Factorising Quadratics

Last modified: 30th September 2015

Factorising Overview

Factorising means :

To turn an expression into a **product** of factors.

Year 8 Factorisation

So what factors can we see here?

$$2x^2 + 4xz \xrightarrow{\text{Factorise}} 2x(x + 2z)$$

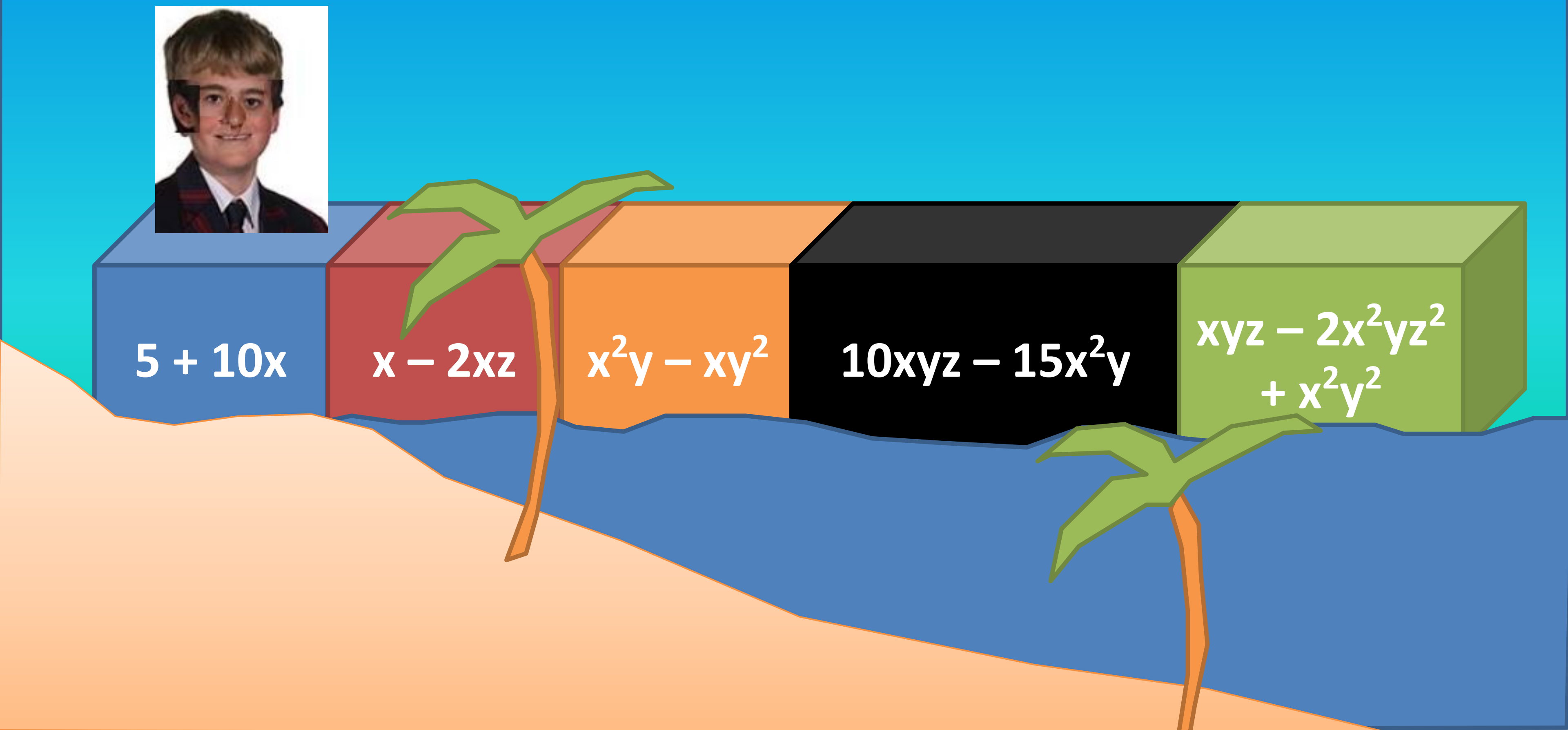
GCSE Factorisation

$$x^2 + 3x + 2 \xrightarrow{\text{Factorise}} (x + 1)(x + 2)$$

A Level Factorisation

$$2x^3 + 3x^2 - 11x - 6 \xrightarrow{\text{Factorise}} (2x + 1)(x - 2)(x + 3)$$

Starter



Exercise 1

1. $2x - 4 =$?
2. $xy + y =$?
3. $qr - 2q =$?
4. $6x - 3y =$?
5. $xyz + yz =$?
6. $x^2y + 2yz =$?
7. $x^3y + xy^2 =$?
8. $5qr + 10r =$?
9. $12pw^2 - 8w^2y =$?
10. $55p^3 + 33p^2 =$?
11. $6p^4 + 8p^3 + 10p^2 =$?
12. $10x^3y^2 + 5x^2y^3 + 15x^2y^2 =$?
13. $x^{100}y^4 + x^{66}y^{70} =$?

Note: We tend to factorise any fraction out, e.g. $\frac{1}{2}x^2 + \frac{1}{4}x = \frac{1}{4}x(2x+1)$

14. $\frac{1}{3}x^2y + \frac{2}{3}xy^2 =$?
15. $\frac{1}{5}x + \frac{1}{10} =$?
16. $\frac{2}{3}ab^2 + \frac{1}{4}ab =$?

☠ $\frac{1}{100!}x^3y^3 + \frac{1}{101!}x^2y^4 =$?

Six different types of factorisation

1. Factoring out a single term

$2x^2 + 4x =$?

2. $x^2 + bx + c$

$x^2 + 4x - 5 =$?

3. Difference of two squares

$4x^2 - 1 =$?

4. $ax^2 + bx + c$

$2x^2 + x - 3 =$?

Strategy: either **split the middle term**, or 'go commando'.

5. Pairwise

$x^3 + 2x^2 - x - 2$

?

6. Intelligent Guesswork

$x^2 + y^2 + 2xy + x + y$

$=$?

TYPE 2: $x^2 + bx + c$

Expand:

$(x+a)(x+b) =$?

a and b add to give 3.

a and b times to give 2.

How does this suggest we can factorise say $x^2 + 3x + 2$?

$x^2 + 3x + 2 =$?

$x^2 - x - 30 =$?

Bro Tip: Think of the factor pairs of 30. You want a pair where the sum or difference of the two numbers is the middle number (-1).

TYPE 2: $x^2 + bx + c$

A few more examples:

$$x^2 + 6x + 5 = \boxed{?}$$

$$x^2 - 12x + 35 = \boxed{?}$$

$$x^2 + 5x - 14 = \boxed{?}$$

$$x^2 + 6x + 9 = \boxed{?}$$

$$x^2 - 6x + 9 = \boxed{?}$$

Exercise 2

1. $x^2 + 4x + 3 =$?
2. $x^2 - 8x + 7 =$?
3. $x^2 + 2x - 8 =$?

π $x^2 + 16x - 36 =$?

4. $y^2 - y - 56 =$?
5. $z^2 + 3z - 54 =$?
6. $z^2 - 3z - 54 =$?
7. $z^2 + 15z + 54 =$?
8. $x^2 + 4x + 4 =$?
9. $x^2 - 14x + 49 =$?
10. $x^2 + 10x - 39 =$?
11. $x^2 + 4x + 4 =$?
12. $x^2 - 17x + 66 =$?
13. $a^2 - 2a - 63 =$?
14. $y^2 - 10y + 25 =$?

Hardcore

☠ $x^4 + 5x^2 + 4 =$?

☠ $x^2 - 2ax + a^2 =$?

☠ $x^4 - 6abx^2 + 9a^2b^2 =$?

☠ $x^9 - x^8 - 2x^7 =$?

☠ $x^{11} + 2x^9 + x^7 =$?

Six different types of factorisation

1. Factoring out a single term

$$2x^2 + 4x = 2x(x+2)$$

2. $x^2 + bx + c$

$$x^2 + 4x - 5 = (x+5)(x-1)$$

3. Difference of two squares

$4x^2 - 1 =$?

4. $ax^2 + bx + c$

$2x^2 + x - 3 =$?

Strategy: either **split the middle term**, or 'go commando'.

5. Pairwise

$x^3 + 2x^2 - x - 2$

$=$?

$=$

6. Intelligent Guesswork

$x^2 + y^2 + 2xy + x + y$

$=$?

TYPE 3: Difference of two squares

Firstly, what is the square root of:

$\sqrt{4x^2} =$?

$\sqrt{25y^2} =$?

$\sqrt{16x^2y^2} =$?

$\sqrt{x^4y^4} =$?

$\sqrt{9(z-6)^2} =$?

TYPE 3: Difference of two squares

$2x \quad 3$

$4x^2 - 9$

$= (\quad + \quad)(\quad - \quad)$

Click to Start Bromanimation

Quickfire Examples

$1 - x^2 =$?

$y^2 - 16 =$?

$x^2y^2 - 9a^2 =$?

$1 - x^4 =$?

$4x^2 - 9y^2 =$?

$x^2 - 3 =$?

Test Your Understanding (Working in Pairs)

Bro Tip: Sometimes you can use one type of factorisation followed by another. Perhaps common term first?

$$x^3 - x = \boxed{?}$$

$$(x+1)^2 - (x-1)^2 = \boxed{?}$$

$$49 - (1-x)^2 = \boxed{?}$$

$$51^2 - 49^2 = \boxed{?}$$

$$18x^2 - 50y^2 = \boxed{?}$$

$$(2t+1)^2 - 9(t-6)^2 = \boxed{?}$$

Exercise 3

1. $4p^2 - 1 =$?
2. $4 - x^2 =$?
3. $144 - b^2 =$?
4. $(x+1)^2 - 25 =$?
5. $7.64^2 - 2.36^2 =$?
6. $2p^2 - 32 =$?
7. $3y^2 - 75x^2 =$?
8. $4a^2 - 64b^2 =$?
9. $9(p+1)^2 - 4p^2$ $=$?
10. $50(2x+1)^2 - 18(1-x)^2$ $=$?

☠ 1: $x^4 - 1 =$?

☠ 2: $32x^8 - 162$ $=$?

☠ 3: [IMO] What is the highest power of 2 that is a factor of $127^2 - 1$?

?

☠ 4: Find four prime numbers less than 100 which are factors of $3^{32} - 2^{32}$

(Hint: you can keep factorising!)

?

TYPE 4: $ax^2 + bx + c$

$$2x^2 + x - 3$$

Factorise using:

a. 'Intelligent Guessing'*

Essentially 'intelligent guessing' of the two brackets, by considering what your guess would expand to.

How would we get the $2x^2$ term in the expansion?

How could we get the -3?

b. Splitting the middle term

$$2x^2 + x - 3 \quad \oplus\ 1 \quad \otimes\ -6$$

'Split the middle term'

Unlike before, we want two numbers which multiply to give the **first times the last number**.

$$2x^2 + 3x - 2x - 3$$

Factorise first and second half separately.

$$= x(2x+3) - 1(2x+3)$$

$$= (2x+3)(x-1)$$

There's a common factor of $(2x + 1)$

* Not official mathematical terminology.

More Examples

$$2x^2 + 11x + 12 =$$?

$$6x^2 - 7x - 3 =$$?

$$2x^2 - 5xy + 3y^2 =$$?

$$6x^2 - 3x - 3 =$$?

Exercise 4

1. $2x^2 + 3x + 1 =$?
2. $3x^2 + 8x + 4 =$?
3. $2x^2 - 3x - 9 =$?
4. $4x^2 - 9x + 2 =$?
5. $2x^2 + x - 15 =$?
6. $2x^2 - 3x - 2 =$?
7. $3x^2 + 4x - 4 =$?
8. $6x^2 - 13x + 6 =$?
9. $15y^2 - 13y - 20 =$?
10. $12x^2 - x - 1 =$?
11. $25y^2 - 20y + 4 =$?

☠ 1: $4x^3 + 12x^2 + 9x =$?

☠ 2: $a^2x^2 - 2abx + b^2 =$?

'Commando' starts to become difficult from this question onwards because the coefficient of x^2 is not prime.

RECAP :: Six different types of factorisation

1. Factoring out a single term

$x^2 - 4x =$?

✔

2. $x^2 + bx + c$

$x^2 + 7x - 30 =$?

✔

3. Difference of two squares

$9 - 16y^2 =$?

✔

4. $ax^2 + bx + c$

$2x^2 + x - 3 = (2x+3)(x-1)$

Strategy: either **split the middle term**, or 'go commando'. ✔

5. Pairwise

$$x^3 + 2x^2 - x - 2$$
$$= x^2(x+2) - 1(x+2)$$
$$= (x^2 - 1)(x+2) = \cdots$$

6. Intelligent Guesswork

$$x^2 + y^2 + 2xy + x + y$$
$$= (x + y + 1)(x + y)$$

RECAP :: $ax^2 + bx + c$

Method A: Guessing the brackets

$$3x^2 + 10x - 8$$

?

Method B: Splitting the middle term

$$3x^2 + 10x - 8$$

?

Both of these methods can be extended to more general expressions.

This method of 'go commando' can be extended to non-quadratics.

After we split the middle term, we **looked at the expression in two pairs and factorised**.
I call more general usage of this 'pairwise factorisation'.

TYPE 5: Intelligent Guessing

Just think what brackets would expand to give you expression. Look at each term one by one.

$$x^2 + ax + bx + ab$$

$$= (x + a)(x + b)$$

It works!

This factorisation will become particularly important when we cover something called 'Diophantine Equations'.

$$ab - a + b - 1$$

$$= ?$$

Test Your Understanding

1 $xy + 3x - 2y - 6 =$?

2 $ac - bc - b + a =$?

3 $a^2b - a + ab - 1 =$?

☠ $x^2 + y^2 + 2xy + xz + yz$
$= $?

Bro Tip: The $2xy$ arose because of collecting like terms in the expansion. It might therefore be easier to first think how we get the 'easier' terms like the y^2, xz, yz (where the coefficient of the term is 1) when we try to fill in the brackets.

Bro Tip: Notice that there's an 'algebraic symmetry' in x and y, as x and y could be swapped without changing the expression. But there's an asymmetry in z.
This gives hints about the factorisation, as the same symmetry must be seen.

TYPE 6: Pairwise Factorisation

We saw earlier with splitting the middle term that we can factorise different parts of the expression separately and hope that a common term emerges.

$$x^2 + ax + bx + ab =$$

$= ?$

$= ?$

$$x^3 - 2x^2 - x + 2$$

$= ?$

$= ?$

$= ?$

$$x^2 - y^2 + 4x + 4y$$

$= ?$

$= ?$

Test Your Understanding

1

$x^2 - xy + 2x - 2y$

$=$?

$=$

2

$ab + a + b + 1$

$=$?

$=$

3

$x^3 - 3x^2 - 4x + 12$

$=$?

$=$

$=$

Can you split the terms into two blocks, where in each block you can factorise?

☠

$a^2 + b^2 + 2ab + ac + bc$

$=$?

$=$

Challenge Wall!

?

Warning: Pairwise factorisation doesn't always work. You sometimes have to resort to 'intelligent guessing'.

4 $x^2 + y^2 + 2xy - 1 =$?

3 $x^3 + 2x^2 - 9x - 18 =$?

2 $xy^2 + 3y^2 + x + 3 =$?

1 $xy - x - y + 1 =$?

Exercise 5

Factorise the following using either 'pairwise factorisation' or 'intelligent guessing'.

1. $ab - 2a + 4b - 8 =$?
2. $ab + bc - 3a - 3c =$?
3. $ac + ad + bc + bd =$?
4. $y^2 + cy + dy + cd =$?
5. $x^2y - x^2 + 2y - 2 =$?
6. $x^3 + x^2 + x + 1 =$?
7. $a^3 - a^2 + a - 1 =$?
8. $x^3 + x^2 - x - 1 =$?
9. $x^3 + x^2 + xy^2 + y^2 =$?
10. $x^3 - 2x^2 + 3x - 6 =$?
11. $x^2 + 2xy + y^2 =$?
12. $a^2x^2 + 4abx + 4b^2 =$?
13. $x^2y^2 - x^2 - y^2 + 1 =$?
14. $a^2 - b^2 - 3a + 3b =$?

☠ 1 $x^4 + y^4 - 2x^2y^2 =$?

☠ 2 $x^2 - y^2 + 2x + 1 =$?

☠ 3 $xyz - xy - xz + yz + x - y - z + 1 =$?

☠ 4 $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca =$?

☠ 5 $x^4 + 6x^3 + 11x^2 + 6x + 1 =$?

Summary

For the following expressions, identify which of the following factorisation techniques that we use, out of: (it may be multiple!)

1	Factorising out single term: $x^2 + 2xy \rightarrow x(x + 2y)$
2	Simple quadratic factorisation: $x^2 + 3x + 2 \rightarrow (x + 2)(x + 1)$
3	Difference Of Two Squares: $9x^2 - 4y^2 = (3x + 2y)(3x - 2y)$
4	Commando/Splitting Middle Term: $2x^2 - 3x + 2 \rightarrow (2x + 1)(x - 2)$
5	Pairwise: $\underline{x^4 - x^3} + \underline{x - 1} \rightarrow (x^3 + 1)(x - 1)$
6	Intelligent Guesswork: $xy + x + y + 1 \rightarrow (x + 1)(y + 1)$

$2x^2y + 4xy^2$	?
$1 - x^2$	?
$x^3 - x$	?
$x^2 - x - 2$	?
$3y^2 - 10y - 8$	?
$x^4 + 2x^2 + 1$	?
$y^3 - y^2 - y + 1$	?
$x^4 - 2x^3 - 8x^2$	?
$xy^2 - x - y^2 + 1$	?
$27 - 3x^2$	?

Factorising out an expression

It's fine to factorise out an entire expression:

$$x(x+2) - 3(x+2) \quad \rightarrow \quad ?$$

$$x(x+1)^2 + 2(x+1)$$

$$\rightarrow \quad ?$$

$$a(2c+1) + b(2c+1) \quad \rightarrow \quad ?$$

$$2(2x-3)^2 + x(2x-3)$$

?