NUFYP Mathematics

5.3 Differentiation 3

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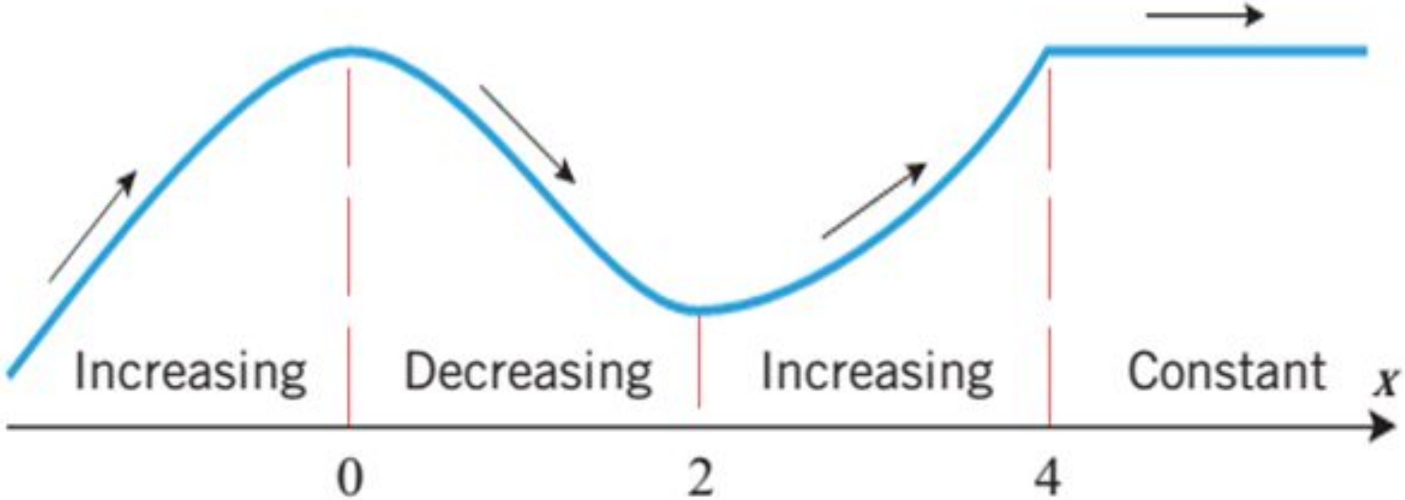
Lecture Outline

- Using
- first derivative
  - Increasing/ decreasing intervals
  - Critical points
    - Stationary points
  - First derivative test
- Using
- second derivative
  - Concavities
  - Inflection points
  - Second derivative test

Introduction

The purpose of this lecture is to develop mathematical tools that can be used to determine the exact shape of a graph and the precise locations of its key features such as local extremes, inflections, intervals of increasing/decreasing, upward/downward concavities.

Increasing and decreasing functions



The terms increasing, decreasing, and constant are used to describe the behavior of a function as we travel left to right along its graph.

The function in the figure can be described as
increasing to the left of $x = 0$, decreasing from $x = 0$ to $x = 2$, increasing from $x = 2$ to $x = 4$, and constant to the right of $x = 4$.

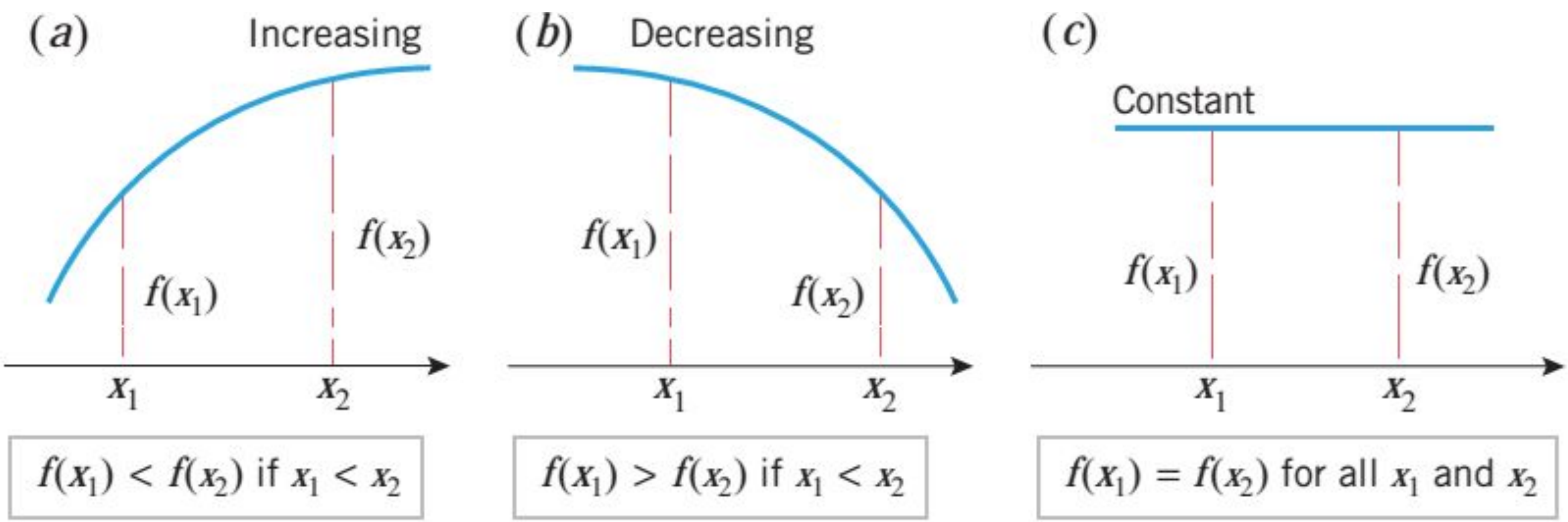
Increasing and decreasing functions

DEFINITION Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

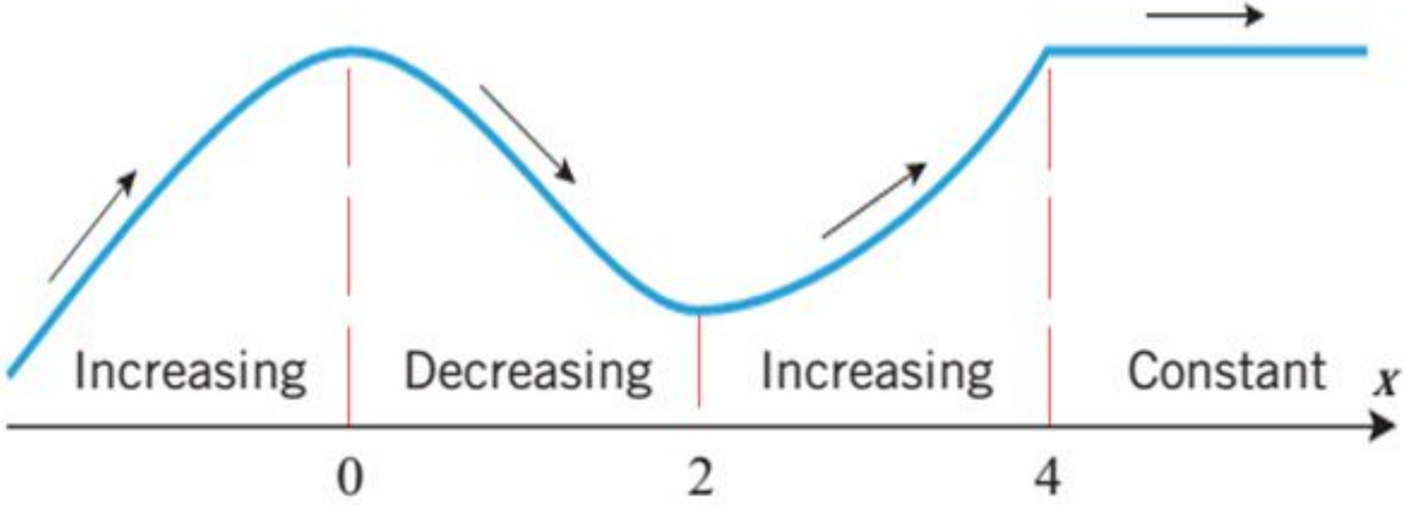
(a) f is ***increasing*** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(b) f is ***decreasing*** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

(c) f is ***constant*** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.



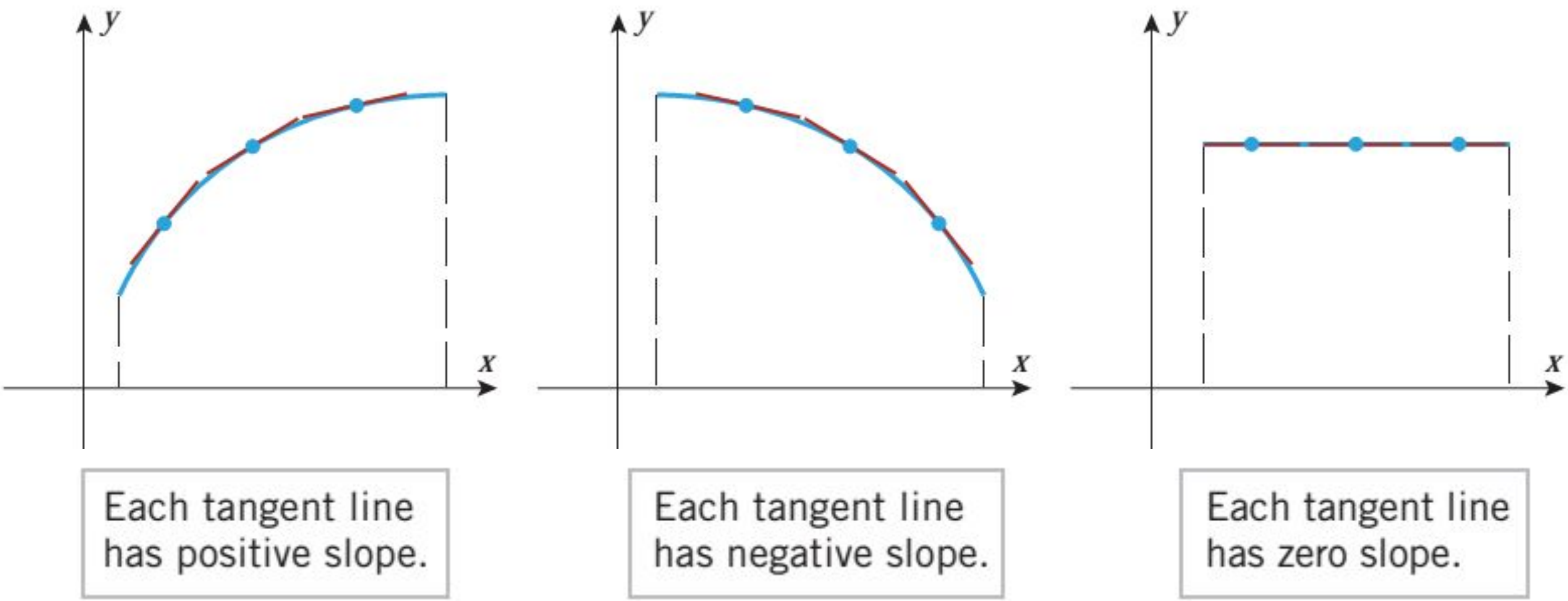
Increasing and decreasing functions



The definitions of "increasing", "decreasing", and "constant" describe the behavior of a function on an interval and not at a point. In particular, it is not inconsistent to say that the function in the Figure is decreasing on the interval [0,2] and increasing on the interval [2,4].

Increasing and decreasing functions

	increasing	on any interval where each tangent line to its graph has	positive slope
	decreasing		negative slope
	constant		zero slope



Increasing and decreasing functions

Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).

(a) If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on$[a, b]$.

(c) If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.

Increasing and decreasing functions

Example 1. Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing and the intervals on which it is decreasing.

Solution

$$f'(x) = 2x - 4 = 2(x - 2)$$

$f'(x) < 0$ if $x < 2$ $\quad$ $f'(x) > 0$ if $x > 2$

Since f is continuous everywhere, f is decreasing on $(-\infty, 2]$ and increasing on $[2, +\infty)$.

Increasing and decreasing functions

Example 1. Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing and the intervals on which it is decreasing.

Solution

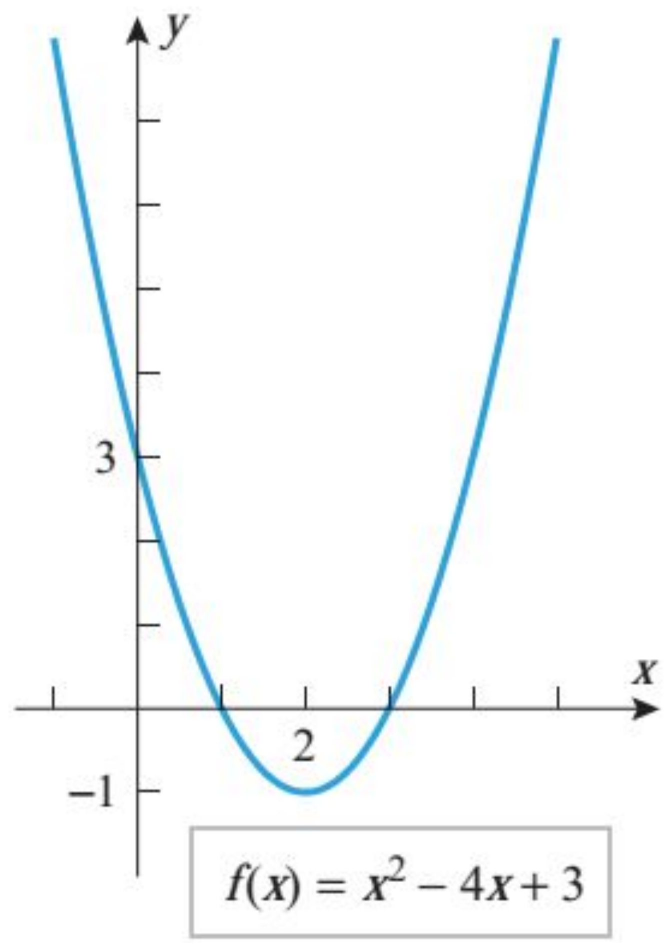
Alternatively, we can determine the vertex of this square parabola:

$$f(x) = x^2 - 4x + 3 = (x-2)^2 - 1$$

Since the parabola opens upward, we can conclude that

$$f(x) = x^2 - 4x + 3 = (x-2)^2 - 1$$

$f(x)$ is decreasing on $(-\infty, 2]$ and increasing on $[2, +\infty)$



Increasing and decreasing functions

Example 2. Find the intervals on which $f(x) = x^3$ is increasing and the intervals on which it is decreasing.

Solution

$f'(x) = 3x^2$

$f'(x) > 0$ if $x < 0$ $\quad$ $f'(x) > 0$ if $x > 0$

Since f is continuous everywhere, f is increasing on $(-\infty, 0]$ and increasing on $[0, +\infty)$.

Since f is increasing on the adjacent intervals $(-\infty, 0]$ and $[0, +\infty)$, it follows that f is increasing on their union $(-\infty, +\infty)$.

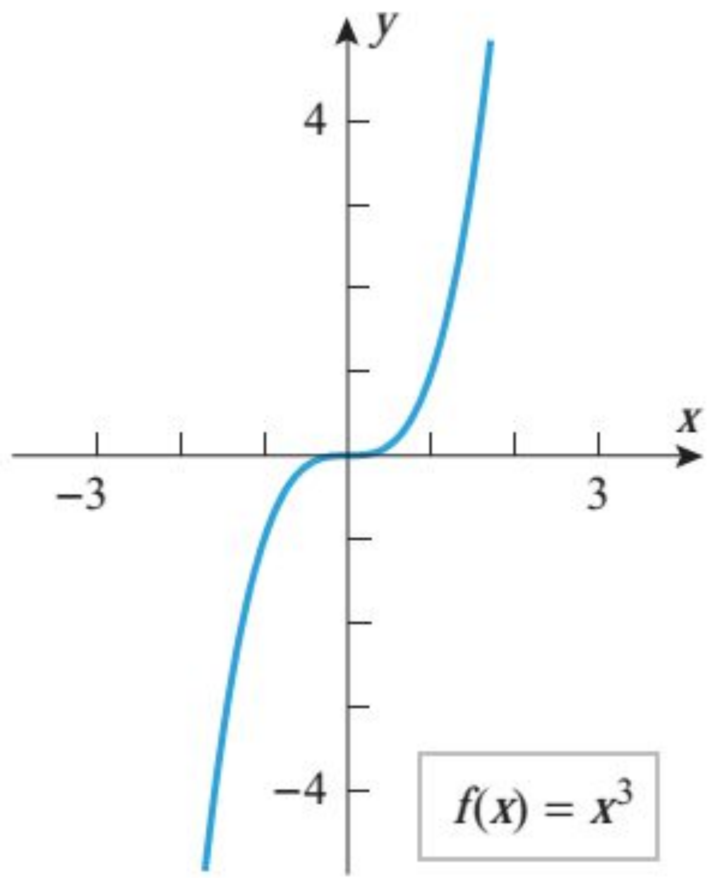
Increasing and decreasing functions

Example 2. Find the intervals on which $f(x) = x^3$ is increasing and the intervals on which it is decreasing.

Solution

f is increasing on $(-\infty, 0]$ and increasing on $[0, +\infty)$.

f is increasing $(-\infty, +\infty)$.



Increasing and decreasing functions

Example 3. Find the intervals on which

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$$

is increasing and the intervals on which it is decreasing.

Solution

Differentiating f we obtain

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2)$$

$$= 12x(x-1)(x+2)$$

Increasing and decreasing functions

Example 3. Find the intervals on which

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$$

is increasing and the intervals on which it is decreasing.

Solution

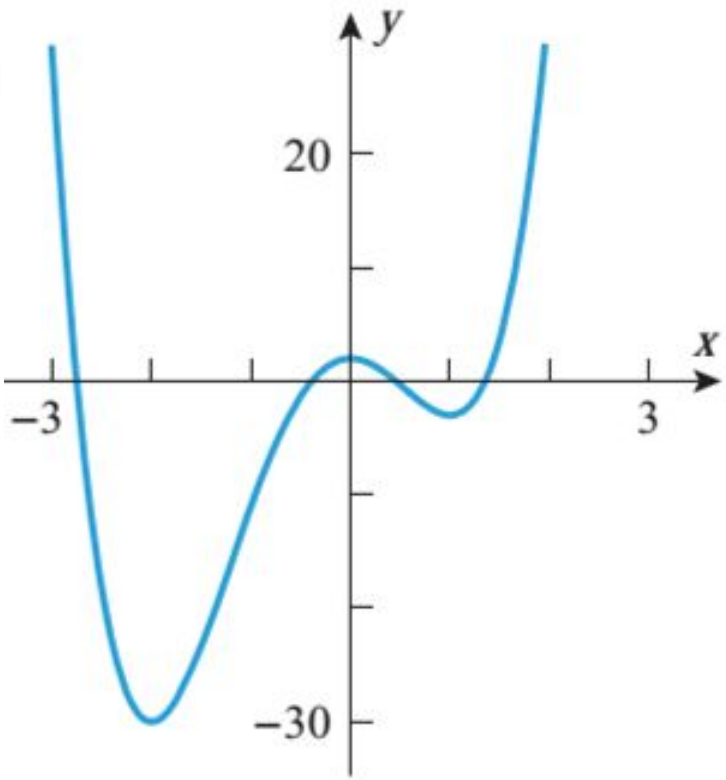
Constructing a following table we conclude:

INTERVAL	$(12x)(x+2)(x-1)$	Sign of $f'(x)$	CONCLUSION
$x < -2$	$(-)(-)(-)$	$-$	f is decreasing on $(-\infty, -2]$
$-2 < x < 0$	$(-)(+)(-)$	$+$	f is increasing on $[-2, 0]$
$0 < x < 1$	$(+)(+)(-)$	$-$	f is decreasing on $[0, 1]$
$1 < x$	$(+)(+)(+)$	$+$	f is increasing on $[1, +\infty)$

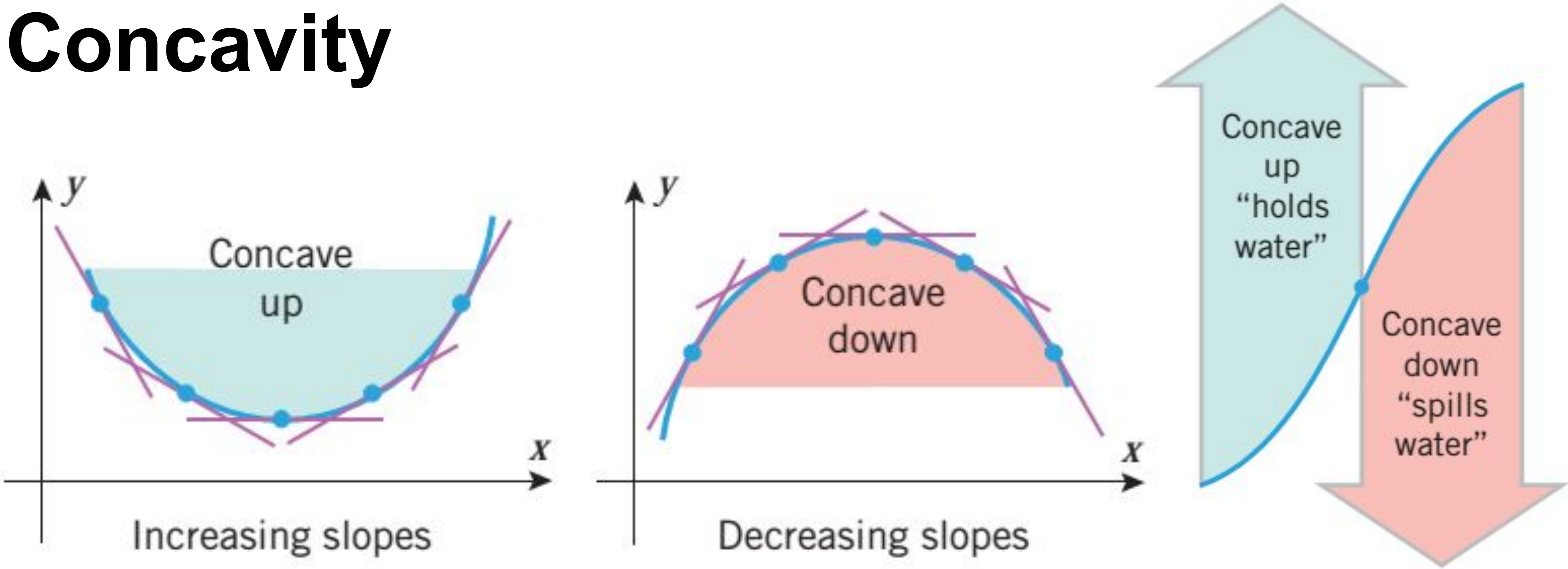
Increasing and decreasing functions

Example 3. $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

INTERVAL	Sign of $f'(x)$	CONCLUSION
$x < -2$	$-$	f is decreasing on $(-\infty, -2]$
$-2 < x < 0$	$+$	f is increasing on $[-2, 0]$
$0 < x < 1$	$-$	f is decreasing on $[0, 1]$
$1 < x$	$+$	f is increasing on $[1, +\infty)$



Concavity



- f is concave up on an open interval if its tangent lines have increasing slopes on that interval and is concave down if they have decreasing slopes.
- f is concave up on an open interval if its graph lies above its tangent lines on that interval and is concave down if it lies below its tangent lines.

Concavity

If f is differentiable on an open interval, then f is said to be ***concave up*** on the open interval if f' is increasing on that interval, and f is said to be ***concave down*** on the open interval if f' is decreasing on that interval

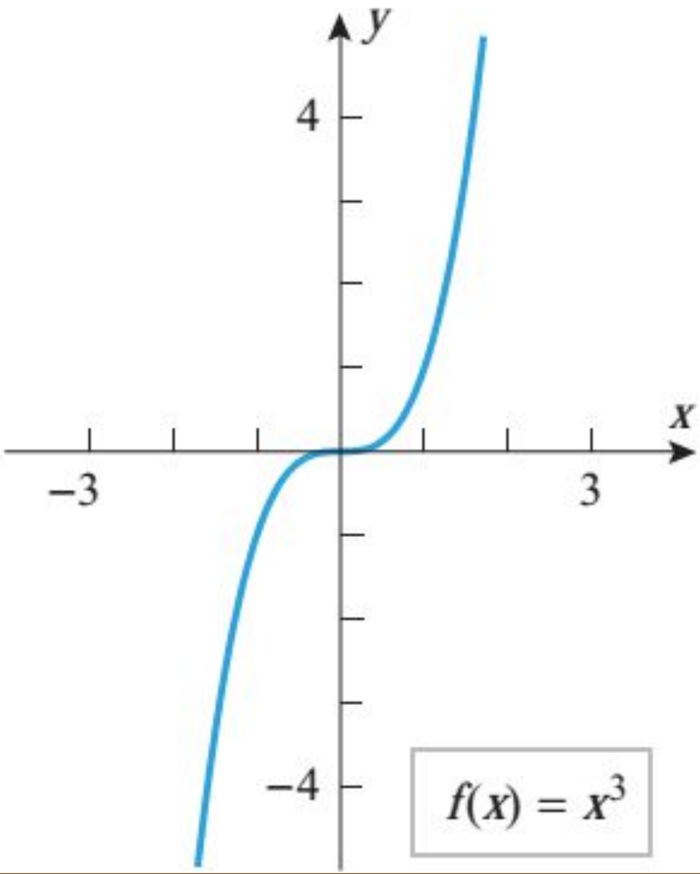
Theorem. Let f be twice differentiable on an open interval.

(a) If $f''(x) > 0$ for every value of x in the open interval, then f is ***concave up*** on that interval.

(b) If $f''(x) < 0$ for every value of x in the open interval, then f is ***concave down*** on that interval.

Concavity

Theorem.
(a) $f''(x) > 0 \therefore$ concave up.
(b) $f''(x) < 0 \therefore$ concave down.



$f(x) = x^3$

is concave down on $(-\infty, 0)$ and concave up on $(0, +\infty)$.

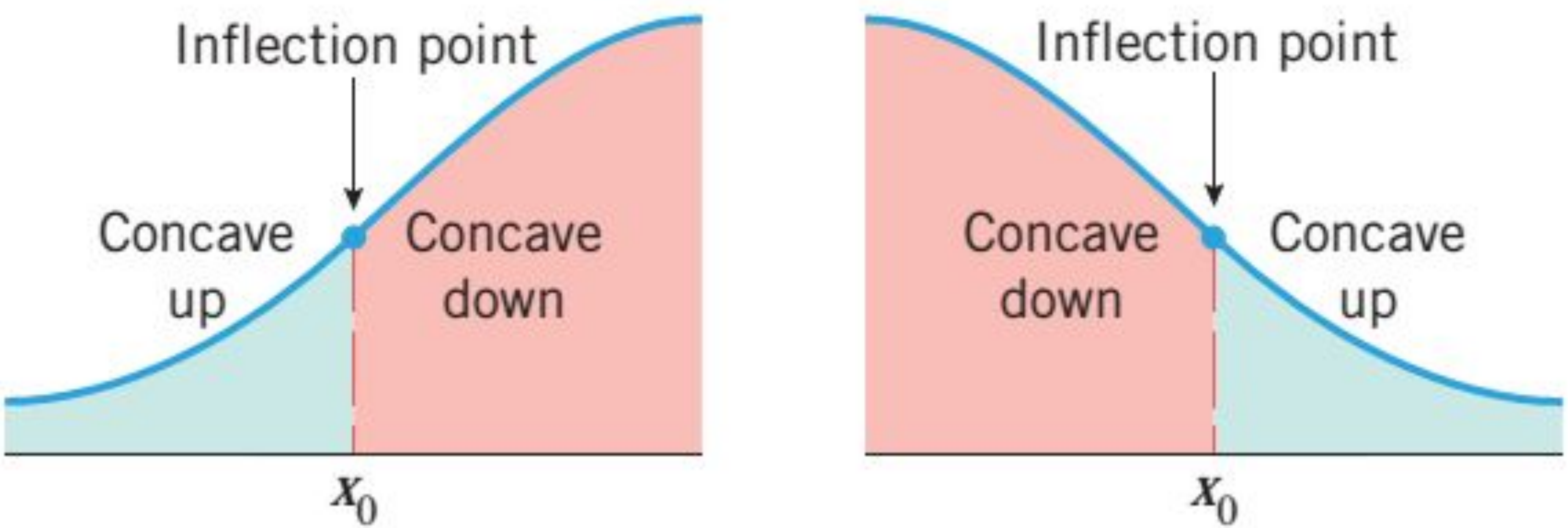
This agrees with the Theorem, since

$f'(x) = 3x^2$ and $f''(x) = 6x$, so

$f''(x) < 0$ if $x < 0$ and
$f''(x) > 0$ if $x > 0$.

Inflection points

If f is continuous on an open interval containing a value x_0, and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an ***inflection point at*** x_0, and we call the point $(x_0, f(x_0))$ on the graph of f an ***inflection point*** of f.



Inflection points

Example 4. $f = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down.

Solution

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

INTERVAL	$(3x)(x-2)$	Sign of $f'(x)$	CONCLUSION
$x < 0$	$(-)(-)$	$+$	f is increasing on $(-\infty, 0]$
$0 < x < 2$	$(+)(-)$	$-$	f is decreasing on $[0, 2]$
$x > 2$	$(+)(+)$	$+$	f is increasing on $[2, +\infty)$

Inflection points

Example 4. $f = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down.

Solution

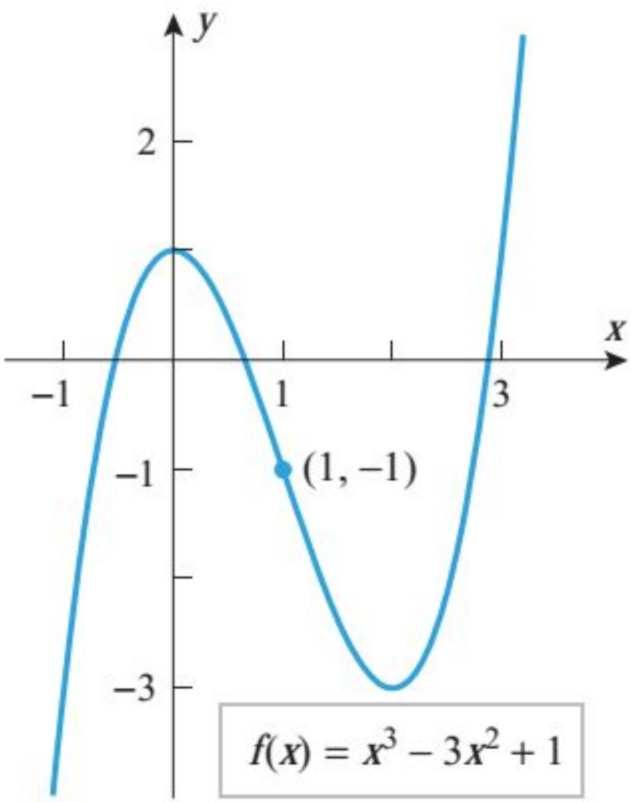
$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

$$f''(x) = 6x - 6 = 6(x-1)$$

INTERVAL	$6(x-1)$	Sign of $f''(x)$	CONCLUSION
$x < 1$	$(-)$	$-$	f is concave down on $(-\infty, 1)$
$x > 1$	$(+)$	$+$	f is concave up on $(1, +\infty)$

Inflection points

Example 4. $f = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down.



INTERVAL	Sign of $f'(x)$	CONCLUSION
$x < 0$	+	f is increasing on $(-\infty, 0]$
$0 < x < 2$	−	f is decreasing on $[0, 2]$
$x > 2$	+	f is increasing on $[2, +\infty)$

INTERVAL	Sign of $f''(x)$	CONCLUSION
$x < 1$	−	f is concave down on $(-\infty, 1)$
$x > 1$	+	f is concave up on $(1, +\infty)$

Relative (Local) Maxima and Minima

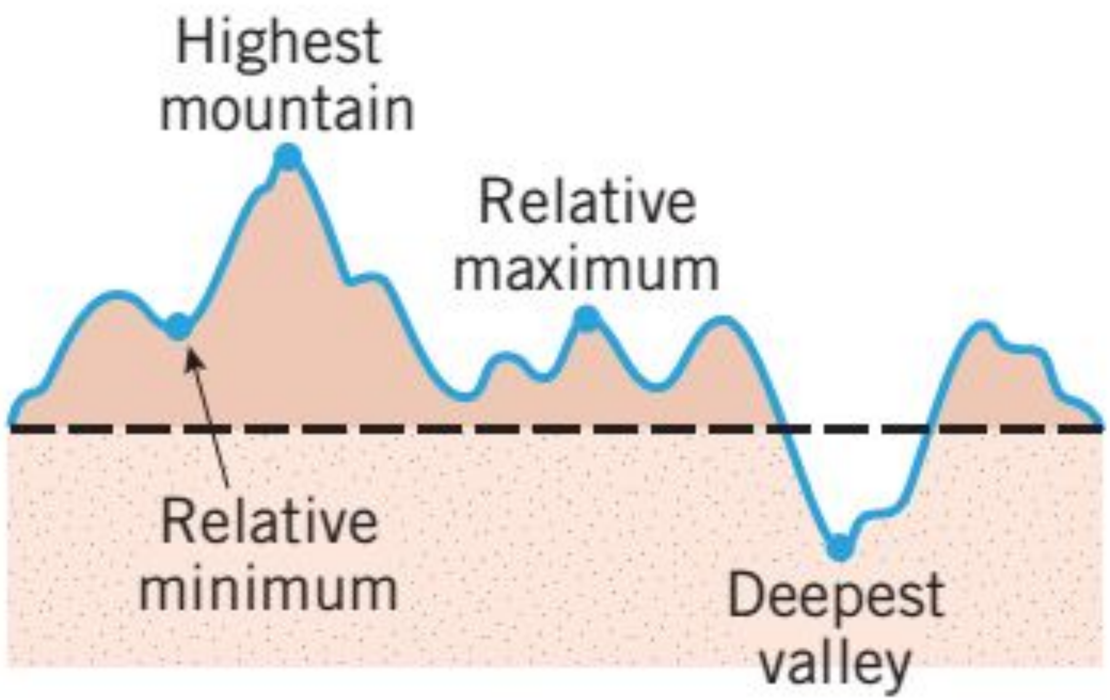
A function f is said to have a ***relative maximum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval.

Similarly, f is said to have a ***relative minimum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest, that is, $f(x_0) \leq f(x)$ for all x in the interval.

If f has either a relative maximum or a relative minimum at x_0, then f is said to have a ***relative extremum*** at x_0.

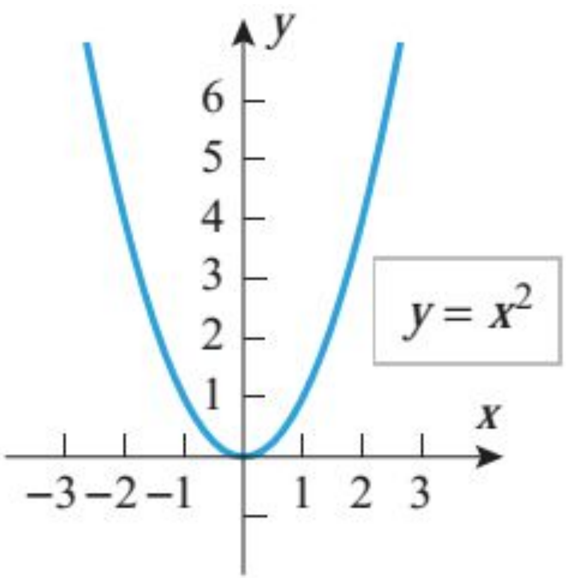
Relative (Local) Maxima and Minima

relative maximum at x_0 if $f(x_0) \geq f(x)$ for all x in the interval.
relative minimum at x_0 if $f(x_0) \leq f(x)$ for all x in the interval.
relative extremum at x_0 if either a relative maximum or a relative minimum

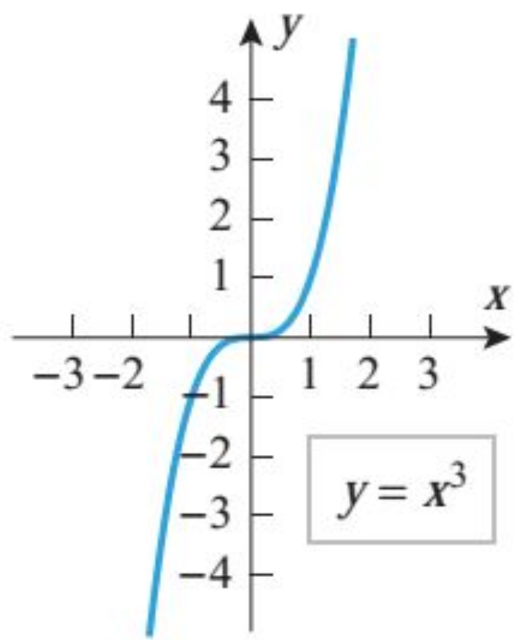


Relative (Local) Maxima and Minima

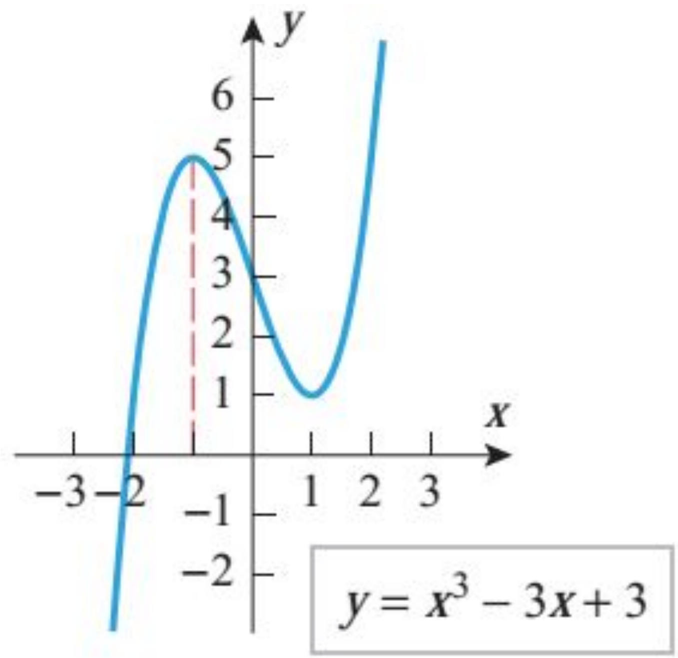
Determine whether the graph has relative extrema.



relative minimum at $x = 0$ but no relative maxima



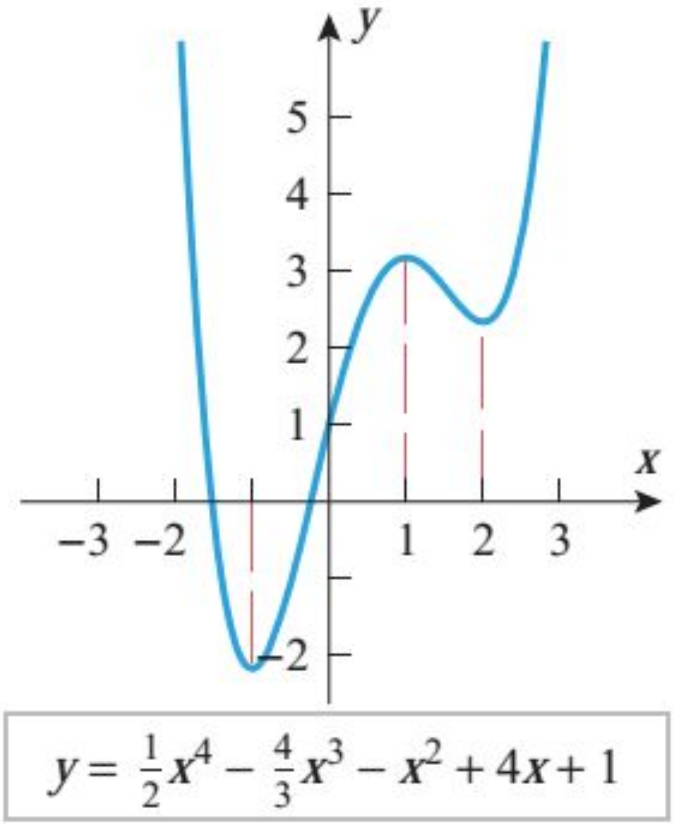
no relative extrema

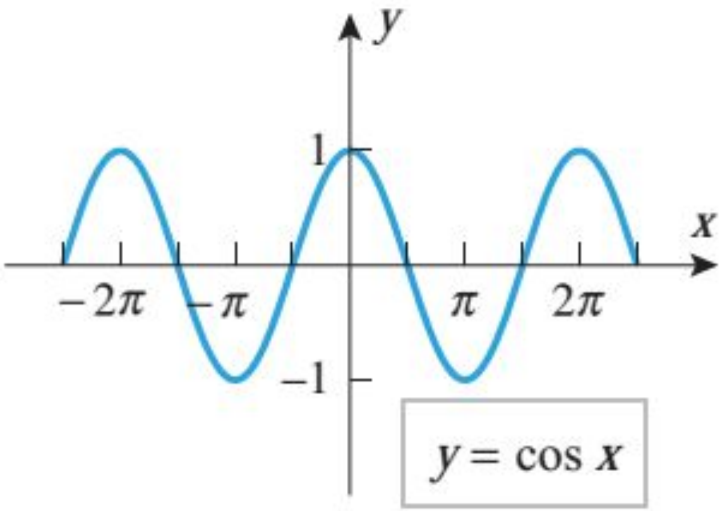


relative maximum at $x = -1$ and a relative minimum $x = 1$

Relative (Local) Maxima and Minima

Determine whether the graph has relative extrema.





relative maxima at all even multiples of π and relative minima at all odd multiples of π

relative maximum at $x = -1$ and a relative minimum at $x = 1$

Relative (Local) Maxima and Minima

Critical and stationary points

In general, a **critical point** for a function f is a point in the domain of f at which:

- either the graph of f has a horizontal tangent line
- or f is not differentiable.

We call x a **stationary point** of f if $f'(x) = 0$.

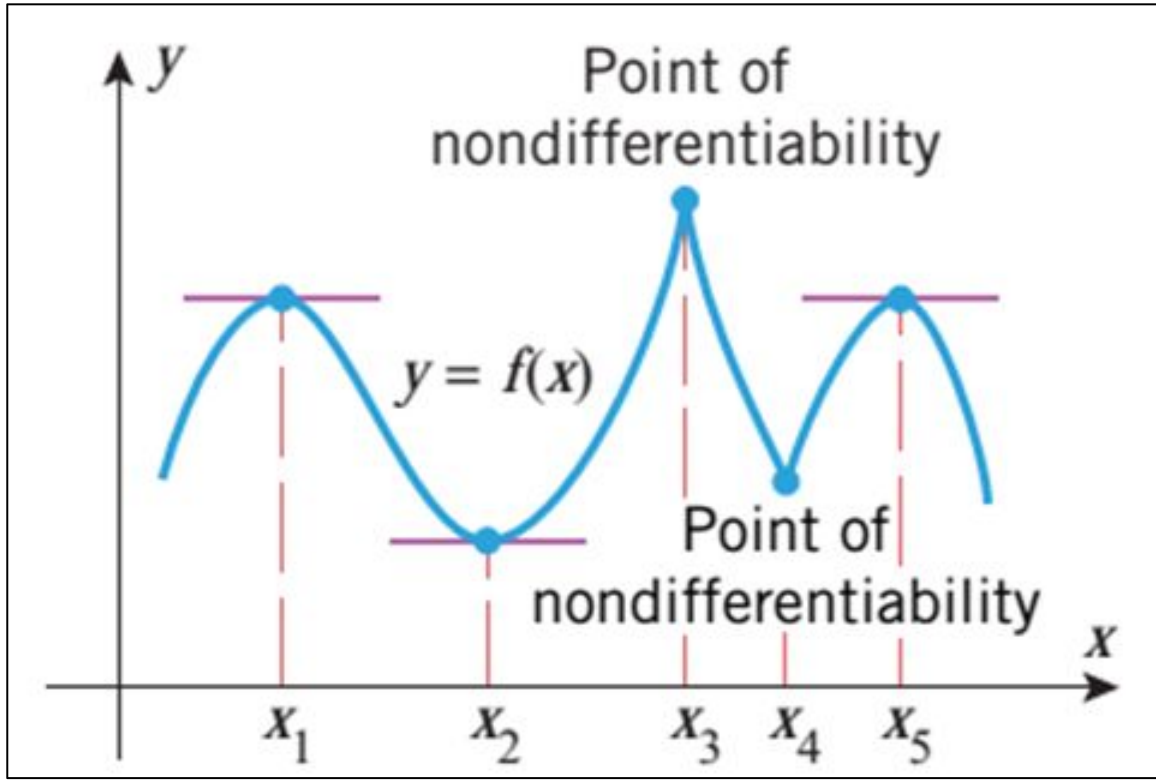
Relative (Local) Maxima and Minima

Critical and stationary points

A **critical point** : horizontal tangent line or not differentiable.

A **stationary point** : $f'(x) = 0$.

Determine critical and stationary points



The points x_1, x_2, x_3, x_4, and x_5 are critical points. Of these, x_1, x_2, and x_5 are stationary points.

Relative (Local) Maxima and Minima

Critical and stationary points

Suppose that f is a function defined on an open interval containing the point x_0.

If f has a relative extremum at $x = x_0$, then $x = x_0$ is a critical point of f;

that is, either $f'(x_0) = 0$ or f is not differentiable at x_0.

Relative (Local) Maxima and Minima

Critical and stationary points

Example 5. Find all critical points of $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$.

Solution

The function f is continuous everywhere and its derivative is

$$f'^{(x)} = 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}} = 5x^{-\frac{1}{3}}(x-2) = \frac{5(x-2)}{x^{\frac{1}{3}}}$$

$f'(x) = 0$ if $x = 2$ and $f'(x)$ is undefined if $x = 0$.

Thus, $x = 2$ and $x = 0$ are critical points,

$x = 2$ is a stationary points.

Relative (Local) Maxima and Minima

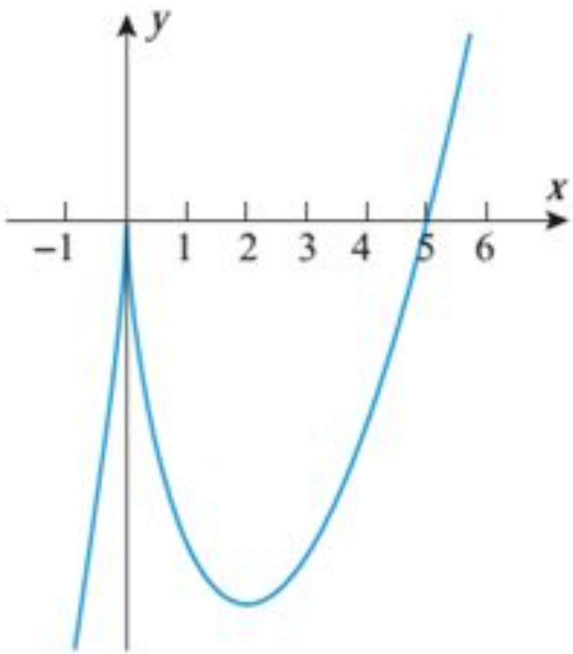
Critical and stationary points

Example 5. Find all critical points of $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$.

Solution

$x = 2$ and $x = 0$ are critical points,

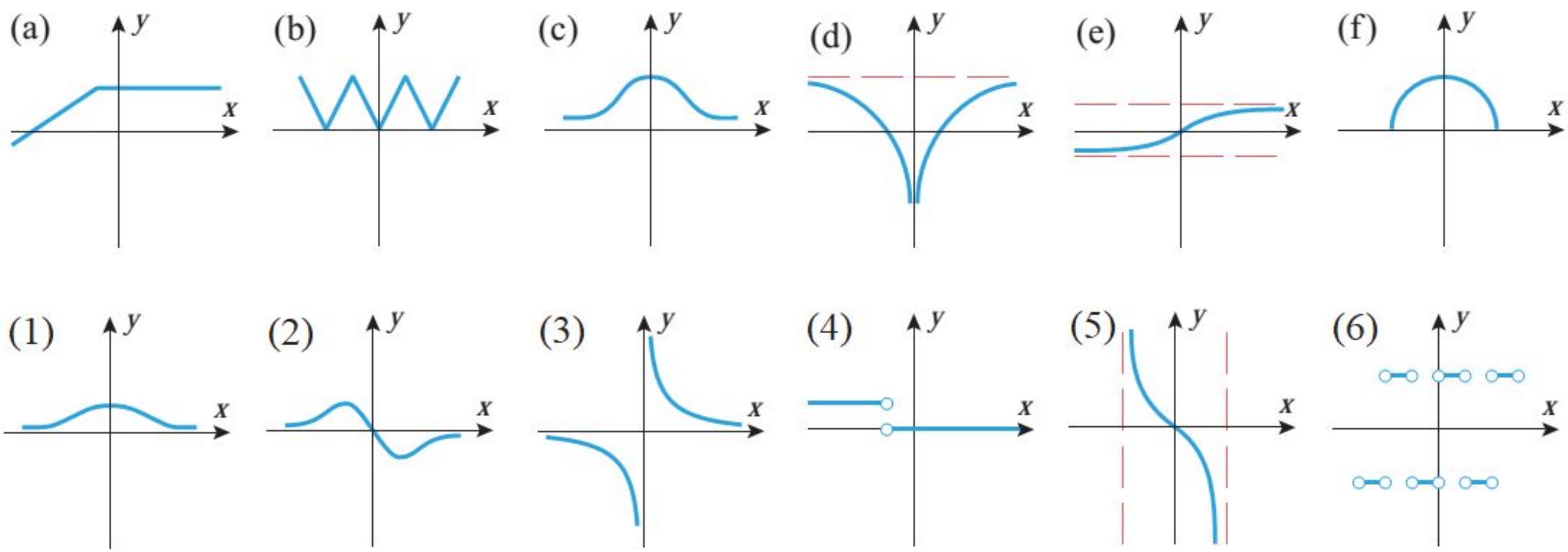
$x = 2$ is a stationary points.



$y = 3x^{5/3} - 15x^{2/3}$

Relative (Local) Maxima and Minima

Match the graphs of the functions (a)-(f) with the graphs of their derivatives (1)-(6)



a-4, b-6, c-2, d-3, e-1, f-5

First derivative test

Theorem (First Derivative Test). Suppose that f is continuous at a critical point x_0.

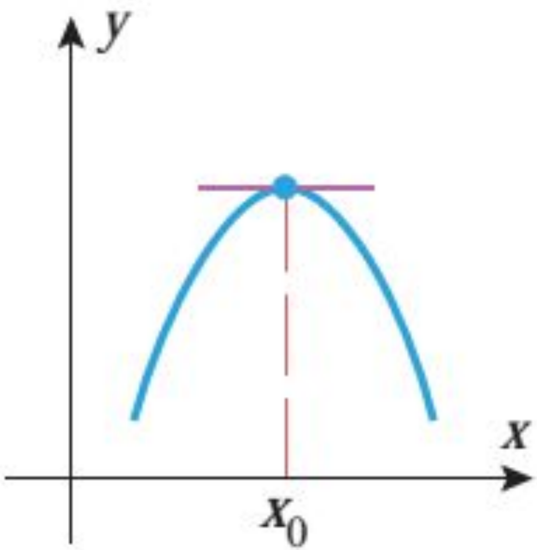
(a) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.

(b) If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.

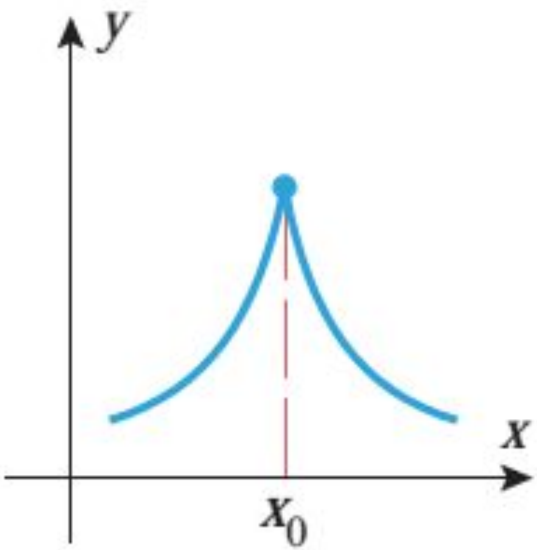
(c) If $f'(x)$ has the same sign on an open interval extending left from x_0 as it does on an open interval extending right from x_0, then f does not have a relative extremum at x_0.

First derivative test

A function f has a relative extremum at those critical points where f' changes sign.



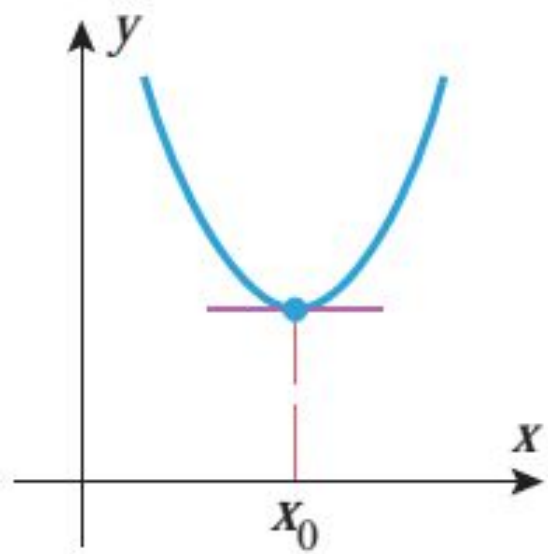
Critical point
Stationary point
Relative maximum



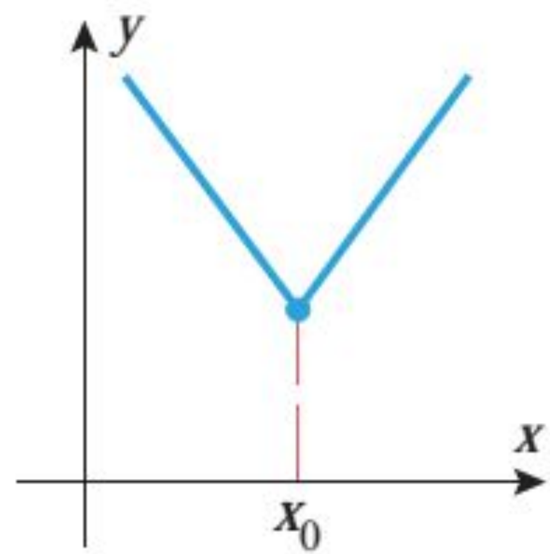
Critical point
Not a stationary point
Relative maximum

First derivative test

A function f has a relative extremum at those critical points where f' changes sign.



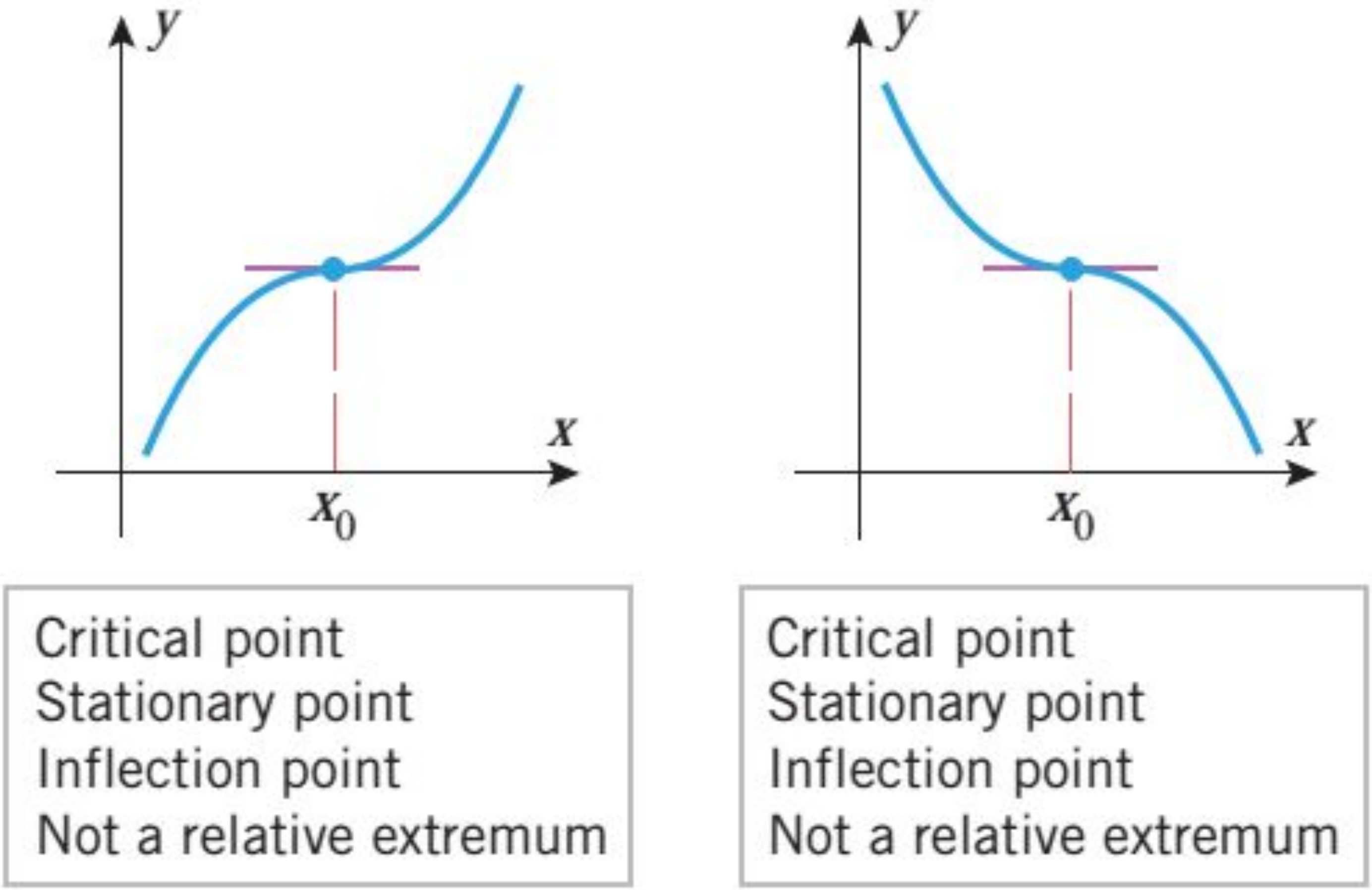
Critical point
Stationary point
Relative minimum



Critical point
Not a stationary point
Relative minimum

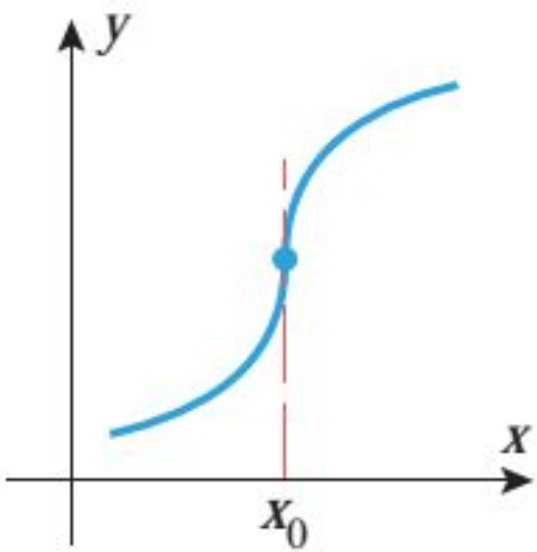
First derivative test

A function f has a relative extremum at those critical points where f' changes sign.

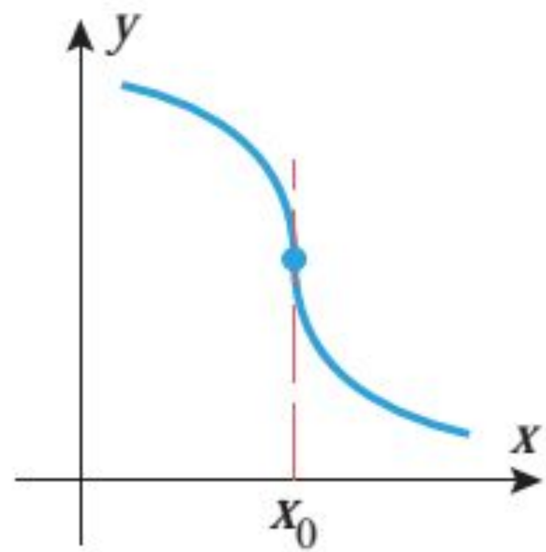


First derivative test

A function f has a relative extremum at those critical points where f' changes sign.



Critical point
Not a stationary point
Inflection point
Not a relative extremum



Critical point
Not a stationary point
Inflection point
Not a relative extremum

Second derivative test

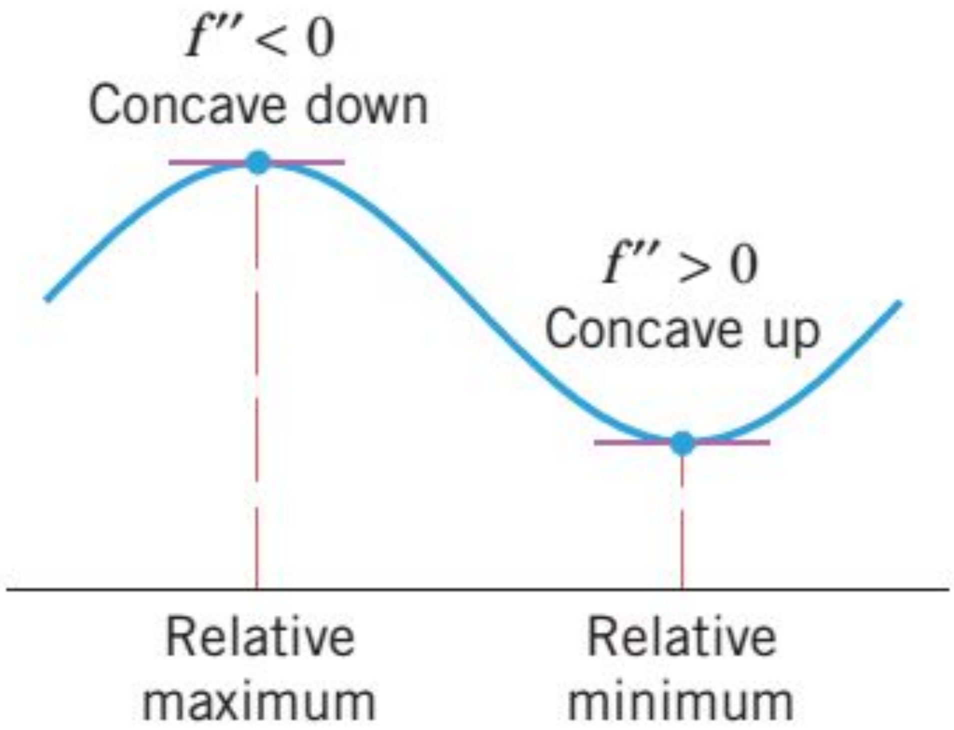
Theorem (Second Derivative Test). Suppose that f is twice differentiable at the point x_0.

(a) If $f'(x) = 0$ and $f''(x) > 0$, then f has a relative minimum at x_0.

(b) If $f'(x) = 0$ and $f''(x) < 0$, then f has a relative maximum at x_0.

(c) If $f'(x) = 0$ and $f''(x) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.

Second derivative test



A function f has a relative maximum at a stationary point if the graph of f is concave down on an open interval containing that point, and it has a relative minimum if it is concave up.

Second derivative test

Example 5. Find the relative extrema of $f(x) = 3x^5 - 5x^3$.

Solution

We have $f'(x) = 15x^4 - 15x^2 = 15x^2(x+1)(x-1)$

$$f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$$

Solving $f'(x) = 0$ yields the stationary points: $x = 0, x = -1, x = 1$.

Implement the Second derivative test:

STATIONARY POINT	$30x(2x^2-1)$	Sign of $f''(x)$	SECOND DERIVATIVE TEST
$x = -1$	-30	$-$	f has a relative maximum
$x = 0$	0	0	Inconclusive
$x = 1$	30	$+$	f has a relative minimum

Second derivative test

Example 5. Find the relative extrema of $f(x) = 3x^5 - 5x^3$.

Solution

Thus $f(x)$ has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

For $x = 0$ implement the First derivative test:

INTERVAL	$15x^2(x+1)(x-1)$	Sign of $f'(x)$
$-1 < x < 0$	$(+)(+)(-)$	$-$
$0 < x < 1$	$(+)(+)(-)$	$-$

Since there is no sign change in f' at $x = 0$, there is neither a relative maximum nor a relative minimum at that point.

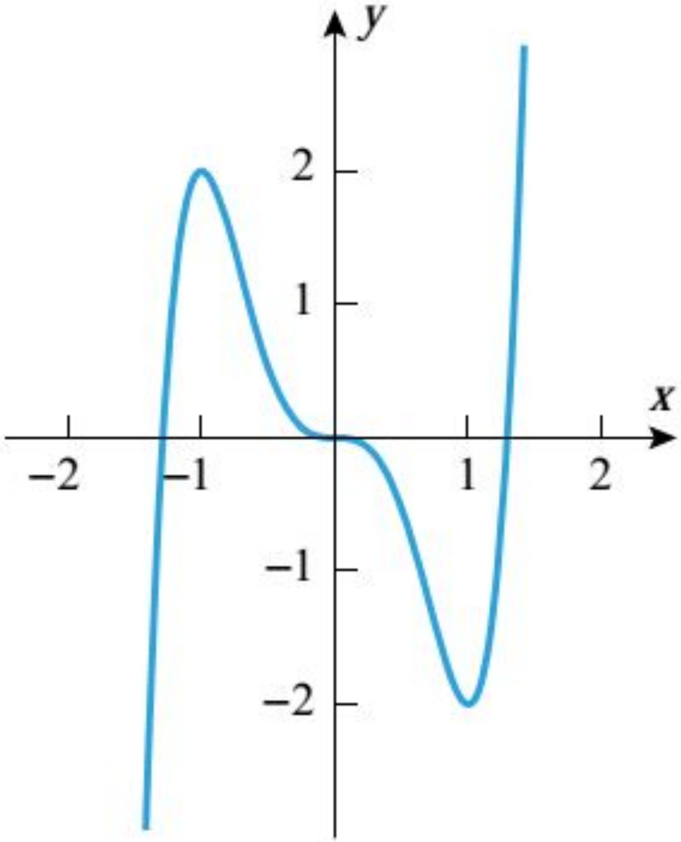
Second derivative test

Example 5. Find the relative extrema of $f(x) = 3x^5 - 5x^3$.

Solution

STATIONARY POINT	Sign of $f''(x)$	SECOND DERIVATIVE TEST
$x = -1$	$-$	relative maximum
$x = 0$	0	Inconclusive
$x = 1$	$+$	relative minimum

INTERVAL	$15x^2(x+1)(x-1)$	$f'(x)$
$-1 < x < 0$	$(+)(+)(-)$	$-$
$0 < x < 1$	$(+)(+)(-)$	$-$



$y = 3x^5 - 5x^3$

Learning outcomes

5.3.1. Define stationary points of a function.

5.3.2. Define intervals on which a function is decreasing or increasing.

5.3.3. Define inflection points and the intervals on which a function is concave upward or downward.

5.3.4. Use first derivative and second derivative tests to define a nature of the stationary points.

Formulae

In general, a **critical point** for a function f is a point in the domain of f at which:

- either the graph of f has a horizontal tangent line
- or f is not differentiable.

We call x a **stationary point** of f if $f'(x) = 0$.

Second Derivative Test. f is twice differentiable at x_0.

(a) If $f'(x) = 0$ and $f''(x) > 0$, then relative minimum at x_0.

(b) If $f'(x) = 0$ and $f''(x) < 0$, then relative maximum at x_0.

(c) If $f'(x) = 0$ and $f''(x) = 0$, then no conclusion about relative extremum at x_0.

Formulae

First Derivative Test. f is continuous at x_0.

(a) If $f'(x) > 0$ on extending left from x_0 and $f'(x) < 0$ on extending right from x_0, then relative maximum at x_0.

(b) If $f'(x) < 0$ on extending left from x_0 and $f'(x) > 0$ on extending right from x_0, then relative minimum at x_0.

(c) If $f'(x)$ has the same sign on extending left from x_0 and on extending right from x_0, then no relative extremum at x_0.

Preview activity: Differentiation 4

In general, for polynomials of degree n (≥ 2),

what can you say about the amount of:
x-intercepts,
relative extrema,
and inflection points?