

# **NUFYP Mathematics & Computing Science Pre-Calculus Course**

## **L21 Confidence interval and Hypothesis testing for population mean ( $\mu$ ) when $\sigma$ is known and $n$ (large)**

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# Lecture overview: Learning outcomes

At the end of this lecture you should be able to:

**7.6.1** Calculate and interpret **confidence intervals** for a population parameter

**7.6.2** Test the **hypothesis** for a **mean** of a normal distribution,

$$H_0: \mu = k,$$

$$H_1: \mu \neq k \text{ or } \mu > k \text{ or } \mu < k$$

# Lecture overview: Learning outcomes

**At the end of this lecture you should be able to:**

**7.6.3** Test the hypothesis for the **difference between means** of two independent normal distributions

$$H_0: \mu_x - \mu_y = 0,$$

$$H_1: \mu_x - \mu_y \neq 0 \text{ or } \mu_x - \mu_y < 0 \text{ or } \mu_x - \mu_y > 0$$

# Textbook Reference

The content of this lecture is from the following textbook:

Chapter 3

Statistics 3 Edexcel AS and A Level Modular  
Mathematics S3 published by Pearson Education  
Limited

ISBN 978 0 435519 14 8

Further examples can be found in the textbook.

# Terminology

## CONFIDENCE INTERVAL

*A range of values constructed so that there is a specified probability of including the true value of a parameter within it*

# Terminology

## CONFIDENCE LEVEL

- *Probability of including the true value of a parameter within a confidence interval*
- *Percentage*

# Terminology

## **CONFIDENCE LIMITS – CRITICAL VALUES**

- *Two extreme measurements within which an observation lies*
- *End points of the confidence interval*
- *Larger confidence – Wider interval*

# Estimation of population parameters

**Point estimate**

**Interval estimate**



*We have covered this  
in previous lecture*



# Point estimate VS Interval estimate

## Point estimate

$$\bar{x} \approx \hat{\mu}$$

$$s^2 \approx \hat{\sigma}^2$$

The sample mean,  $\bar{x}$ , is the **best point estimate** of the population mean,  $\mu$ .

But, sample mean is still an approximation, and how close (**ERROR**) it is to true population mean value we do not consider in the Point estimate.

# Point estimate VS Interval estimate

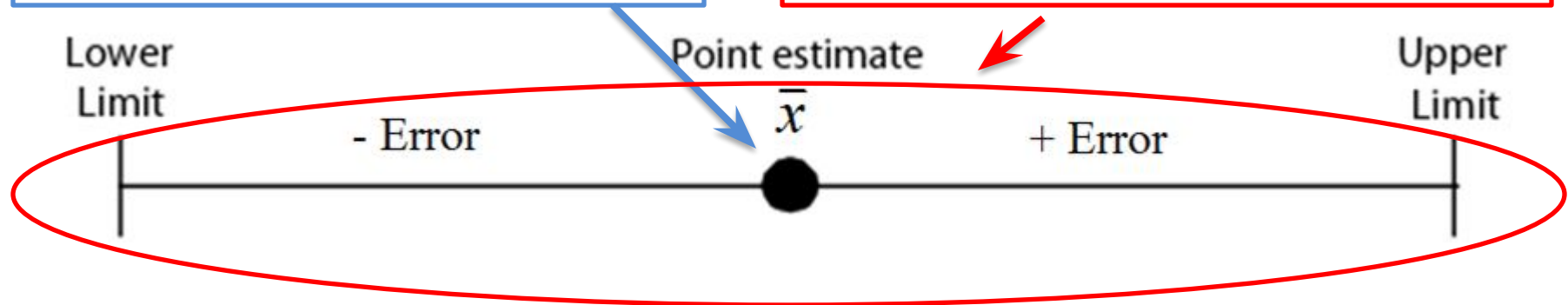
## Point estimate

$$\bar{x} \approx \hat{\mu}$$

$$s^2 \approx \hat{\sigma}^2$$

## Interval estimate

$$\bar{x} - Error \leq \mu \leq \bar{x} + Error$$



# Point estimate VS Interval estimate

## Interval estimate

$$\bar{x} - Error \leq \mu \leq \bar{x} + Error$$

In interval estimate we do consider **ERROR**

Interval estimate is a range of numbers around the point estimate within which the parameter is believed to fall

# Point estimate VS Interval estimate

## Point estimate

$$\bar{x} \approx \hat{\mu}$$

$$s^2 \approx \hat{\sigma}^2$$

## Interval estimate

$$\bar{x} - \text{Error} \leq \mu \leq \bar{x} + \text{Error}$$

Until now we didn't specify what is meant by error

# Point estimate VS Interval estimate

## Point estimate

$$\bar{x} \approx \hat{\mu}$$

$$s^2 \approx \hat{\sigma}^2$$

## Interval estimate

$$\bar{x} - \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + \frac{\sigma}{\sqrt{n}}$$

Standard error

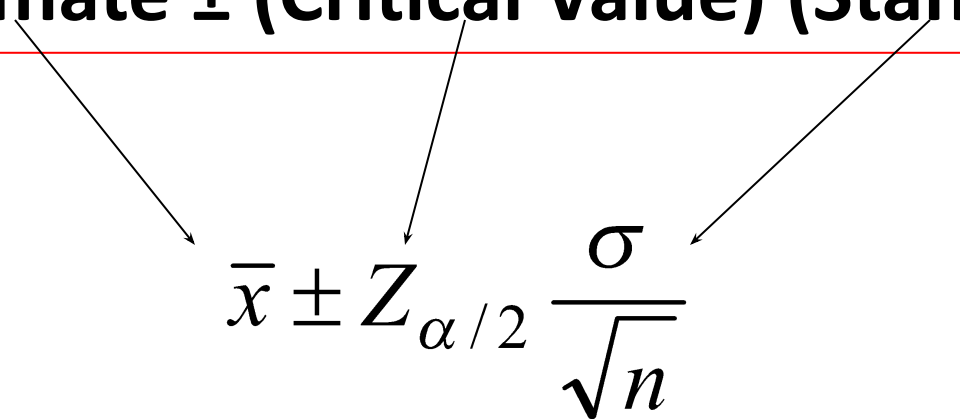
## 7.5.1 Calculate and interpret **confidence intervals** for a population parameter

- Interval estimate provides us interval within which we believe value of true population mean falls
- Then by using Standard Normal Distribution we can consider specific level of confidence that  $\mu$  is really there by adjusting critical coefficient

## 7.5.1 Calculate and interpret **confidence intervals** for a population parameter

The general formula for **all** confidence intervals are:

**Point Estimate  $\pm$  (Critical Value) (Standard Error)**



$$\bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

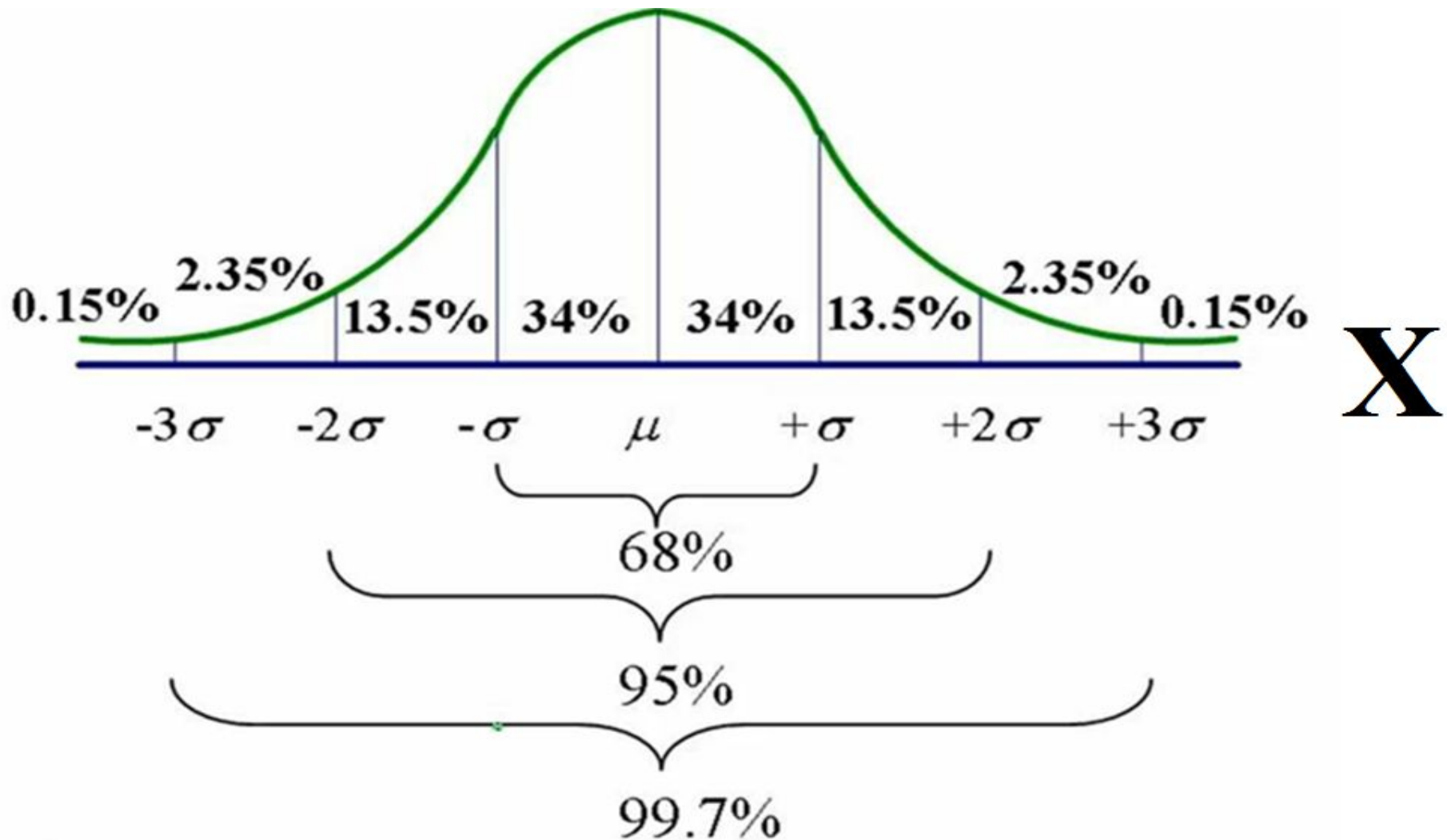
## 7.5.1 Calculate and interpret **confidence intervals** for a population parameter

$$\bar{x} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

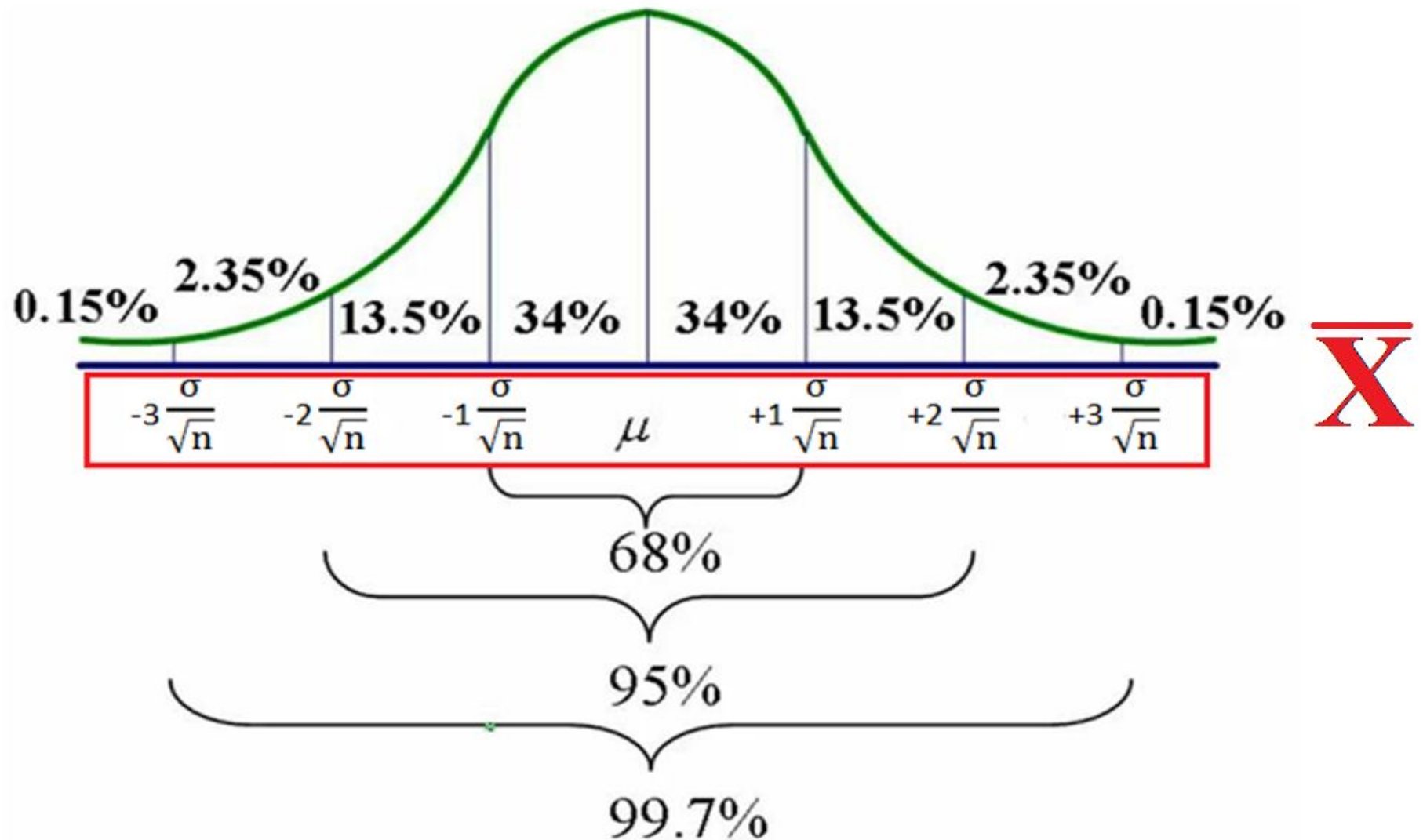
**Critical values**



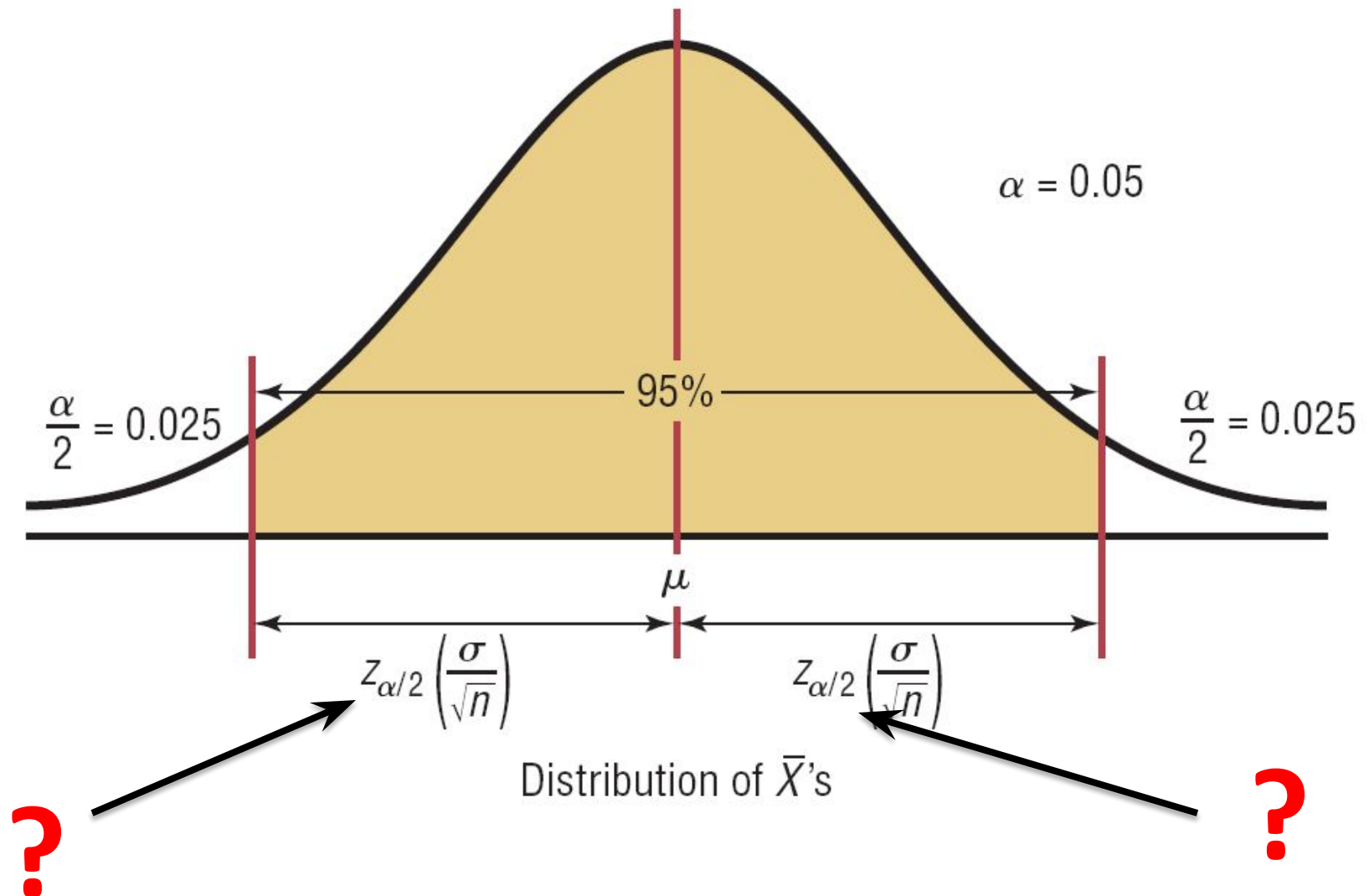
# Empirical rule



# Empirical rule

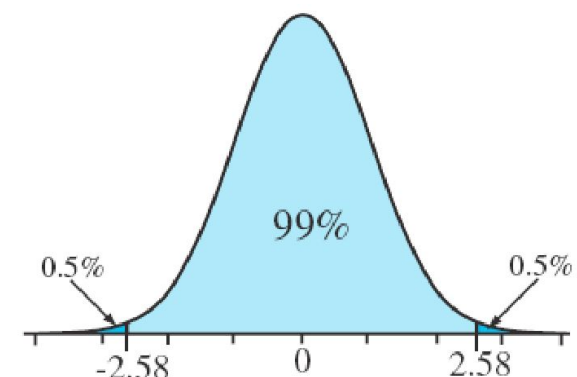
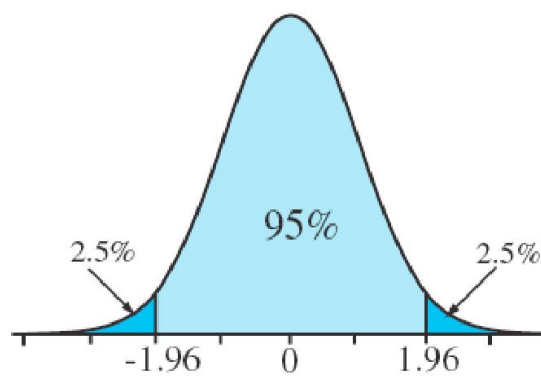
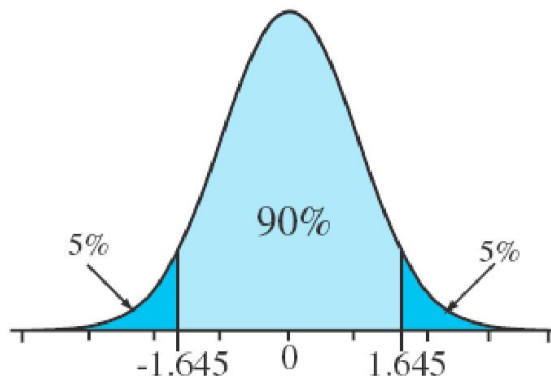


# 95% Confidence Interval of the Mean



# Common Levels of Confidence

Confidence level	$\alpha$ level	Z value
$1 - \alpha$	$\alpha$	$z_{1-(\alpha/2)}$
.90	.10	1.645
.95	.05	1.960
.99	.01	2.576



# Formula for the Confidence Interval of the Mean for a Specific $\alpha$

$$\bar{X} - z_{\alpha/2} \left( \frac{\sigma}{\sqrt{n}} \right) < \mu < \bar{X} + z_{\alpha/2} \left( \frac{\sigma}{\sqrt{n}} \right)$$

For a 90% confidence interval:  $z_{\alpha/2} = 1.65$

For a 95% confidence interval:  $z_{\alpha/2} = 1.96$

For a 99% confidence interval:  $z_{\alpha/2} = 2.58$

## 7.5.1 Calculate and interpret **confidence intervals** for a population parameter

### Example 1

A random sample of size 25 is taken from a normal population with standard deviation of 2.5.

The mean of the sample was 17.8.

**a** Find a 99% C.I. for the population mean  $\mu$ .

## 7.5.1 Calculate and interpret **confidence intervals** for a population parameter

### Example 1

a 99% C.I. is

$$\begin{aligned}\bar{x} \pm 2.5758 \times \frac{\sigma}{\sqrt{n}} &= 17.8 \pm 2.5758 \times \frac{2.5}{\sqrt{25}} \\ &= (16.51, 19.09)\end{aligned}$$

## 7.5.1 Calculate and interpret **confidence intervals** for a population parameter

### Example 2

A random sample of size 9 is taken from a normal distribution with variance 36.

The sample mean is 128.

- a** Find a 95% confidence interval for the mean  $\mu$  of the distribution.
- b** Find a 99% confidence interval for the mean  $\mu$  of the distribution.



## 7.5.1 Calculate and interpret **confidence intervals** for a population parameter

### Example 2

$$n = 9, \sigma^2 = 36, \bar{x} = 128$$

$$\begin{aligned} \text{a } 95\% \text{ C.I. for } \mu \text{ is } 128 \pm 1.96 \times \frac{6}{\sqrt{9}} &= (124.08, 131.92...) \\ &= (124, 132) \text{ (3 s.f.)} \end{aligned}$$

$$\begin{aligned} \text{b } 99\% \text{ C.I. for } \mu \text{ is } 128 \pm 2.5758 \times \frac{6}{\sqrt{9}} &= (122.84..., 133.15...) \\ &= (123, 133) \text{ (3 s.f.)} \end{aligned}$$

## 7.5.2 Test the **hypothesis** for a **mean** of a normal distribution

### Hypothesis testing

- **Hypothesis testing** as well as estimation is a method used to reach a conclusion on population parameter by using sample statistics.

## 7.5.2 Test the hypothesis for a mean of a normal distribution

### Hypothesis testing

- In Hypothesis testing beside sample statistics level of significance ( $\alpha$ ) is used to make a meaningful conclusion.

# Level of Significance ( $\alpha$ )

- The **level of significance**,  $\alpha$ , is a **probability** and is, in reality, the probability of rejecting a true null hypothesis.

Confidence level

$$C = (1 - \alpha)$$

Level of Significance

$$\alpha = 1 - C$$

## 7.5.2 Test the **hypothesis** for a **mean** of a normal distribution

### Hypothesis testing

- In Hypothesis testing we **compare** a sample statistic to a population parameter to see if there is a *significant difference*.

## 7.5.1 Calculate and interpret **confidence intervals** for a population parameter

### Hypothesis testing

1. Hypothesis testing can be used to determine whether a statement about the value of a population parameter should or should not be rejected.
2. The null hypothesis, denoted by  $H_0$ , is a tentative assumption about a population parameter.

## 7.5.1 Calculate and interpret **confidence intervals** for a population parameter

### **Hypothesis testing**

3. The alternative hypothesis, denoted by  $H_a$ , is the opposite of what is stated in the null hypothesis.

4. The hypothesis testing procedure uses data from a sample to test the two competing statements indicated by  $H_0$  and  $H_a$ .

# Types of Hypothesis testing

- **Null Hypothesis ( $H_0$ )**
- **Alternative Hypothesis ( $H_a$  or  $H_1$ )**

Each of the following statements is an example of a null hypothesis and corresponding alternative hypothesis.

$H_0 : \mu = \mu_0$	$H_a : \mu \neq \mu_0$
$H_0 : \mu \leq \mu_0$	$H_a : \mu > \mu_0$
$H_0 : \mu \geq \mu_0$	$H_a : \mu < \mu_0$



# Steps of Hypothesis Testing

- ▶ Step 1. Develop the null and alternative hypotheses.
- ▶ Step 2. Specify the level of significance  $\alpha$ .
- ▶ Step 3. Collect the sample data and compute the value of the test statistic.

## *p*-Value Approach

- ▶ Step 4. Use the value of the test statistic to compute the *p*-value.
- ▶ Step 5. Reject  $H_0$  if *p*-value  $\leq \alpha$ .

# Steps of Hypothesis Testing

## Critical Value Approach

- ▶ Step 4. Use the level of significance to determine the critical value and the rejection rule.
- ▶ Step 5. Use the value of the test statistic and the rejection rule to determine whether to reject  $H_0$ .

# *p*-Value Approach to One-Tailed Hypothesis Testing

- ▶ The *p*-value is the probability, computed using the test statistic, that measures the support (or lack of support) provided by the sample for the null hypothesis.
- ▶ If the *p*-value is less than or equal to the level of significance  $\alpha$ , the value of the test statistic is in the rejection region.
- ▶ Reject  $H_0$  if the *p*-value  $\leq \alpha$ .

# Critical Value Approach to One-Tailed Hypothesis Testing

- ▶ The test statistic  $z$  has a standard normal probability distribution.
- ▶ We can use the standard normal probability distribution table to find the  $z$ -value with an area of  $\alpha$  in the lower (or upper) tail of the distribution.
- ▶ The value of the test statistic that established the boundary of the rejection region is called the critical value for the test.
- ▶ The rejection rule is:
  - Lower tail: Reject  $H_0$  if  $z \leq -z_\alpha$
  - Upper tail: Reject  $H_0$  if  $z \geq z_\alpha$

# One-tailed test (left-tailed)

Acceptance and Rejection  
regions in case of a left tailed  
test

Suitable When  $H_0: \mu = \mu_0$   
 $H_a: \mu < \mu_0$

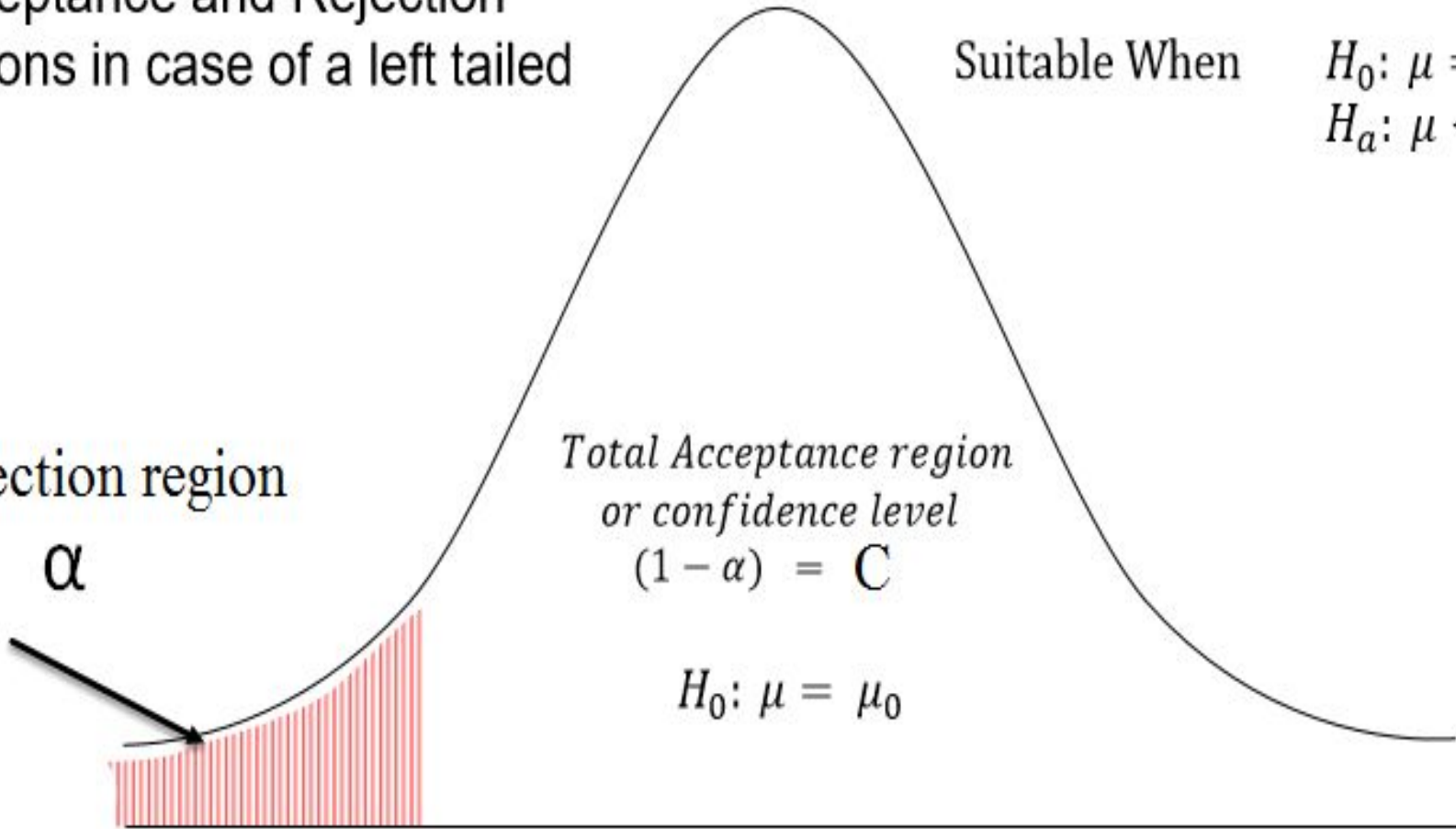
Rejection region

$\alpha$

Total Acceptance region  
or confidence level

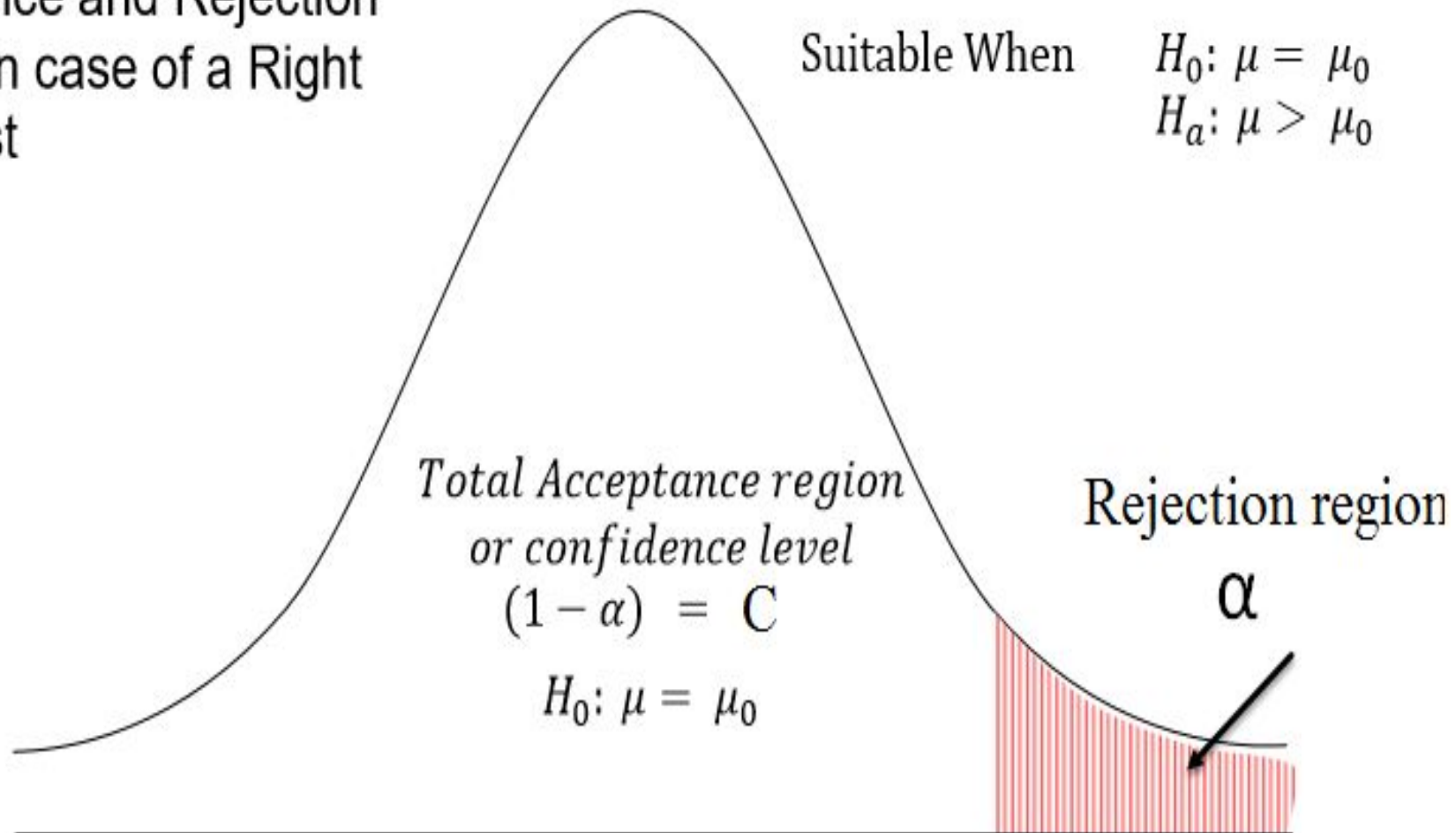
$$(1 - \alpha) = C$$

$$H_0: \mu = \mu_0$$



# One-tailed test (right-tailed)

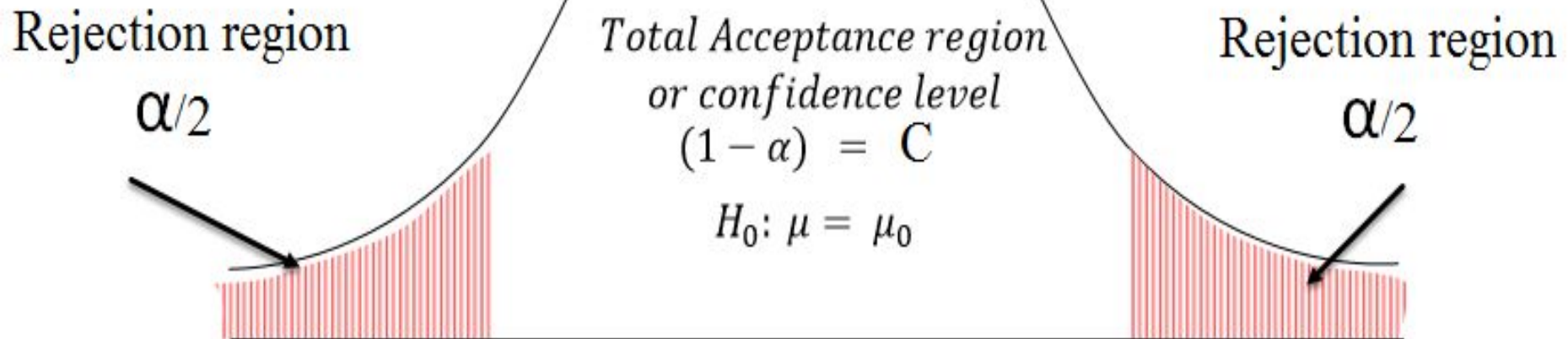
Acceptance and Rejection  
regions in case of a Right  
tailed test



# Two-tailed test

Acceptance and Rejection regions in case of a Two tailed test

Suitable When  $H_0: \mu = \mu_0$   
 $H_a: \mu \neq \mu_0$



## 7.5.2 Test the **hypothesis** for a **mean** of a normal distribution, $H_0: \mu=k$ , $H_1: \mu \neq k$ or $\mu > k$ or $\mu < k$

### Example 3

At a certain college new students are weighed when they join the college. The distribution of weights of students at the college when they enrol has a standard deviation of 7.5 kg and a mean of 70 kg. A random sample of 90 students from the new entry were weighed and their mean weight was 71.6 kg. Assuming that the standard deviation has not changed

**a** test, at the 5% level, whether there is evidence that the mean of the new entry is more than 70 kg.



## Example 3

a  $H_0: \mu = 70$   $H_1: \mu > 70$  •  $\mu > 70$ , so a one-tailed test

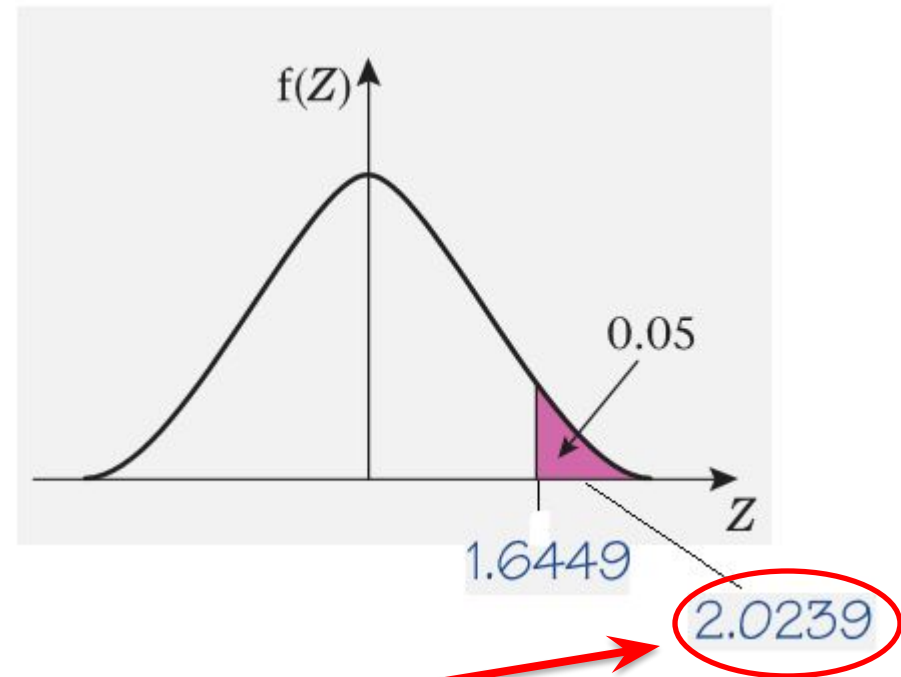
$$\sigma = 7.5$$

A 5% significance level

The sample gives  $n = 90$ ,  $\bar{x} = 71.6$

## Example 3

$$z = \frac{71.6 - 70}{\frac{7.5}{\sqrt{90}}} = 2.0239$$



This value is in the critical region, so you reject  $H_0$  and conclude that there is evidence that the new class have a higher mean weight.

# Example 4

A machine produces bolts of diameter  $D$  where  $D$  has a normal distribution with mean 0.580 cm and standard deviation 0.015 cm.

The machine is serviced and after the service a random sample of 50 bolts from the next production run is taken to see if the mean diameter of the bolts has changed from 0.580 cm. The distribution of the diameters of bolts after the service is still normal with a standard deviation of 0.015 cm. The mean diameter of the 50 bolts is 0.577 cm.

- a** Stating your hypotheses clearly test, at the 1% level, whether or not there is evidence that the mean diameter of the bolts has changed.

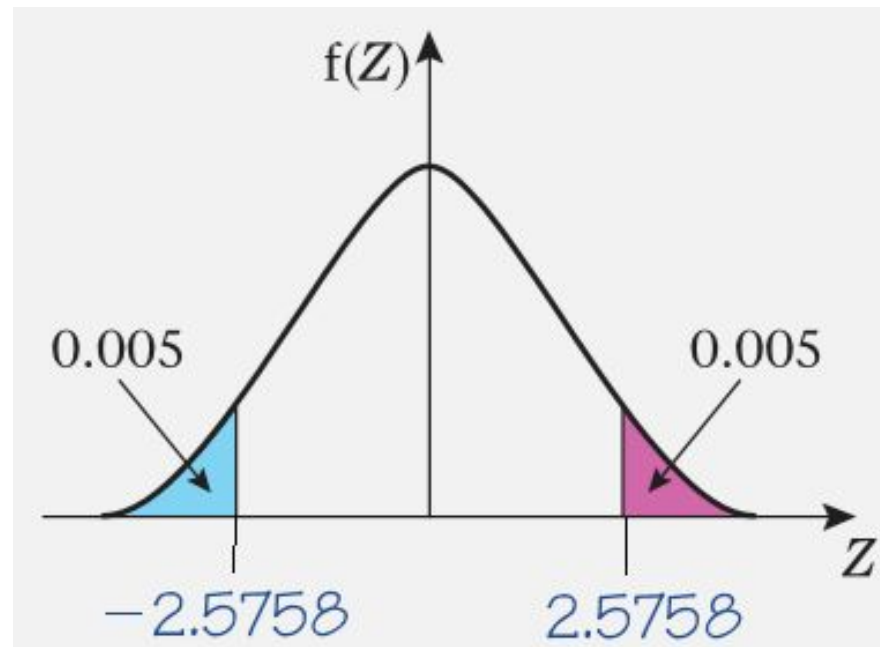
# Example 4

$$a \quad H_0: \mu = 0.580 \quad H_1: \mu \neq 0.580$$

$$\sigma = 0.015$$

The word 'changed' in the question suggests that the alternative hypothesis is  $\mu \neq 0.580$  so a two-tailed test is needed.

# Example 4



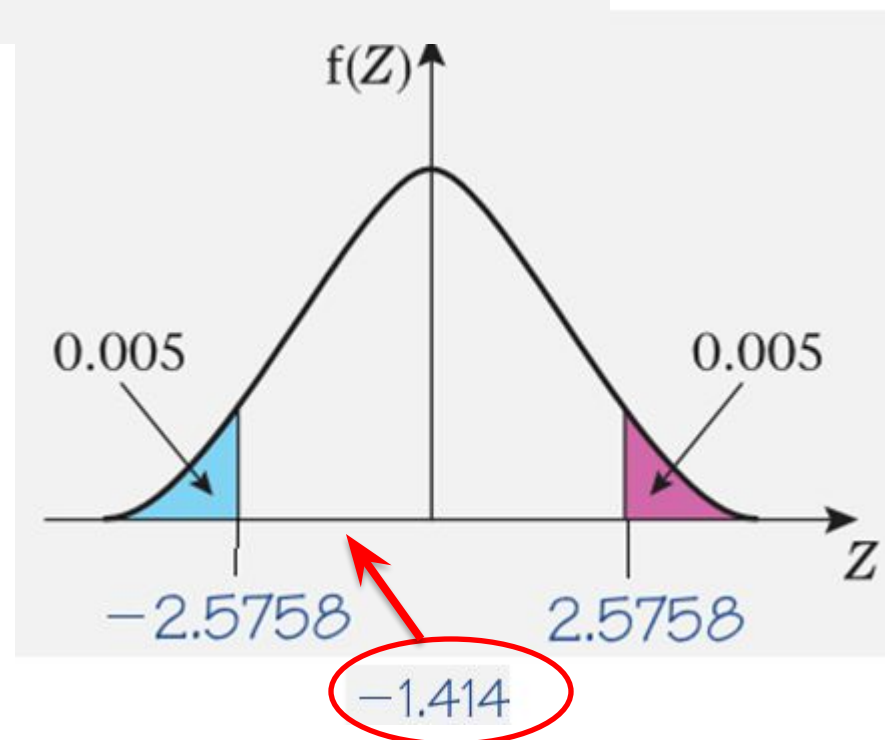
A 1% significance test the critical region of  $Z$  is

$$Z \leq -2.5758 \text{ or } Z \geq 2.5758.$$

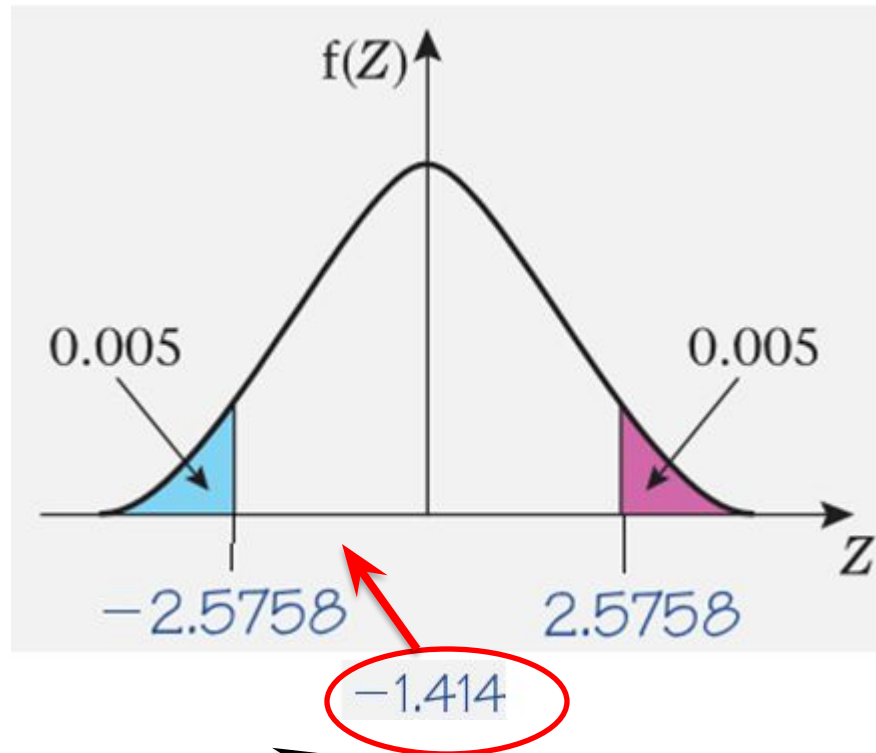
# Example 4

The sample gives  $n = 50$ ,  $\bar{x} = 0.577$  so the value of the test statistic is

$$z = \frac{0.577 - 0.580}{\frac{0.015}{\sqrt{50}}} = -1.414\dots,$$



# Example 4



This is not in the critical region so you accept  $H_0$  and conclude that there is no significant evidence that the mean diameter has changed.



## 7.5.2 Test the **hypothesis** for a **mean** of a normal distribution, $H_0: \mu=k$ , $H_1: \mu \neq k$ or $\mu > k$ or $\mu < k$

### Example 5

A certain company sells fruit juice in cartons. The amount of juice in a carton has a normal distribution with a standard deviation of 3 ml.

The company claims that the mean amount of juice per carton,  $\mu$ , is 60 ml. A trading inspector has received complaints that the company is overstating the mean amount of juice per carton and he wishes to investigate this complaint. The trading inspector took a random sample of 16 cartons which gave a mean of 59.1 ml.

Using a 5% level of significance, and stating your hypotheses clearly, test whether or not there is evidence to uphold this complaint.



# Example 5

The hypotheses are:

$$H_0: \mu = 60$$

$$H_1: \mu < 60$$

Remember  $H_0$  must specify a particular value of  $\mu$ . The inspector therefore will assume that the company is innocent and wish to formulate a null hypothesis to express this idea in terms of the parameter  $\mu$ .

If the company is guilty then  $\mu$  must be less than 60 (there would be few complaints if the cartons contained on average more than 60 ml) and so the alternative hypothesis  $H_1$  is  $H_1: \mu < 60$ . **This means the test is one-tailed.**

## Example 5

This is like the 'evidence' presented at a trial.

The sample gives  $n = 16$  and  $\bar{x} = 59.1$

$$\begin{aligned} P(\bar{X} \leq 59.1 | \mu = 60) &= P\left(Z \leq \frac{59.1 - 60}{\frac{3}{4}}\right) \\ &= P(Z \leq -1.2) \\ &= 0.1151 \end{aligned}$$

## Example 5

$0.1151 > 0.05$  so the result is not significant and there is insufficient evidence to reject  $H_0$ , that  $\mu = 60$ .

This probability is greater than the 5% significance level so there is no reason to suspect the validity of  $H_0$ .

The conclusion should incorporate two statements:

- 1 State whether or not the test is significant.
- 2 Interpret this in the context of the question.

There is insufficient evidence to support the complaint.

## 7.5.3 Test the hypothesis for the difference between means of two independent normal distributions

If, instead of one population, you now have two independent populations then you can test hypotheses about the differences in the population means.

In Chapter 1 you saw that if  $X$  and  $Y$  are two independent normal distributions with means of  $\mu_x$  and  $\mu_y$  and standard deviations  $\sigma_x$  and  $\sigma_y$  respectively then

$$X - Y \sim N(\mu_x - \mu_y, \sigma_x^2 + \sigma_y^2)$$

Now if  $\bar{X}$  and  $\bar{Y}$  are sample means based on samples of size  $n_x$  and  $n_y$  respectively from the above two normal populations then:

$$\bar{X} - \bar{Y} \sim N\left(\mu_x - \mu_y, \frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}\right)$$

and the statistic  $\bar{X} - \bar{Y}$  can be used to test hypotheses about the values of  $\mu_x$  and  $\mu_y$ .

## 7.5.3 Test the hypothesis for the difference between means of two independent normal distributions

The test statistic you will need to use is based upon the distribution of  $\bar{X} - \bar{Y}$  and is

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{\sqrt{\left(\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}\right)}}$$

## 7.5.3 Test the hypothesis for the **difference between means** of two independent normal distributions

### Example 6

The weights of boys and girls in a certain school are known to be normally distributed with standard deviations of 5 kg and 8 kg respectively. A random sample of 25 boys had a mean weight of 48 kg and a random sample of 30 girls had a mean weight of 45 kg.

Stating your hypotheses clearly test, at the 5% level of significance, whether or not there is evidence that the mean weight of boys in the school is greater than the mean weight of the girls.



# Example 6

$$H_0: \mu_{\text{boy}} = \mu_{\text{girl}} \quad H_1: \mu_{\text{boy}} > \mu_{\text{girl}}$$

$$\sigma_1 = 5, n_1 = 25, \sigma_2 = 8 \text{ and } n_2 = 30:$$

The test statistic is

$$z = \frac{\bar{x} - \bar{y} - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}}}$$

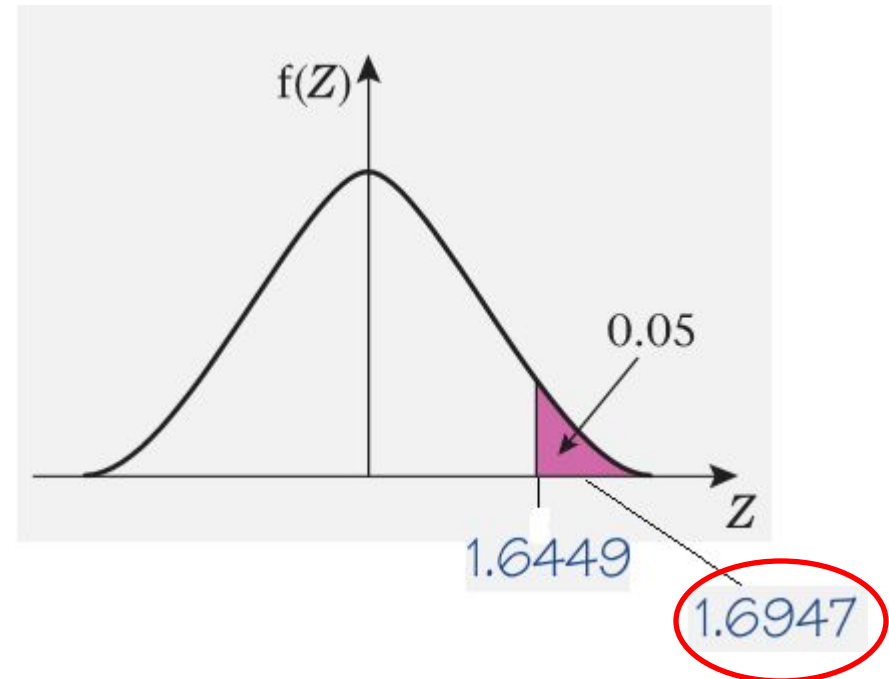
$$= \frac{48 - 45}{\sqrt{\frac{25}{25} + \frac{64}{30}}}$$

$$= \frac{3}{\sqrt{3.1333...}}$$

$$= 1.6947...$$

## Example 6

$$1.6449 < 1.6947$$



The 5% (one-tailed) critical value for  $Z$  is  $z = 1.6449$  so this value is significant and you can reject  $H_0$  and conclude that there is evidence that the mean weight of boys is greater than the mean weight of the girls.