

# General problem of mathematical programming

$$f(y) \rightarrow \min, y \in Q \subseteq R^N \quad (1)$$

$$Q = \{y \in D : g_j(y) \leq 0, 1 \leq j \leq m\} \quad (2)$$

$$D = \{y \in R^N : y_i \in [a_i, b_i], 1 \leq i \leq N\} \quad (3)$$

$$h(y) = 0 \sim |h(y)| \leq \delta, \quad \delta > 0$$

$$h(y) = 0 \Leftrightarrow \begin{cases} h(y) \leq 0, \\ h(y) \geq 0, \end{cases}$$

$$\chi(Q) = \begin{cases} 0, & y \in Q, \\ 1, & y \notin Q, \end{cases} \quad \begin{array}{l} \text{- характеристическая функция множества} \\ Q \end{array}$$

$$\chi(Q) \leq 0$$

# Auxiliary functions and sets

Feasibility function  $G(y), y \in D$ :

$$G(y) \leq 0, y \in Q, \quad (4)$$

$$G(y) > 0, y \notin Q.$$

$$G(y) = \max\{g_1(y), \dots, g_m(y)\} \quad (5)$$

$$G(y) = \max\{0; g_1(y), \dots, g_m(y)\} \quad (6)$$

$$u_i = (y_1, \dots, y_i), \quad v_i = (y_{i+1}, \dots, y_N), \quad 1 \leq i \leq N-1,$$

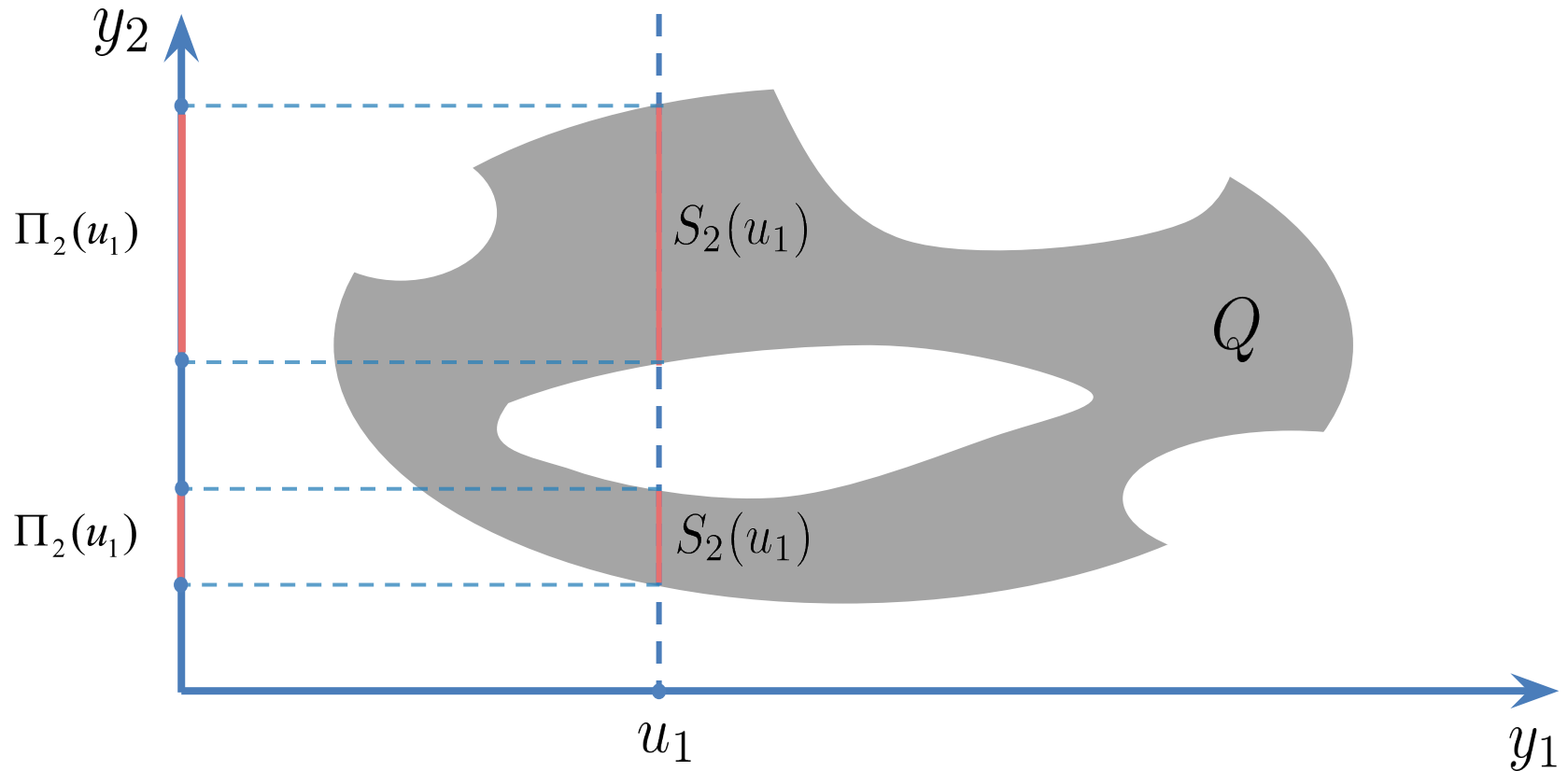
$$u_N = y, \quad i = N,$$

$$v_0 = y, \quad i = 0.$$

$$S_1 = Q, \quad S_{i+1}(u_i) = \{v_i \in R^{N-i} : (u_i, v_i) \in Q\}, \quad 1 \leq i \leq N-1$$

$$\Pi_{i+1}(u_i) = \{y_{i+1} \in R^1 : \exists (y_{i+1}, v_{i+1}) \in S_{i+1}(u_i)\}$$

# Sections and projections



# Reducing the feasibility function

$$G^N(y) \equiv G(y)$$



$$(10)$$

$$D_i = \{u_i \in R^i : y_j \in [a_j, b_j], 1 \leq j \leq i\} \quad (11)$$

## LEMMA 1

$$G^i(u_i) = \min \{G(u_i, v_i) : y_j \in [a_j, b_j], i+1 \leq j \leq N\}$$

Projecting  $Q$  onto coordinate axes  $y_1, \dots, y_i$  (13)

$$Q_i = \{u_i \in R^i : \exists (u_i, v_i) \in Q\}, 1 \leq i \leq N. \quad (14)$$

## LEMMA 2

$$Q_i = \{u_i \in R^i : G^i(u_i) \leq 0\} \quad (15)$$

## LEMMA 3

$$\Pi_{i+1}(u_i) = \{y_{i+1} \in [a_{i+1}, b_{i+1}] : G^{i+1}(u_i, y_{i+1}) \leq 0\}$$

# Nested optimization scheme

$$f^N(y) \equiv f(y)$$

$$f^i(u_i) = \min \{ f^{i+1}(u_i, y_{i+1}) : y_{i+1} \in \Pi_{i+1}(u_i) \}, u_i \in Q_i$$

## Main relation

$$\min_{y \in Q} f(y) = \min_{y_1 \in \Pi_1} \min_{y_2 \in \Pi_2(u_1)} \dots \min_{y_N \in \Pi_N(u_{N-1})} f(y)$$

$$f^1(y_1) \rightarrow \min, y_1 \in \Pi_1 \subseteq R^1$$

$$\Pi_1 = \{y_1 \in [a_1, b_1] : G^1(y_1) \leq 0\}$$

$$f^2(y_1, y_2) \rightarrow \min, y_2 \in \Pi_2(y_1) \subseteq R^1,$$

$$\Pi_2(y_1) = \{y_2 \in [a_2, b_2] : G^2(y_1, y_2) \leq 0\}$$

$$f^3(u_2, y_3) \rightarrow \min$$

. . .

$$f^N(u_{N-1}, y_N) = f(u_{N-1}, y_N) \rightarrow \min, y_N \in \Pi_N(u_{N-1}).$$

# Nested optimization scheme

$$f^i(u_{i-1}, y_i) \rightarrow \min, y_i \in \Pi_i(u_{i-1}) \quad (19)$$

$$u_{i-1} \in Q_{i-1} - \text{fixed}$$

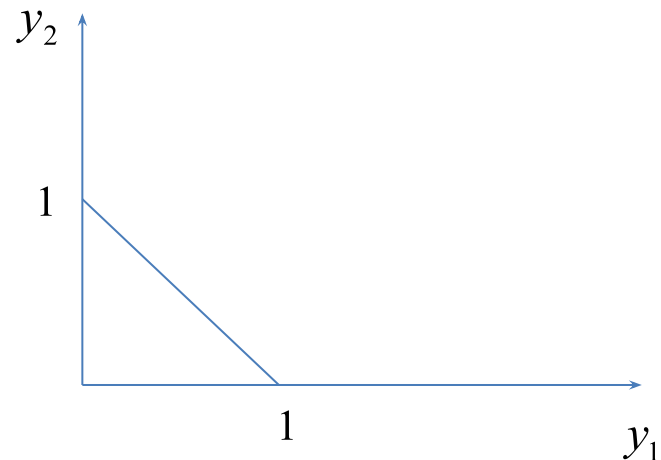
$$\underset{y_1 \in \Pi_1}{extr} \underset{y_2 \in \Pi_2(u_1)}{extr} \dots \underset{y_N \in \Pi_N(u_{N-1})}{extr} \varphi(y)$$

## Example

$$f(y_1, y_2) = (y_1 - 1)^2 + (y_2 - 1)^2$$

$$g_1(y) = y_1 + y_2 - 1$$

$$0 \leq y_1 \leq 2, 0 \leq y_2 \leq 2$$



# Nested optimization scheme

$$G^2(y) = y_1 + y_2 - 1$$

$$G^1(y_1) = \min\{y_1 + y_2 - 1, y_2 \in [0, 2]\} = y_1 - 1$$

$$\Pi_2(y_1) = \{y_2 \in [0, 2] : y_1 + y_2 - 1 \leq 0\} = [0, 1 - y_1]$$

$$\Pi_1 = \{y_1 \in [0, 2] : y_1 - 1 \leq 0\} = [0, 1]$$

$$f^2(y_1, y_2) = (y_1 - 1)^2 + (y_2 - 1)^2$$

$$f^1(y_1) = \min\{(y_1 - 1)^2 + (y_2 - 1)^2 : y_2 \in \Pi_2(y_1) = [0, 1 - y_1]\}$$

Функция  $(y_1 - 1)^2 + (y_2 - 1)^2$  достигает минимума по  $y_2$  в единице, а слева от 1 убывает, поэтому на отрезке  $[0, 1 - y_1]$  ее минимум достигается в точке  $y_2 = 1 - y_1$ , а значение равно  $y_1^2 - 2y_1 + 1$ , т.е.

$$f^1(y_1) = 2y_1^2 - 2y_1 + 1$$

Ее минимум на отрезке  $[0, 1]$  достигается в точке  $y_1^* = 1/2$

$$f^2(0.5, y_2) = 1/4 + (y_2 - 1)^2 \quad \text{Т.к. } \Pi_2(1/2) = [0, 1 - 1/2] = [0, 1/2] \text{ убывает, то } y_2^* = 1/2$$