Combinatorics

Irina Prosvirnina

- Permutations
- Combinations
- The binomial theorem

Permutations

Example 1

In how many ways can we select three students from a group of five students to stand in line for a picture?

Solution:

First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line.

By the product rule, there are $5 \cdot 4 \cdot 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

Permutations

<u>Example 2</u>

In how many ways can we arrange five students to stand in line for a picture?

<u>Solution:</u>

To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way.

Consequently, there are

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

ways to arrange all five students in a line for a picture.

Permutations

<u>Definition 1</u>

A **permutation** of a set of distinct objects is an ordered arrangement of these objects.

We also are interested in ordered arrangements of some of the elements of a set.

An ordered arrangement of r elements of a set is called an **r-permutation**.

Permutations

<u>Example 3</u>

Let $S = \{1, 2, 3\}$.

The ordered arrangement $3, 1, 2$ is a permutation of S.

The ordered arrangement $3, 2$ is a 2-permutation of S.

Permutations

- The number of r-permutations of a set with n elements is denoted by $P(n, r)$.
  We can find $P(n, r)$ using the product rule.

Permutations

<u>Example 4</u>

Let $S = \{a, b, c\}$.

The 2-permutations of S are the ordered arrangements:

$$a, b;$$
$$a, c;$$
$$b, a;$$
$$b, c;$$
$$c, a;$$
$$c, b.$$

Consequently, there are six 2-permutations of this set with three elements.

Permutations

Example 4

Let $S = \{a, b, c\}$.

The 2-permutations of S are the ordered arrangements:

$$a, b;$$
$$a, c;$$
$$b, a;$$
$$b, c;$$
$$c, a;$$
$$c, b.$$

There are three ways to choose the first element of the arrangement. There are two ways to choose the second element of the arrangement, because it must be different from the first element. By the product rule, it follows that $P(3, 2) = 3 \cdot 2 = 6$.

Permutations

<u>Theorem 1</u>

If n is a positive integer and r is an integer with $1 \leq r \leq n$ then there are

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1)$$

r-permutations of a set with n distinct elements.

<u>Proof:</u>

We will use the product rule to prove that this formula is correct.

Permutations

Theorem 1

If n is a positive integer and r is an integer with $1 \leq r \leq n$ then there are

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$$

r-permutations of a set with n distinct elements.

Proof:

The **first** element of the permutation can be chosen in $\boldsymbol{n}$ ways because there are n elements in the set.

There are $\boldsymbol{n-1}$ ways to choose the **second** element of the permutation, because there are $n-1$ elements left in the set after using the element picked for the first position.

Permutations

<u>Theorem 1</u>

If n is a positive integer and r is an integer with $1 \leq r \leq n$ then there are

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$$

r-permutations of a set with n distinct elements.

<u>Proof:</u>

Similarly, there are $\boldsymbol{n-2}$ ways to choose the **third** element, and so on, until there are exactly $\boldsymbol{n-(r-1)=n-r+1}$ ways to choose the $\boldsymbol{r}$th element.

Consequently, by the product rule, there are

$$n(n-1)(n-2) \cdots (n-r+1)$$

r-permutations of the set. ■

Permutations

<u>Corollary 1</u>

If n and r are integers with $0 \leq r \leq n$ then

$$P(n,r) = \frac{n!}{(n-r)!}$$

<u>Proof:</u>

When n and r are integers with $1 \leq r \leq n$, by Theorem 1 we have

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}.$$

Because

$$\frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$$

whenever n is a nonnegative integer, we see that the formula $P(n,r) = \frac{n!}{(n-r)!}$ also holds when $r = 0$.■

Permutations

<u>Example 5</u>

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

<u>Solution:</u> Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is

$$\begin{aligned} P(100, 3) &= 100 \cdot 99 \cdot 98 \\ &= 970\,200. \end{aligned}$$

Permutations

<u>Example 6</u>

How many permutations of the letters

$$ABCDEFGH$$

contain the string ABC?

<u>Solution:</u>

Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters D, E, F, G, and H.

Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters $ABCDEFGH$ in which ABC occurs as a block.

Combinations

<u>Example 7</u>

How many different committees of three students can be formed from a group of four students?

<u>Solution:</u>

To answer this question, we need only find the number of subsets with three elements from the set containing the four students.

We see that there are four such subsets, one for each of the four students, because choosing three students is the same as choosing one of the four students to leave out of the group.

This means that there are four ways to choose the three students for the committee, where the order in which these students are chosen does not matter.

Combinations

<u>Definition 2</u>

An **r-combination** of elements of a set is an unordered selection of r elements from the set.

Thus, an **r-combination** is simply a subset of the set with r elements.

Combinations

<u>Example 8</u>

Let be the set $S = \{1, 2, 3, 4\}$.

Then $\{4, 1, 3\}$ is a 3-combination from S.

Note that $\{4, 1, 3\}$ is the same 3-combination as $\{1, 3, 4\}$, because the order in which the elements of a set are listed does not matter.

Combinations

- The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.
Note that $C(n, r)$ is also denoted by $\binom{n}{r}$ and is called a **binomial coefficient**.

Combinations

Example 9

We see that $C(4,2) = 6$, because the 2-combinations of $\{a, b, c, d\}$ are the six subsets

$$\{a, b\},$$
$$\{a, c\},$$
$$\{a, d\},$$
$$\{b, c\},$$
$$\{b, d\},$$
$$\{c, d\}.$$

Combinations

<u>Theorem 2</u>

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n, r) = \frac{n!}{(n-r)!\, r!}$$

<u>Proof:</u>

The $P(n, r)$ r-permutations of the set can be obtained by forming the $C(n, r)$ r-combinations of the set, and then ordering the elements in each r-combination, which can be done in $P(r, r)$ ways.

Consequently, by the product rule,

$$P(n, r) = C(n, r)\, P(r, r)$$

Combinations

Theorem 2

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Proof:

$$P(n,r) = C(n,r)\,P(r,r)$$

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!\,r!} \blacksquare$$

Corollary 1

$$C(n,r) = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$$

Combinations

<u>Example 10</u>

a) How many poker hands of five cards can be dealt from a standard deck of 52 cards?

b) Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

<u>Solution:</u>

a) $C(52, 5) = \frac{52!}{5!47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2\,598\,960$

b) $C(52, 47) = \frac{52!}{47!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2\,598\,960$

Note that $C(52, 5) = C(52, 47)$.

Combinations

Theorem 2

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Corollary 2

Let n and r be nonnegative integers with $r \leq n$. Then

$$C(n,r) = C(n,n-r)$$

Proof:

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

$$= \frac{n!}{(n-r)!\,\big(n-(n-r)\big)!} = C(n,n-r) \;\blacksquare$$

Combinations

Example 11

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution:

$C(9,3) \cdot C(11,4) =$

$\frac{9!}{3!\,6!} \cdot \frac{11!}{4!\,7!} =$

$84 \cdot 330 =$

$27\,720$

The binomial theorem

- The binomial theorem gives the coefficients of the expansion of powers of binomial expressions.
A binomial expression is simply the sum of two terms, such as $x + y$.
(The terms can be products of constants and variables, but that does not concern us here.)

The binomial theorem

- Example 12

$$(x+y)^3 = (x+y)(x+y)(x+y)$$
$$= \boldsymbol{xxx} + \boldsymbol{xxy} + \boldsymbol{xyx} + \boldsymbol{xyy} + \boldsymbol{yxx} + \boldsymbol{yxy} + \boldsymbol{yyx} + \boldsymbol{yyy}$$

When $(x+y)^3 = (x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, and a term in the third sum are added.

Terms of the form $\boldsymbol{x^3}$, $\boldsymbol{x^2y}$, $\boldsymbol{xy^2}$, $\boldsymbol{y^3}$ arise.

$$(x+y)^3 = (x+y)(x+y)(x+y)$$
$$= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

To obtain a term of the form x^3, an x must be chosen in each of the sums $(x+y)$, and this can be done in only one way. Thus, the x^3 term in the product has a coefficient of 1: $\binom{3}{3} = 1$.

To obtain a term of the form x^2y, an x must be chosen in two of the three sums $(x+y)$ (and consequently a y in the other sum). Hence, the number of such terms is the number of 2-combinations of three objects, namely, $\binom{3}{2} = 3$.

Similarly, the number of terms of the form xy^2 is the number of ways to pick one of the three sums $(x+y)$ to obtain an x (and consequently take a y from each of the other two sums). This can be done in $\binom{3}{1} = 3$ ways.

Finally, the only way to obtain a y^3 term is to choose the y for each of the three sums $(x+y)$ in the product, and this can be done in exactly one way: $\binom{3}{0} = 1$.

The binomial theorem

Example 12

Consequently, it follows that

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

$$= xxx + xxy + xyx + xyy + yxx + yxy$$

$$+ yyx + yyy$$

$$= x^3 + 3\,x^2y + 3\,xy^2 + y^3$$

The binomial theorem

Theorem 3

Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Proof:

The terms in the product $(x+y)^n$ when it is expanded are of the form $x^{n-j}y^j$, $j = 0, 1, 2, \dots, n$.

To count the number of terms of the form $x^{n-j}y^j$, note that to obtain such a term it is necessary to choose $n - j$ x's from the n sums (so that the other j terms in the product are y's).

Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$, which is equal to $\binom{n}{j}$ ■

The binomial theorem

Example 13

What is the expansion of $(x+y)^4$?

Solution:

From the binomial theorem it follows that

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4.$$

The binomial theorem

<u>Example 14</u>

What is the coefficient of $x^{12}y^{13}$ in the expansion $(x+y)^{25}$?

<u>Solution:</u>

From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} =$$

$$= 5\,200\,300.$$

The binomial theorem

Example 15

Determine the coefficient of $x^{12}y^{13}$ in the expansion $(2x-3y)^{25}$.

Solution:

The coefficient in the expansion $(2x-3y)^{25}$ is

$$\binom{25}{13}2^{12}(-3)^{13} = -\frac{25!}{13!\,12!}2^{12}3^{13}$$

The binomial theorem

Theorem 3

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Corollary 1

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof:

Using the binomial theorem with $x = 1$ and $y = 1$, we see that

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} \blacksquare$$

The binomial theorem

- There is also a nice combinatorial proof of Corollary 1, which we now present.
A set with n elements has a total of 2^n different subsets.
Each subset has zero elements, one element, two elements, ..., or n elements in it.
There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ subsets with one element, $\binom{n}{2}$ subsets with two elements, ..., and $\binom{n}{n}$ subsets with n elements.
Therefore, $\sum_{k=0}^{n}\binom{n}{k}$ counts the total number of subsets of a set with n elements.
By equating the two formulas we have for the number of subsets of a set with n elements, we see that
$\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

The binomial theorem

Theorem 3

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Corollary 2

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof:

Using the binomial theorem with $x = 1$ and $y = -1$, we see that

$$0 = \left(1 + (-1)\right)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \blacksquare$$

The binomial theorem

Example 16

Let n be a nonnegative integer.

Proof that

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n .$$