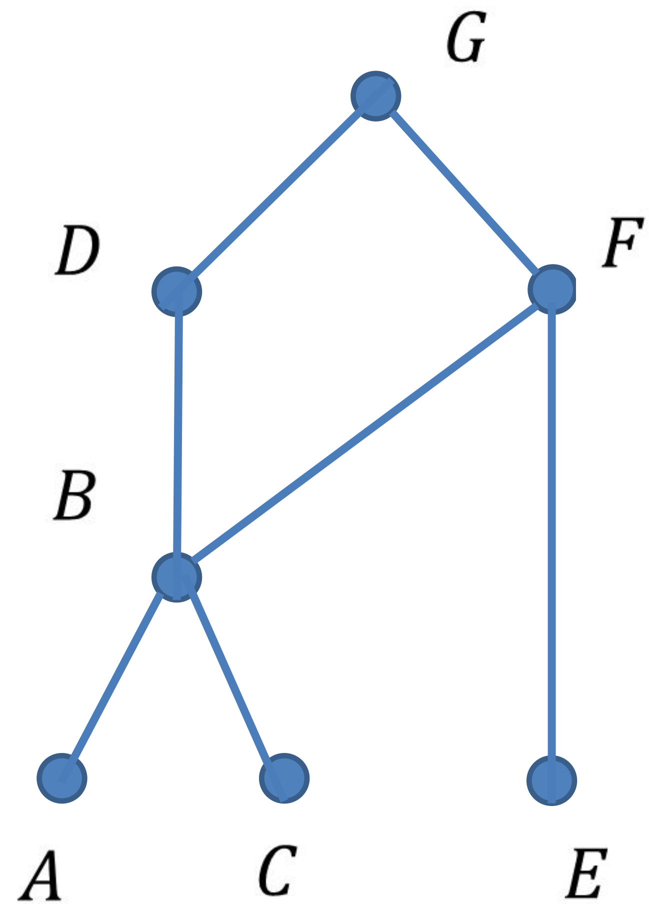
The Hasse diagram for the seven tasks, with respect to this partial ordering, is shown in the figure.

Find an order in which these tasks can be carried out to complete the project.

The topological sorting algorithm

<u>Example 8</u>

Find a compatible total ordering for the poset.



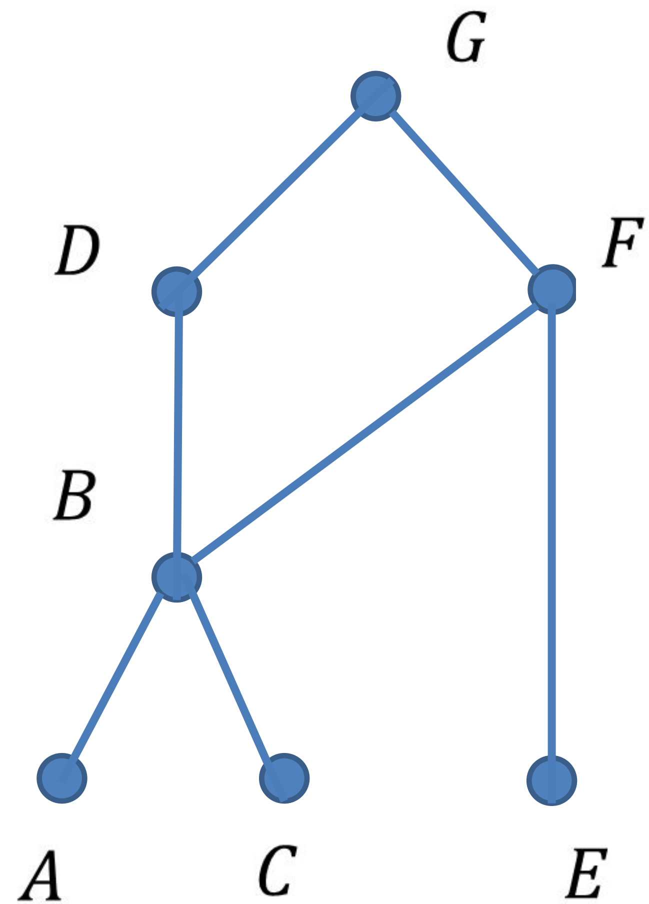
The topological sorting algorithm

Example 8

Find a compatible total ordering for the poset.

Solution

A



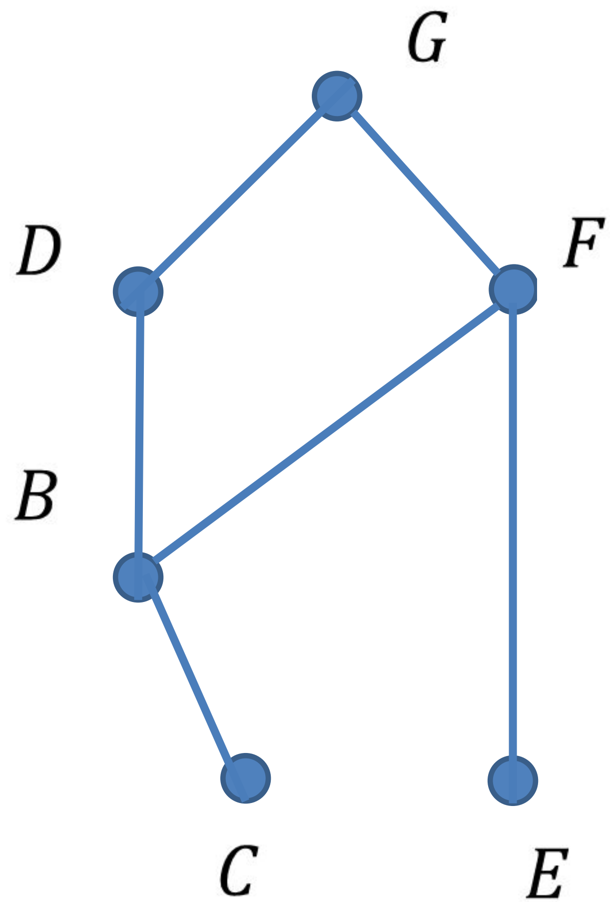
The topological sorting algorithm

Example 8

Find a compatible total ordering for the poset.

Solution

A



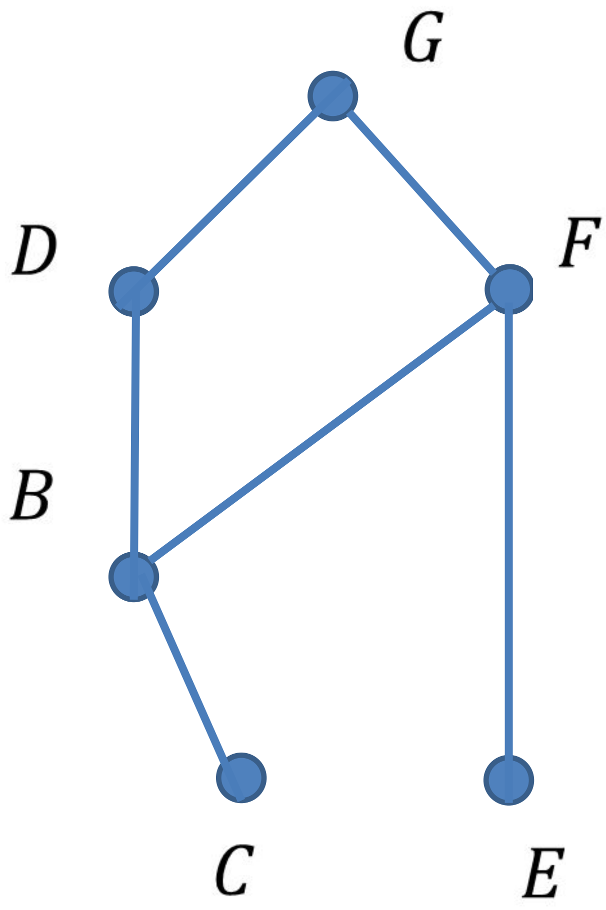
The topological sorting algorithm

Example 8

Find a compatible total ordering for the poset.

Solution

$A \prec C$



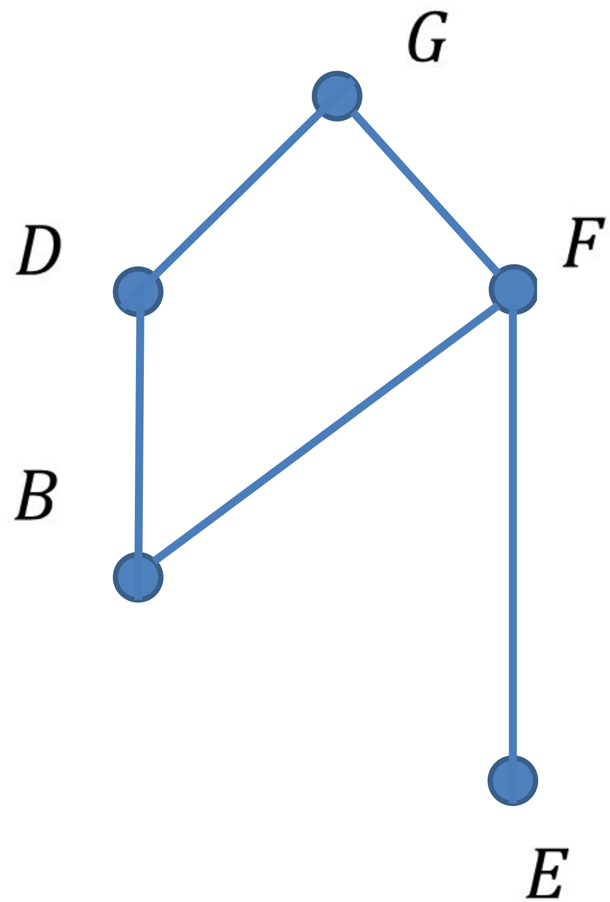
The topological sorting algorithm

Example 8

Find a compatible total ordering for the poset.

Solution

$A \prec C$



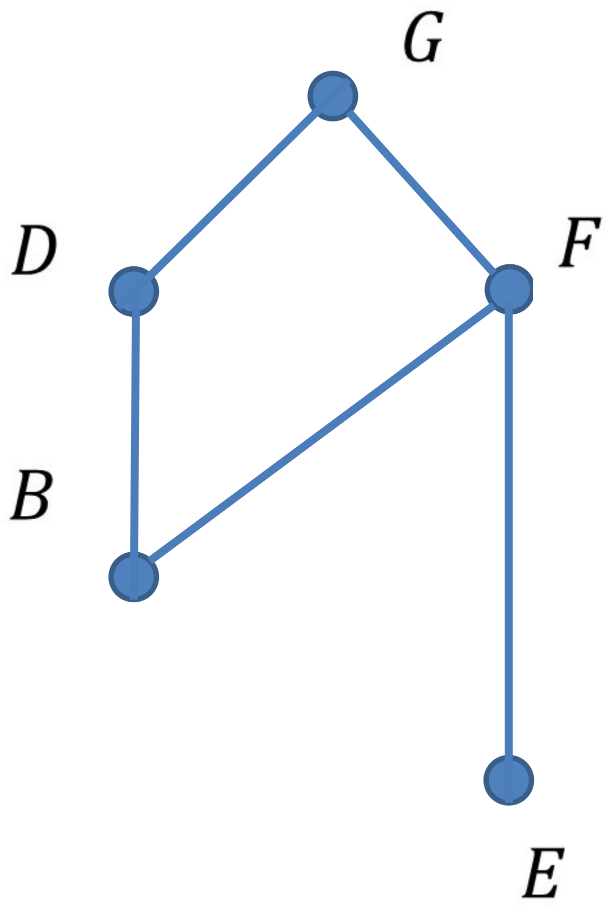
The topological sorting algorithm

<u>Example 8</u>

Find a compatible total ordering for the poset.

<u>Solution</u>

$A \prec C \prec B$



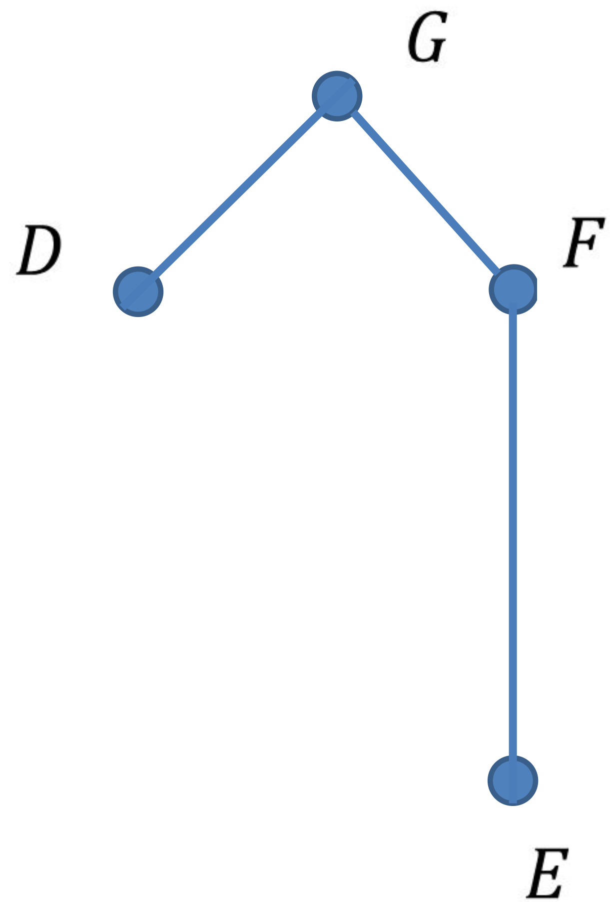
The topological sorting algorithm

<u>Example 8</u>

Find a compatible total ordering for the poset.

<u>Solution</u>

$A \prec C \prec B$



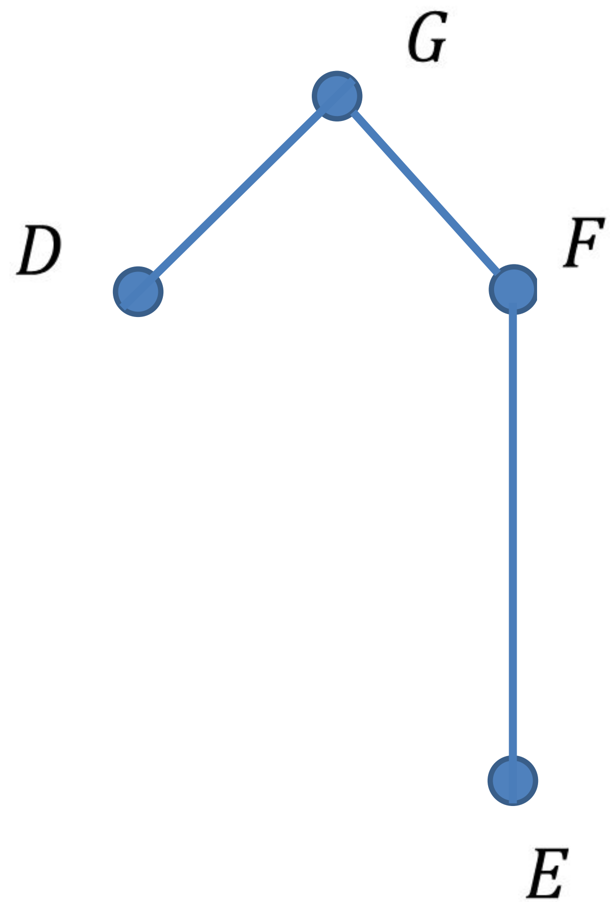
The topological sorting algorithm

Example 8

Find a compatible total ordering for the poset.

Solution

$A \prec C \prec B \prec E$

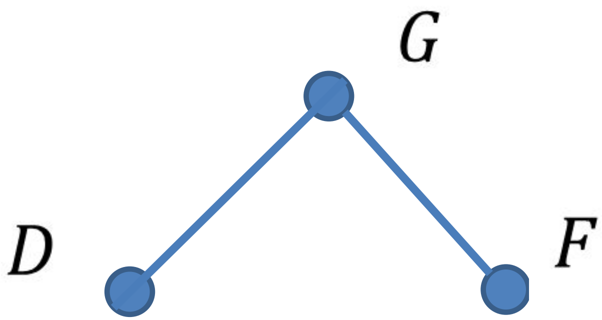


The topological sorting algorithm

<u>Example 8</u>

Find a compatible total ordering for the poset.

<u>Solution</u>



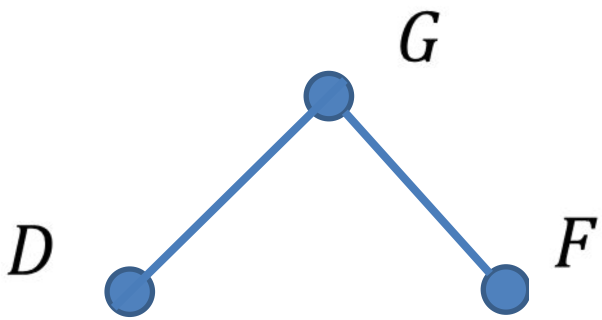
$A \prec C \prec B \prec E$

The topological sorting algorithm

<u>Example 8</u>

Find a compatible total ordering for the poset.

<u>Solution</u>



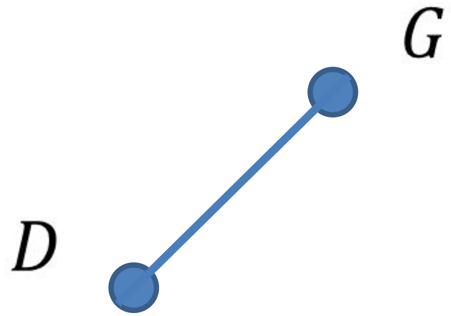
$A \prec C \prec B \prec E \prec F$

The topological sorting algorithm

<u>Example 8</u>

Find a compatible total ordering for the poset.

<u>Solution</u>



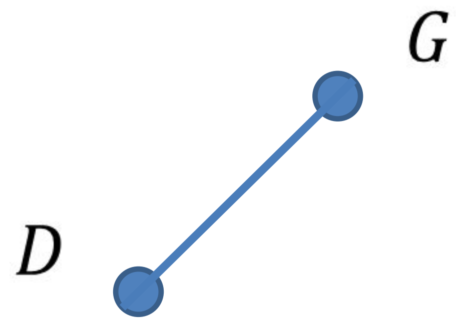
$$A \prec C \prec B \prec E \prec F$$

The topological sorting algorithm

<u>Example 8</u>

Find a compatible total ordering for the poset.

<u>Solution</u>

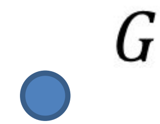


$A \prec C \prec B \prec E \prec F \prec D$

The topological sorting algorithm

<u>Example 8</u>

Find a compatible total ordering for the poset.



<u>Solution</u>

$$A \prec C \prec B \prec E \prec F \prec D$$

The topological sorting algorithm

Example 8

Find a compatible total ordering for the poset.

Solution

$A \prec C \prec B \prec E \prec F \prec D \prec G$ ■