

Set Theory

Rosen 6th ed., §2.1-2.2

Introduction to Set Theory

- A *set* is a structure, representing an *unordered* collection (group, plurality) of zero or more *distinct* (different) objects.
- Set theory deals with operations between, relations among, and statements about sets.

Basic notations for sets

- For sets, we'll use variables S, T, U, \dots
- We can denote a set S in writing by listing all of its elements in curly braces:
 - $\{a, b, c\}$ is the set of whatever 3 objects are denoted by a, b, c .
- *Set builder notation*: For any proposition $P(x)$ over any universe of discourse, $\{x|P(x)\}$ is *the set of all x such that $P(x)$* .
 - e.g., $\{x \mid x \text{ is an integer where } x > 0 \text{ and } x < 5 \}$

Basic properties of sets

- Sets are inherently unordered:
 - No matter what objects a, b, and c denote,
 $\{a, b, c\} = \{a, c, b\} = \{b, a, c\} =$
 $\{b, c, a\} = \{c, a, b\} = \{c, b, a\}.$
- All elements are distinct (unequal); multiple listings make no difference!
 - $\{a, b, c\} = \{a, a, b, a, b, c, c, c, c\}.$
 - This set contains at most 3 elements!

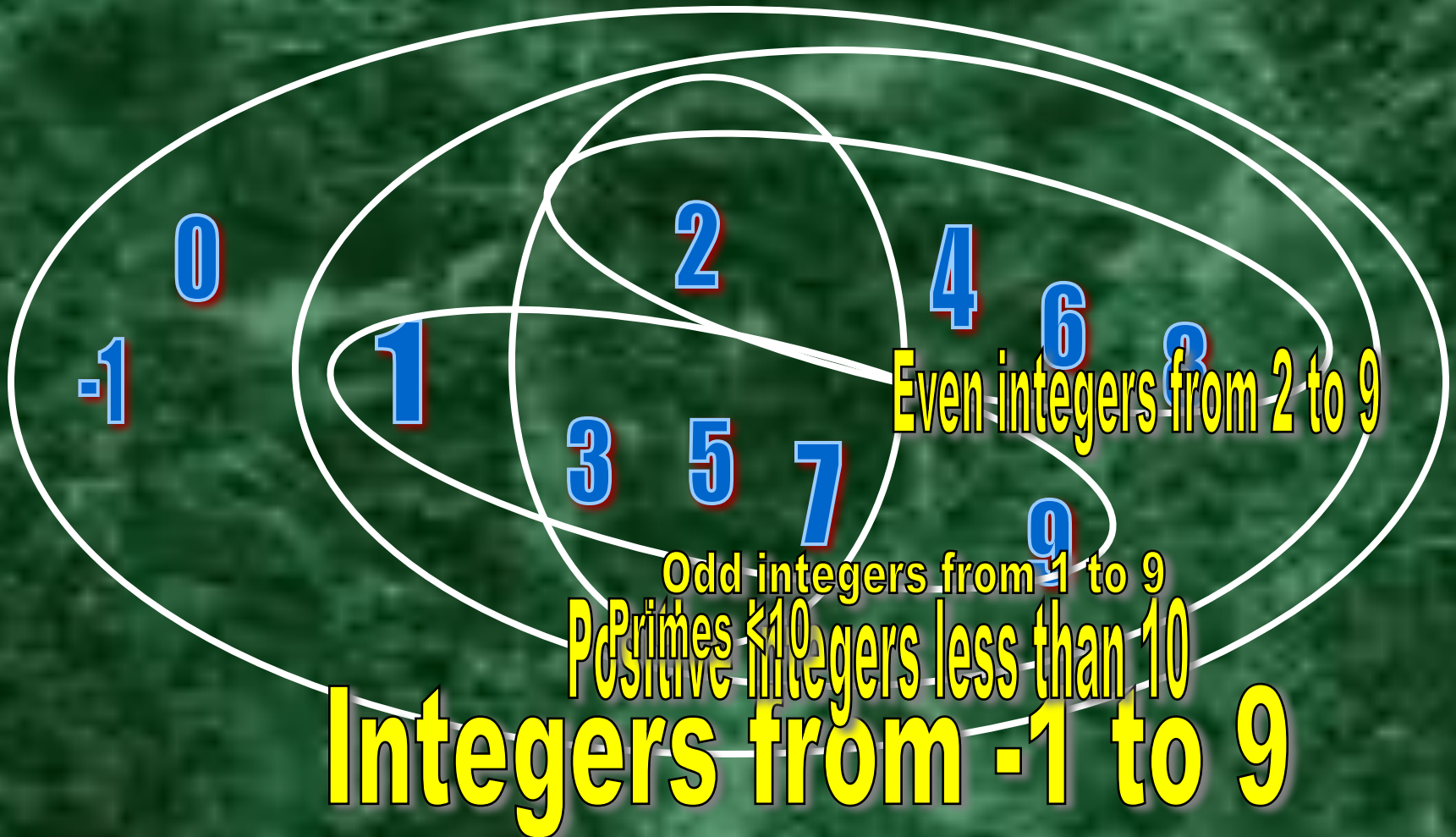
Definition of Set Equality

- Two sets are declared to be equal *if and only if* they contain exactly the same elements.
- In particular, it does not matter *how the set is defined or denoted*.
- For example: The set $\{1, 2, 3, 4\} =$
 $\{x \mid x \text{ is an integer where } x > 0 \text{ and } x < 5\} =$
 $\{x \mid x \text{ is a positive integer whose square}$
 $\text{is } > 0 \text{ and } < 25\}$

Infinite Sets

- Conceptually, sets may be *infinite* (i.e., not *finite*, without end, unending).
- Symbols for some special infinite sets:
 $\mathbf{N} = \{0, 1, 2, \dots\}$ The **n**atural numbers.
 $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ The integers.
 $\mathbf{R} =$ The “**r**real” numbers, such as
374.1828471929498181917281943125...
- Infinite sets come in different sizes!

Venn Diagrams



Basic Set Relations: Member of

- $x \in S$ (“ x is in S ”) is the proposition that object x is an *element* or *member* of set S .
 - e.g. $3 \in \mathbf{N}$, “a” $\in \{x \mid x \text{ is a letter of the alphabet}\}$
- Can define set equality in terms of \in relation:
$$\forall S, T: S=T \leftrightarrow (\forall x: x \in S \leftrightarrow x \in T)$$
“Two sets are equal **iff** they have all the same members.”
- $x \notin S := \neg(x \in S)$ “ x is not in S ”

The Empty Set

- \emptyset (“null”, “the empty set”) is the unique set that contains no elements whatsoever.
- $\emptyset = \{\} = \{x | \mathbf{False}\}$
- No matter the domain of discourse, we have the axiom

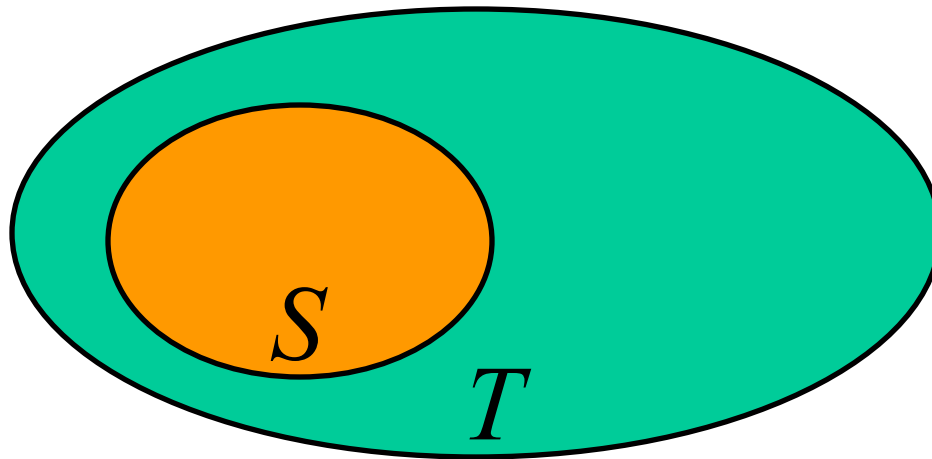
$$\neg \exists x: x \in \emptyset.$$

Subset and Superset Relations

- $S \subseteq T$ (“ S is a subset of T ”) means that every element of S is also an element of T .
- $S \subseteq T \Leftrightarrow \forall x (x \in S \rightarrow x \in T)$
- $\emptyset \subseteq S, S \subseteq S$.
- $S \supseteq T$ (“ S is a superset of T ”) means $T \subseteq S$.
- Note $S = T \Leftrightarrow S \subseteq T \wedge S \supseteq T$.
- $S \not\subseteq T$ means $\neg(S \subseteq T)$, *i.e.* $\exists x(x \in S \wedge x \notin T)$

Proper (Strict) Subsets & Supersets

- $S \subset T$ (“ S is a proper subset of T ”) means that $S \subseteq T$ but $T \not\subseteq S$. Similar for $S \supset T$.



Example:

$$\{1,2\} \subset \{1,2,3\}$$

Venn Diagram equivalent of $S \subset T$

Sets Are Objects, Too!

- The objects that are elements of a set may *themselves* be sets.
- *E.g.* let $S = \{x \mid x \subseteq \{1,2,3\}\}$
then $S = \{\emptyset,$
 $\{1\}, \{2\}, \{3\},$
 $\{1,2\}, \{1,3\}, \{2,3\},$
 $\{1,2,3\}\}$
- Note that $1 \neq \{1\} \neq \{\{1\}\} !!!!$

 Very Important!

Cardinality and Finiteness

- $|S|$ (read “the *cardinality* of S ”) is a measure of how many different elements S has.
- *E.g.*, $|\emptyset|=0$, $|\{1,2,3\}|=3$, $|\{a,b\}|=2$,
 $|\{\{1,2,3\},\{4,5\}\}|= \underline{2}$
- We say S is *infinite* if it is not *finite*.
- What are some infinite sets we’ve seen?

N Z R

The *Power Set* Operation

- The *power set* $P(S)$ of a set S is the set of all subsets of S . $P(S) = \{x \mid x \subseteq S\}$.
- *E.g.* $P(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.
- Sometimes $P(S)$ is written 2^S .
Note that for finite S , $|P(S)| = 2^{|S|}$.
- It turns out that $|P(\mathbf{N})| > |\mathbf{N}|$.
There are different sizes of infinite sets!

Cartesian Products of Sets

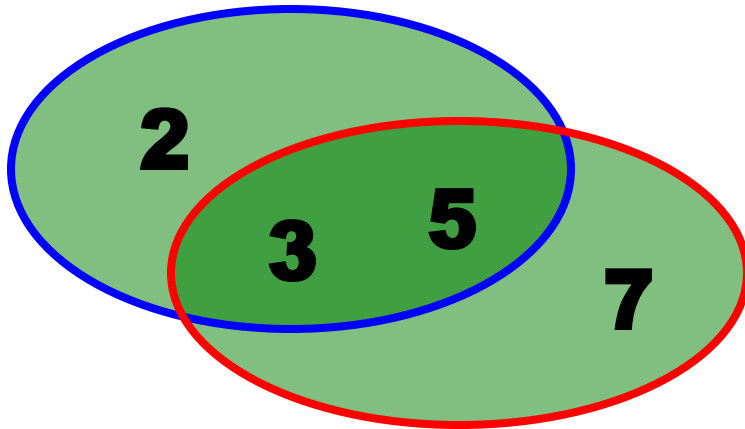
- For sets A, B , their *Cartesian product* $A \times B := \{(a, b) \mid a \in A \wedge b \in B\}$.
- *E.g.* $\{a, b\} \times \{1, 2\} = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$
- Note that for finite A, B , $|A \times B| = |A| |B|$.
- Note that the Cartesian product is **not** commutative: $\neg \forall AB: A \times B = B \times A$.
- Extends to $A_1 \times A_2 \times \dots \times A_n \dots$

The Union Operator

- For sets A , B , their *union* $A \cup B$ is the set containing all elements that are either in A , **or** (“ \vee ”) in B (or, of course, in both).
- Formally, $\forall A, B: A \cup B = \{x \mid x \in A \vee x \in B\}$.
- Note that $A \cup B$ contains all the elements of A **and** it contains all the elements of B :
 $\forall A, B: (A \cup B \supseteq A) \wedge (A \cup B \supseteq B)$

Union Examples

- $\{a,b,c\} \cup \{2,3\} = \{a,b,c,2,3\}$ **Required Form**
- $\{2,3,5\} \cup \{3,5,7\} = \{2,3,5,3,5,7\} = \{2,3,5,7\}$

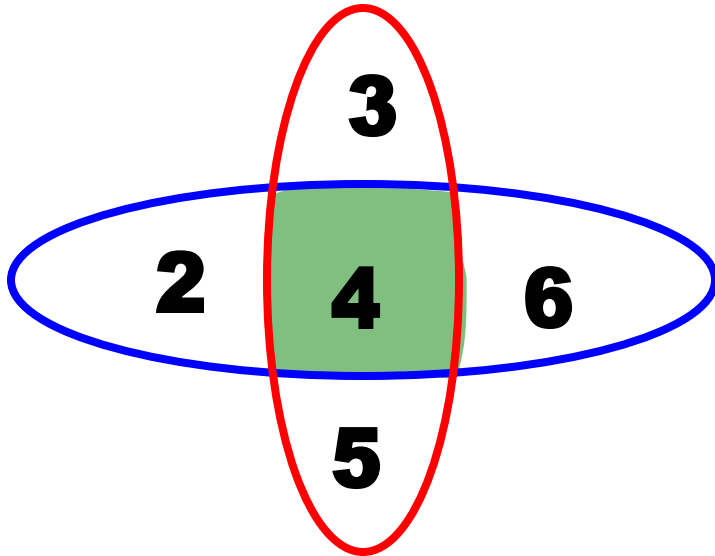


The Intersection Operator

- For sets A, B , their *intersection* $A \cap B$ is the set containing all elements that are simultaneously in A **and** (“ \wedge ”) in B .
- Formally, $\forall A, B: A \cap B \equiv \{x \mid x \in A \wedge x \in B\}$.
- Note that $A \cap B$ is a subset of A **and** it is a subset of B :
$$\forall A, B: (A \cap B \subseteq A) \wedge (A \cap B \subseteq B)$$

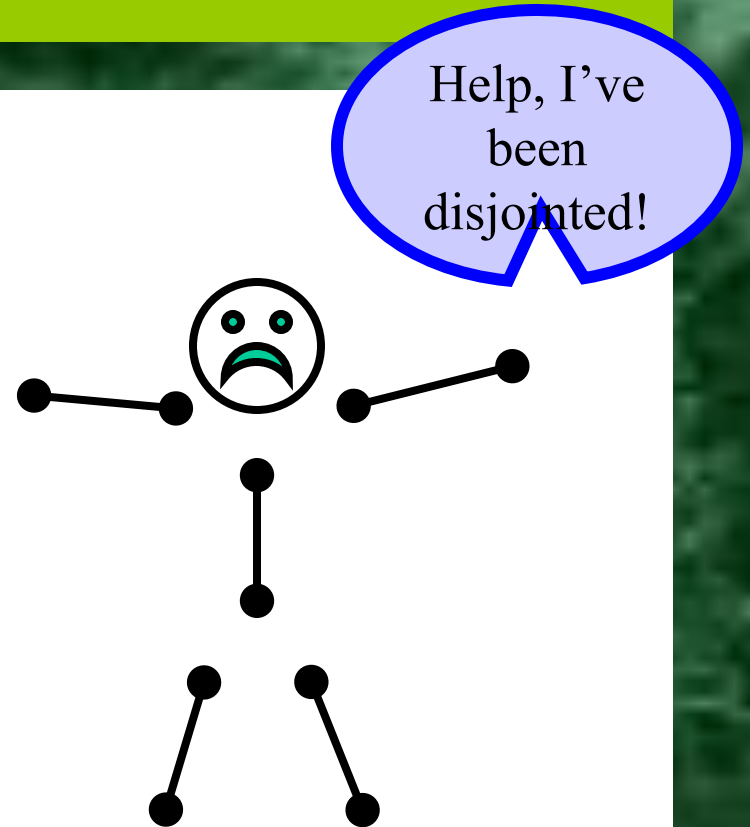
Intersection Examples

- $\{a,b,c\} \cap \{2,3\} = \underline{\emptyset}$
- $\{2,4,6\} \cap \{3,4,5\} = \underline{\{4\}}$



Disjointedness

- Two sets A , B are called *disjoint* (i.e., unjoined) iff their intersection is empty. ($A \cap B = \emptyset$)
- Example: the set of even integers is disjoint with the set of odd integers.



Inclusion-Exclusion Principle

- How many elements are in $A \cup B$?

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Example:

$$\{2, 3, 5\} \cup \{3, 5, 7\} = \{2, 3, 5, 3, 5, 7\} = \{2, 3, 5, 7\}$$

Set Difference

- For sets A, B , the *difference of A and B* , written $A - B$, is the set of all elements that are in A but not B .
- $A - B := \{x \mid x \in A \wedge x \notin B\}$
 $= \{x \mid \neg(x \in A \rightarrow x \in B)\}$
- Also called:
The *complement of B with respect to A* .

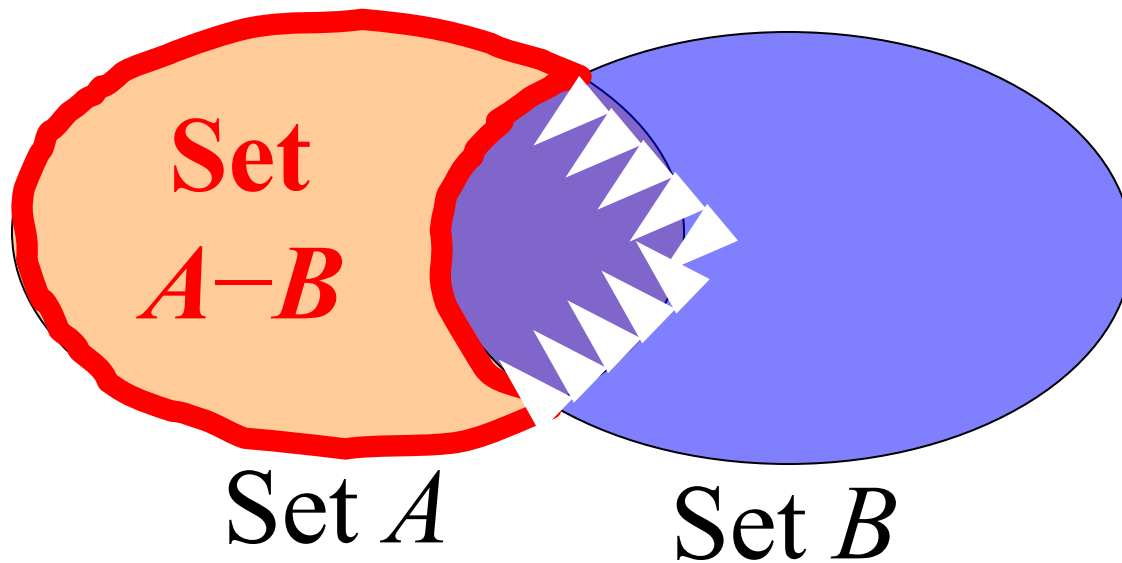
Set Difference Examples

- $\{\cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}\} - \{2, 3, 5, 7, 9, 11\} =$
 $\{1, 4, 6\}$

- $\mathbf{Z} - \mathbf{N} = \{\dots, -1, 0, 1, 2, \dots\} - \{0, 1, \dots\}$
 $= \{x \mid x \text{ is an integer but not a nat. \#}\}$
 $= \{x \mid x \text{ is a negative integer}\}$
 $= \{\dots, -3, -2, -1\}$

Set Difference - Venn Diagram

- $A-B$ is what's left after B
“takes a bite out of A ”



Set Complements

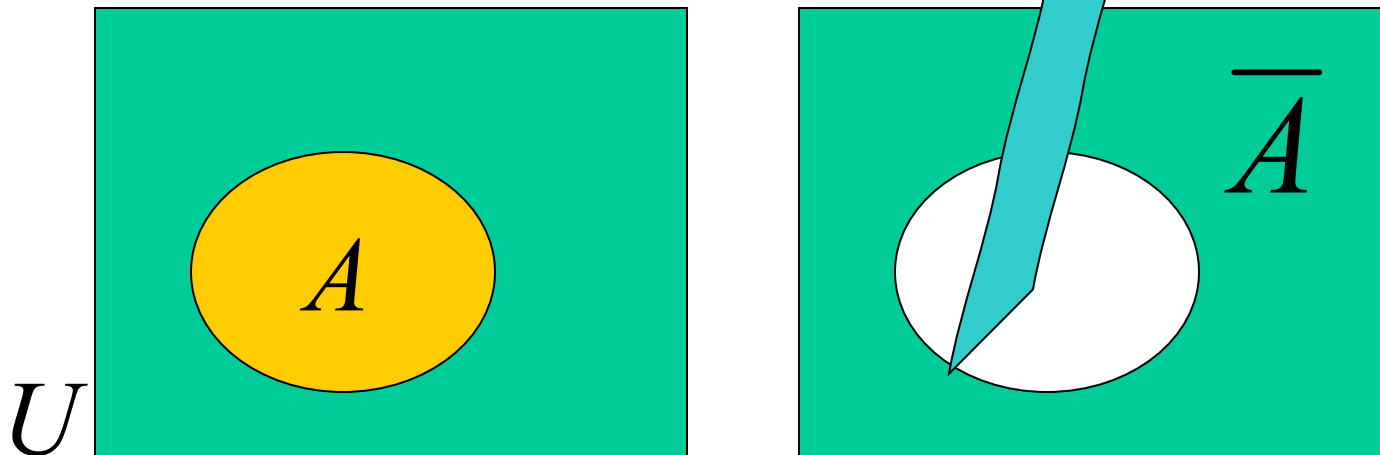
- The *universe of discourse* can itself be considered a set, call it U .
- The *complement* of A , written \overline{A} , is the complement of A w.r.t. U , *i.e.*, it is $U-A$.
- *E.g.*, If $U=\mathbf{N}$,

$$\overline{\{3,5\}} = \{0,1,2,4,6,7,\dots\}$$

More on Set Complements

- An equivalent definition, when U is clear:

$$\overline{A} = \{x \mid x \notin A\}$$



Set Identities

- Identity: $A \cup \emptyset = A$ $A \cap U = A$
- Domination: $A \cup U = U$ $A \cap \emptyset = \emptyset$
- Idempotent: $A \cup A = A = A \cap A$
- Double complement: $\overline{\overline{A}} = A$
- Commutative: $A \cup B = B \cup A$ $A \cap B = B \cap A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$
 $A \cap (B \cap C) = (A \cap B) \cap C$

DeMorgan's Law for Sets

- Exactly analogous to (and derivable from) DeMorgan's Law for propositions.

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Proving Set Identities

To prove statements about sets, of the form $E_1 = E_2$ (where E s are set expressions), here are three useful techniques:

- Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.
- Use logical equivalences.
- Use a *membership table*.

Method 1: Mutual subsets

Example: Show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

- Show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
 - Assume $x \in A \cap (B \cup C)$, & show $x \in (A \cap B) \cup (A \cap C)$.
 - We know that $x \in A$, and either $x \in B$ or $x \in C$.
 - Case 1: $x \in B$. Then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$.
 - Case 2: $x \in C$. Then $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$.
 - Therefore, $x \in (A \cap B) \cup (A \cap C)$.
 - Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
- Show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Method 3: Membership Tables

- Just like truth tables for propositional logic.
- Columns for different set expressions.
- Rows for all combinations of memberships in constituent sets.
- Use “1” to indicate membership in the derived set, “0” for non-membership.
- Prove equivalence with identical columns.

Membership Table Example

Prove $(A \cup B) - B = A - B$.

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Membership Table Exercise

Prove $(A \cup B) - C = (A - C) \cup (B - C)$.

A	B	C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A - C) \cup (B - C)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

Generalized Union

- Binary union operator: $A \cup B$

- n -ary union:

$$A \cup A_2 \cup \dots \cup A_n \equiv (((\dots((A_1 \cup A_2) \cup \dots) \cup A_n))$$

(grouping & order isⁿ irrelevant)

- “Big U” notation: $\bigcup_{i=1}^n A_i$

$$\bigcup A$$

- Or for infinite sets of sets: $\bigcup_{A \in X} A$

Generalized Intersection

- Binary intersection operator: $A \cap B$

- n -ary intersection:

$$A \cap A_2 \cap \dots \cap A_n \equiv (((A_1 \cap A_2) \cap \dots) \cap A_n)$$

(grouping & order is irrelevant)

- “Big Arch” notation: $\bigcap_{i=1}^n A_i$

- Or for infinite sets of sets: $\bigcap_{A \in X} A$