Engineering Mathematics III

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Finite Mathematics: An Applied Approach by Michael Sullivan

1.3 Separable ODEs. Modeling

Many practically useful ODEs can be reduced to the form

(1) $$g(y)\,y' = f(x)$$

by purely algebraic manipulations. Then we can integrate on both sides with respect to x, obtaining

(2) $$\int g(y)\,y'\,dx = \int f(x)\,dx + c.$$

On the left we can switch to y as the variable of integration. By calculus, $y'dx = dy$ so that

(3) $$\int g(y)\,dy = \int f(x)\,dx + c.$$

If f and g are continuous functions, the integrals in (3) exist, and by evaluating them we obtain a general solution of (1).
This method of solving ODEs is called the **method of separating variables**, and (1) is called a **separable equation**, because in (3) the variables are now separated: x appears only on the right and y only on the left.

$$y' = 1 + y^2$$

Separable ODE

The ODE $y' = 1 + y^2$ is separable because it can be written

$$\frac{dy}{1 + y^2} = dx. \qquad \text{By integration,} \qquad \arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c).$$

It is very important to introduce the constant of integration immediately when the integration is performed. If we wrote $\arctan y = x$, then $y = \tan x$, and *then* introduced c, we would have obtained $y = \tan x + c$, which is not a solution (when $c \neq 0$). Verify this. ■

$$y' = (x + 1)e^{-x}y^2$$

Separable ODE

The ODE $y' = (x + 1)e^{-x}y^2$ is separable; we obtain $y^{-2}\,dy = (x + 1)e^{-x}\,dx$.

By integration, $$-y^{-1} = -(x + 2)e^{-x} + c, \qquad y = \frac{1}{(x + 2)e^{-x} - c}.$$

Solve $y' = -2xy$,

Solve $y' = -2xy$, $y(0) = 1.8$.

Solution. By separation and integration,

$$\frac{dy}{y} = -2x\,dx, \qquad \ln y = -x^2 + \tilde{c}, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$. Hence the IVP has the solution $y = 1.8e^{-x^2}$. This is a particular solution, representing a bell-shaped curve (Fig. 10). ■

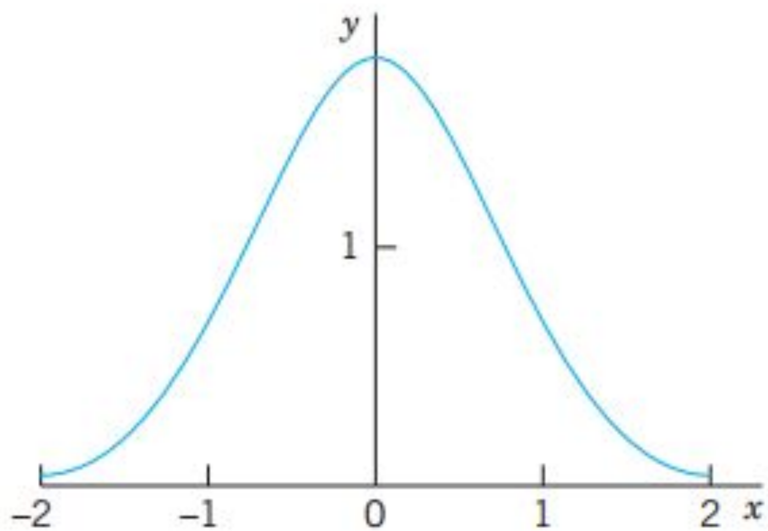

Fig. 10. Solution in Example 3 (bell-shaped curve)

Example

Solve the differential equation

$$9yy' + 4x = 0$$

Example
Solve the differential equation

$$9yy' + 4x = 0$$

On separating the variables

$$9ydy = -4xdx$$

Integrating both sides

$$\frac{9}{2}y^2 = -2x^2 + c \qquad \text{giving} \qquad \frac{x^2}{9} + \frac{y^2}{4} = \frac{c}{18}$$

the solution represents a family of ellipses.

Example – Initial Value problem

Solve the initial problem

$$y' = -\frac{y}{x}, \qquad y(1) = 1$$

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Solve the initial problem

$$y' = -\frac{y}{x}, \qquad y(1) = 1$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\ln|y| = -\ln|x| + \tilde{c} = \ln\frac{1}{|x|} + \tilde{c}$$

$$y = \frac{c}{x}$$

$$1 = \frac{c}{1} \qquad \Rightarrow \qquad c = 1$$

$$y = \frac{1}{x}$$

EXAMPLE 5

Mixing Problem

Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 11 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time t.

EXAMPLE 5 (continued)

Solution.

Step 1. Setting up a model.

Let $y(t)$ denote the amount of salt in the tank at time t. Its time rate of change is

$$y' = \text{Salt inflow rate - Salt outflow rate} \qquad \textbf{Balance law.}$$

5 lb times 10 gal gives an inflow of 50 lb of salt. Now, the outflow is 10 gal of brine.

This is $10/1000 = 0.01(= 1\%)$ of the total brine content in the tank, hence 0.01 of the salt content $y(t)$, that is, $0.01\ y(t)$.

Thus the model is the ODE

$$(4) \qquad y' = 50 - 0.01y = -0.01(y - 5000).$$

EXAMPLE 5 (continued)

Step 2. Solution of the model.

The ODE (4) is separable. Separation, integration, and taking exponents on both sides gives

$$\frac{dy}{y-5000} = -0.01\,dt, \quad \ln|y-5000| = -0.01t + c^*, \quad y - 5000 \quad ce^{-0.01t}.$$

Initially the tank contains 100 lb of salt. Hence $y(0) = 100$ is the initial condition that will give the unique solution.

Substituting $y = 100$ and $t = 0$ in the last equation gives

$$100 - 5000 = ce^{0} = c.$$

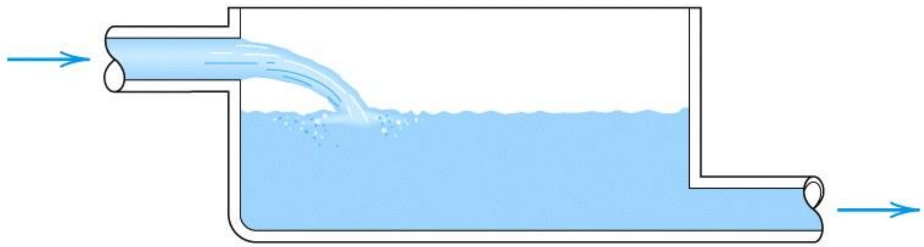
Hence $c = -4900$. Hence the amount of salt in the tank at time t is

(5) $\qquad y(t) = 5000 - 4900e^{-0.01t}$

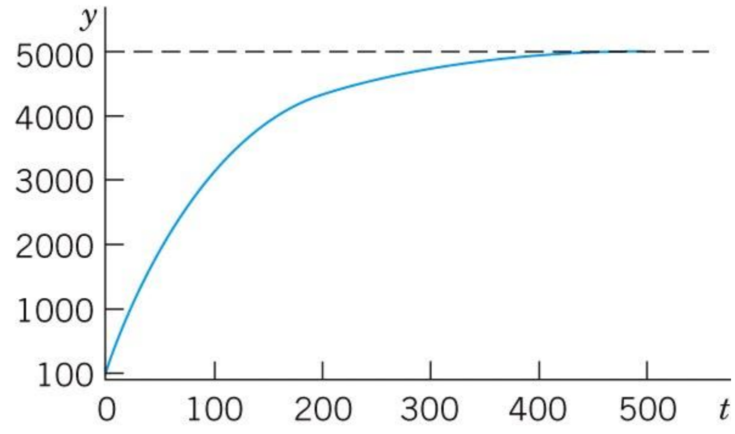
This function shows an exponential approach to the limit 5000 lb; see Fig. 11. Can you explain physically that $y(t)$ should increase with time? That its limit is 5000 lb? Can you see the limit directly from the ODE?

EXAMPLE 5 (continued)

The model discussed becomes more realistic in problems on pollutants in lakes (see Problem Set 1.5, Prob. 35) or drugs in organs. These types of problems are more difficult because the mixing may be imperfect and the flow rates (in and out) may be different and known only very roughly.



Tank



Salt content $y(t)$

Extended Method: Reduction to Separable Form

Certain non separable ODEs can be made separable by transformations that introduce for y a new unknown function. We discuss this technique for a class of ODEs of practical importance, namely, for equations

(8) $$y' = f\left(\frac{y}{x}\right).$$

Here, f is any (differentiable) function of y/x such as sin (y/x), $(y/x)^4$, and so on. (Such an ODE is sometimes called a *homogeneous ODE,* a term we shall not use but reserve for a more important purpose in Sec. 1.5.)

Extended Method: Reduction to Separable Form (continued)

The form of such an ODE suggests that we set $y/x = u$; thus,

(9) $y = ux$ and by product differentiation $y' = u'x + u$.

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

(10)
$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

$$2xyy' = y^2 - x^2.$$

Solve

$$2xyy' = y^2 - x^2.$$

Solution. To get the usual explicit form, divide the given equation by $2xy$,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' from (9) and then simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

You see that in the last equation you can now separate the variables,

$$\frac{2u\,du}{1 + u^2} = -\frac{dx}{x}. \qquad \text{By integration,} \qquad \ln(1 + u^2) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

Take exponents on both sides to get $1 + u^2 = c/x$ or $1 + (y/x)^2 = c/x$. Multiply the last equation by x^2
obtain (Fig. 14)

$$x^2 + y^2 = cx. \qquad \text{Thus} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centers on the x-axis.

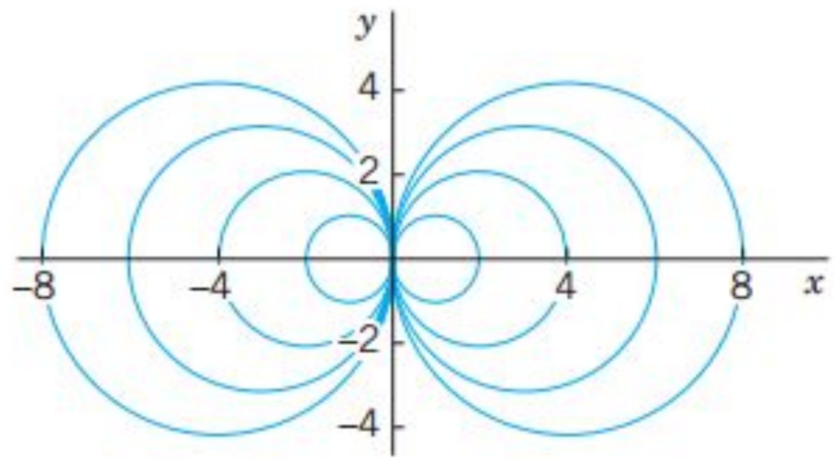


Fig. 14. General solution (family of circles) in Example 8

1.4 Exact ODEs. Integrating Factors

We recall from calculus that if a function $u(x, y)$ has continuous partial derivatives, its **differential** (also called its *total differential*) is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

From this it follows that if $u(x, y) = c = const$, $du = 0$.

A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as (use $dy = y'dx$ as in Sec. 1.3)

(1) $$M(x, y)\, dx + N(x, y)\, dy = 0$$

is called an **exact differential equation** if the **differential form** $M(x, y)\, dx + N(x, y)\, dy$ is **exact**, that is, this form is the differential

(2) $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

of some function $u(x, y)$. Then (1) can be written

$$du = 0.$$

By integration we immediately obtain the general solution of (1) in the form

(3) $$u(x, y) = c.$$

This is called an **implicit solution**, in contrast to a solution $y = h(x)$ as defined in Sec. 1.1, which is also called an *explicit solution,* for distinction. Sometimes an implicit solution can be converted to explicit form. (Do this for $x^2 + y^2 = 1$.) If this is not possible, your CAS may graph a figure of the **contour lines** (3) of the function $u(x, y)$ and help you in understanding the solution.

Comparing (1) and (2), we see that (1) is an exact differential equation if there is some function $u(x, y)$ such that

(4) (a) $\frac{\partial u}{\partial x} = M$ (b) $\frac{\partial u}{\partial y} = N.$

From this we can derive a formula for checking whether (1) is exact or not, as follows.

Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections. Then by partial differentiation of (4) (see App. 3.2 for notation),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\, \partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\, \partial y}.$$

By the assumption of continuity the two second partial derivatives are equal. Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{5}$$

This condition is not only necessary but also sufficient for (1) to be an exact differential equation. (See book for proof)

If (1) is exact, the function $u(x, y)$ can be found by inspection or in the following systematic way. From (4a) we have by integration with respect to x

(6) $$u = \int M \, dx + k(y);$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a "constant" of integration.
To determine $k(y)$, we derive $\partial u/\partial y$ from (6), use (4b) to get dk/dy, and integrate dk/dy to get k.

$$\cos(x + y)\, dx + (3y^2 + 2y + \cos(x + y))\, dy = 0.$$

Example

An Exact ODE

Solve

$$\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0. \tag{7}$$

Solution. ***Step 1. Test for exactness.*** Our equation is of the form (1) with

$$M = \cos(x + y),$$

$$N = 3y^2 + 2y + \cos(x + y).$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$

$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

From this and (5) we see that (7) is exact.

Step 2. Implicit general solution. From (6) we obtain by integration

$$u = \int M\,dx + k(y) = \int \cos(x + y)\,dx + k(y) = \sin(x + y) + k(y). \tag{8}$$

To find $k(y)$, we differentiate this formula with respect to y and use formula (4b), obtaining

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y).$$

Hence $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into (8) and observing (3), we obtain the *answer*

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c.$$

Step 3. Checking an implicit solution. We can check by differentiating the implicit solution $u(x, y) = c$ implicitly and see whether this leads to the given ODE (7):

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \cos(x + y)\, dx + (\cos(x + y) + 3y^2 + 2y)\, dy = 0. \tag{9}$$

This completes the check. ■

Reduction to Exact Form. Integrating Factors

The ODE in Example 3 is $-y\,dx + x\,dy = 0$. It is not exact.

Reduction to Exact Form. Integrating Factors

The ODE in Example 3 is $-y\,dx + x\,dy = 0$. It is not exact. However, if we multiply by $1/x^2$, we get an exact equation [check exactness by (5)!],

$$\frac{-y\,dx + x\,dy}{x^2} = -\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = d\left(\frac{y}{x}\right) = 0. \tag{11}$$

Integration of (11) then gives the general solution $y/x = c = \text{const}$.

The previous examples gives the idea

We multiply a given nonexact equation,

(12) $$P(x, y)\, dx + Q(x, y)\, dy = 0,$$

by a function F that, in general, will be a function of both x and y. The result was an equation

(13) $$FP\, dx + FQ\, dy = 0$$

that is exact, so we can solve it as just discussed. Such a function is then called an **integrating factor** of (12).

How to Find Integrating Factors

For $M\,dx + N\,dy = 0$ the exactness condition (5) is $\partial M/\partial y = \partial N/\partial x$. Hence for (13), $FP\,dx + FQ\,dy = 0$, the exactness condition is

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ). \tag{15}$$

By the product rule, with subscripts denoting partial derivatives, this gives

$$F_y P + F P_y = F_x Q + F Q_x.$$

How to Find Integrating Factors (continued)

Let $F = F(x)$. Then $F_y = 0$, and $F_x = F' = dF/dx$, so that (15) becomes

$$FP_y = F'Q + FQ_x.$$

Dividing by FQ and reshuffling terms, we have

(16) $$\frac{1}{F}\frac{dF}{dx} = R, \qquad \text{where} \qquad R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

Theorem 1

Integrating Factor $F(x)$

If (12) is such that the right side R of (16) depends only on x, then (12) has an integrating factor F = F(x), which is obtained by integrating (16) and taking exponents on both sides.

(17) $$F(x) = \exp \int R(x)dx.$$

Note:

(12) $$P(x, y)\,dx + Q(x, y)\,dy = 0,$$

(16) $$\frac{1}{F}\frac{dF}{dx} = R, \qquad \text{where} \qquad R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

Similarly, if $F^* = F^*(y)$, then instead of (16) we get

$$\text{(18)} \quad \frac{1}{F^*}\frac{dF^*}{dy} = R^*, \qquad \text{where} \qquad R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

Theorem 2

Integrating Factor $F^*(y)$

If (12) is such that the right side R^* of (18) depends only on y, then (12) has an integrating factor $F^* = F^*(y)$, which is obtained from (18) and taking exponents on both sides.

(19)
$$F^*(y) = \exp \int R^*(y)\,dy$$

Note:

(12)
$$P(x, y)\,dx + Q(x, y)\,dy = 0,$$

(18)
$$\frac{1}{F^*}\frac{dF^*}{dy} = R^*, \qquad \text{where} \qquad R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

$$(e^{x+y} + ye^{y})\,dx + (xe^{y} - 1)\,dy = 0, \quad y(0) = -1$$

Using Theorem 1 or 2, find an integrating factor and solve the initial value problem

$$(e^{x+y} + ye^{y})\,dx + (xe^{y} - 1)\,dy = 0, \quad y(0) = -1 \tag{20}$$

Solution. ***Step 1. Nonexactness.*** The exactness check fails:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^{x+y} + ye^{y}) = e^{x+y} + e^{y} + ye^{y} \qquad \text{but} \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xe^{y} - 1) = e^{y}.$$

Step 2. Integrating factor. General solution. Theorem 1 fails because R [the right side of (16)] depends on both x and y.

$$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{xe^y - 1}(e^{x+y} + e^y + ye^y - e^y).$$

Try Theorem 2. The right side of (18) is

$$R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{e^{x+y} + ye^y}(e^y - e^{x+y} - e^y - ye^y) = -1.$$

Hence (19) gives the integrating factor $F^*(y) = e^{-y}$. From this result and (20) you get the exact equation

$$(e^x + y)\,dx + (x - e^{-y})\,dy = 0.$$

Test for exactness; you will get 1 on both sides of the exactness condition. By integration, using (4a),

$$u = \int (e^x + y)\, dx = e^x + xy + k(y).$$

Differentiate this with respect to y and use (4b) to get

$$\frac{\partial u}{\partial y} = x + \frac{dk}{dy} = N = x - e^{-y}, \qquad \frac{dk}{dy} = -e^{-y}, \qquad k = e^{-y} + c^*.$$

Hence the general solution is

$$u(x, y) = e^x + xy + e^{-y} = c.$$

Setp 3. Particular solution. The initial condition $y(0) = -1$ gives $u(0, -1) = 1 + 0 + e = 3.72$. Hence the answer is $e^x + xy + e^{-y} = 1 + e = 3.72$. Figure 18 shows several particular solutions obtained as level curves of $u(x, y) = c$, obtained by a CAS, a convenient way in cases in which it is impossible or difficult to cast a solution into explicit form. Note the curve that (nearly) satisfies the initial condition.

Step 4. Checking. Check by substitution that the answer satisfies the given equation as well as the initial condition. ■

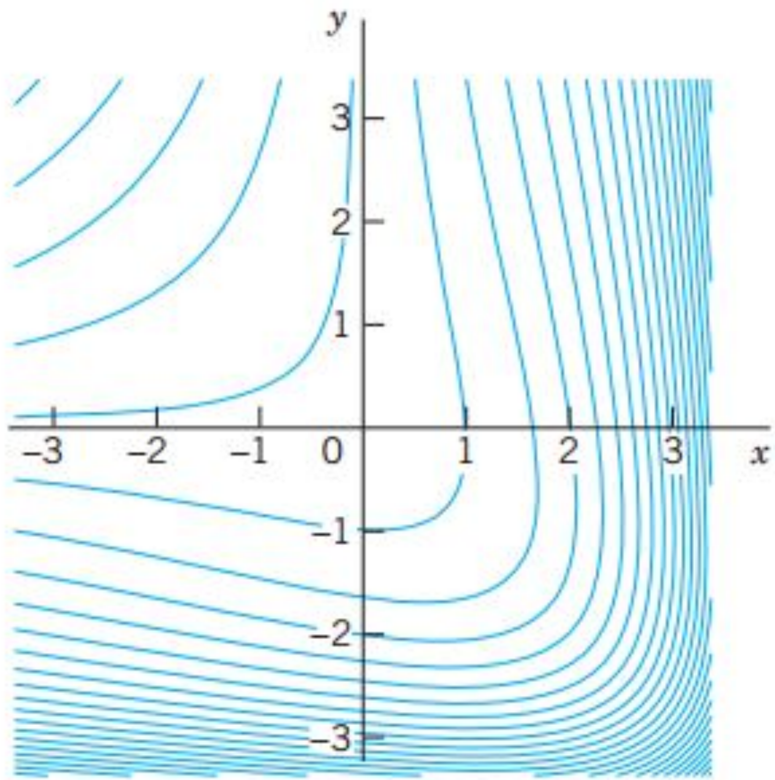


Fig. 18. Particular solutions in Example 5

Find an integrating factor using either Theorem 1 or 2 and solve the initial value problem

$$2\sin(y^2)dx + xy\cos(y^2)dy = 0, \qquad y(2) = \sqrt{\pi/2}$$

Solution. 1st Step Check for exactness

$$P = 2\sin(y^2) \qquad Q = xy\cos(y^2)$$

The equation is not exact because

$$P_y = 4y\cos(y^2) \neq Q_x = y\cos(y^2)$$

2nd Step Integrating factor

$$R = \frac{1}{Q}\left(P_y - Q_x\right) = \frac{1}{xy\cos(y^2)}[4y\cos(y^2) - y\cos(y^2)] = \frac{3y}{xy} = \frac{3}{x}$$

This shows that Theorem 1 applies and thus we obtain the integrating factor

$$F(x) = \exp \int R(x)dx = \exp \int \frac{3}{x}dx = x^3$$

Multiplying the given equation by x^3 we get

$$2x^3 \sin(y^2)dx + x^4 y \cos(y^2)dy = 0 \qquad (20)$$

This equation is exact because

$$\frac{\partial}{\partial y}[2x^3 \sin(y^2)] = 4x^3 y \cos(y^2) = \frac{\partial}{\partial x}[x^4 y \cos(y^2)]$$

3rd Step General solution

From $$u = \int M dx + k(y)$$

now written as $u_x dx + u_y dy$

$$u = \int 2x^3 \sin(y^2) dx = \frac{1}{2} x^4 \sin(y^2) + k(y)$$

From this and the second term in (20)

$$u_y = x^4 y \cos(y^2) + k'(y) = x^4 y \cos(y^2)$$

Hence $k'(y) = 0$ and $k = const$. This gives the general solution

$$u(x, y) = \frac{1}{2} x^4 \sin(y^2) = c = const.$$

4th Step Particular solution

$$y(2) = \sqrt{\pi/2}$$

$$\frac{1}{2} \cdot 2^4 \sin\frac{\pi}{2} = 8 = c$$

$$\frac{1}{2} x^4 \sin(y^2) = 8 \qquad \Rightarrow \qquad x^4 \sin(y^2) = 16$$

1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

A first-order ODE is said to be **linear** if it can be brought into the form

(1) $$y' + p(x)y = r(x),$$

by algebra, and **nonlinear** if it cannot be brought into this form.

Homogeneous Linear ODE.

We want to solve (1) on some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) \equiv 0$.) Then the ODE (1) becomes

(2) $$y' + p(x)y = 0$$

and is called **homogeneous.**

The general solution of the homogeneous ODE (2) is

(3) $$y(x) = ce^{-\int p(x)dx} \qquad (c = \pm e^{c^*} \text{ when } y >/< 0);$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

Nonhomogeneous Linear ODE

We now solve (1) in the case that $r(x)$ in (1) is not everywhere zero on the interval J considered. Then the ODE (1) is called **nonhomogeneous**.

Solution of nonhomogeneous linear ODE (1):

(4) $$y(x) = e^{-h}\left(\int e^{h} r\, dx + c\right), \qquad h = \int p(x)\, dx.$$

The structure of (4) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing (4) as a sum of two terms,

(4*) $$y(x) = e^{-h}\int e^{h} r\, dx + c\, e^{-h},$$

we see the following:

(5) Total Output = Response to the Input r + Response to the Initial Data.

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

EXAMPLE 1

First-Order ODE, General Solution, Initial Value Problem

Solve the initial value problem

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

Solution.

Here $p = \tan x$, $r = \sin 2x = 2 \sin x \cos x$ and

$$h = \int p\, dx = \int \tan x\, dx = \ln |\sec x|.$$

From this we see that in (4),

$$e^h = \sec x,\ e^{-h} = \cos x,\ e^h r = (\sec x)(2 \sin x \cos x) = 2 \sin x,$$

and the general solution of our equation is

$$y(x) = \cos x\, (2 \int \sin x\, dx + c) = c \cos x - 2 \cos^2 x.$$

From this and the initial condition, $1 = c \cdot 1 - 2 \cdot 1^2$, thus $c = 3$ and the solution of our initial value problem is $y = 3 \cos x - 2 \cos^2 x$. Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$.

Reduction to Linear Form. Bernoulli Equation

Numerous applications can be modeled by ODEs that are nonlinear but can be transformed to linear ODEs. One of the most useful ones of these is the **Bernoulli equation**

(9) $y' + p(x)y = g(x)y^a \qquad (a \text{ any real number}).$

If $a = 0$ or $a = 1$, Equation (9) is linear. Otherwise it is nonlinear. Then we set

$$u(x) = [y(x)]^{1-a}.$$

We differentiate this and substitute y' from (9), obtaining

$$u' = (1-a)y^{-a}y' = (1-a)y^{-a}(gy^a - py).$$

Simplification gives

$$u' = (1-a)(g - py^{1-a}),$$

where $y^{1-a} = u$ on the right, so that we get the linear ODE

(10) $u' + (1-a)pu = (1-a)g.$

Example 4

Logistic Equation

Solve the following Bernoulli equation, known as the **logistic equation** (or **Verhulst equation**)

(11) $y' = Ay - By^2$

Solution. Write (11) in the form (9), that is,

$$y' - Ay = -By^2$$

to see that $a = 2$ so that $u = y^{1-a} = y^{-1}$. Differentiate this u and substitute y' from (11), $u' = -y^{2}y' = -y^{-2}(Ay - By^2) = B - Ay^{-1}$.

The last term is $-Ay^{-1} = -Au$. Hence we have obtained the linear ODE

$$u' + Au = B.$$

The general solution is [by (4)]

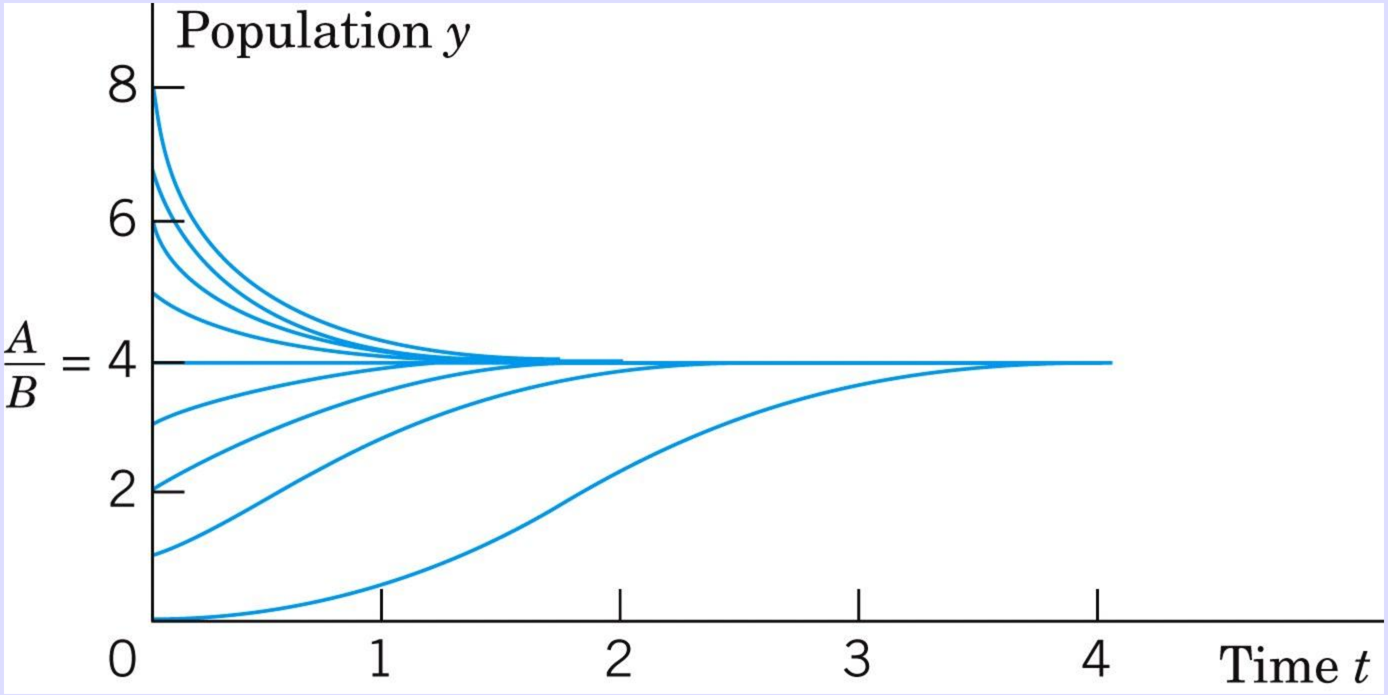
$$u = ce^{-At} + B/A.$$

Since $u = 1/y$, this gives the general solution of (11),

(12) $$y = \frac{1}{u} = \frac{1}{ce^{-At} + B/A}$$ (Fig. 21)

Directly from (11) we see that $y = 0$ ($y(t) = 0$ for all t) is also a solution.

Example 4 (continued)



SUMMARY OF CHAPTER 1

First-Order ODEs

First-Order ODEs

This chapter concerns **ordinary differential equations (ODEs) of first order** and their applications. These are equations of the form

(1) $F(x, y, y') = 0$ or in explicit form $y' = f(x, y)$

involving the derivative $y' = dy/dx$ of an unknown function y, given functions of x, and, perhaps, y itself. If the independent variable x is time, we denote it by t.

In Sec. 1.1 we explained the basic concepts and the process of **modeling**, that is, of expressing a physical or other problem in some mathematical form and solving it. Then we discussed the method of direction fields (See. 1.2), solution methods and models (Sees. 1.3–1.6), and, finally, ideas on existence and uniqueness of solutions (Sec. 1.7).

(continued 1)

A first-order ODE usually has a **general solution**, that is, a solution involving an arbitrary constant, which we denote by c. In applications we usually have to find a unique solution by determining a value of c from an **initial condition** $y(x_0) = y_0$. Together with the ODE this is called an **initial value problem**

(2) $y' = f(x, y) \quad y(x_0) = y_0 \quad (x_0, y_0$ given numbers)

and its solution is a **particular solution** of the ODE. Geometrically, a general solution represents a family of curves, which can be graphed by using **direction fields** (Sec. 1.2). And each particular solution corresponds to one of these curves.

(continued 2)

A **separable ODE** is one that we can put into the form

(3) $g(y)\,dy = f(x)\,dx$ (Sec. 1.3)

by algebraic manipulations (possibly combined with transformations, such as $y/x = u$) and solve by integrating on both sides.

An **exact ODE** is of the form

(4) $M(x, y)\,dx + N(x, y)\,dy = 0$ (Sec. 1.4)

where $M\,dx + N\,dy$ is the **differential**

$$du = u_x\,dx + u_y\,dy$$

of a function $u(x, y)$, so that from $du = 0$ we immediately get the implicit general solution $u(x, y) = c$. This method extends to nonexact ODEs that can be made exact by multiplying them by some function $F(x, y)$, called an **integrating factor** (Sec. 1.4).

(continued 3) **Linear ODEs**

(5) $y' + p(x)y = r(x)$

are very important. Their solutions are given by the integral formula (4). Sec. 1.5. Certain nonlinear ODEs can be transformed to linear form in terms of new variables.

This holds for the **Bernoulli equation**

$$y' + p(x)y = g(x)y^a \qquad \text{(Sec. 1.5).}$$

Applications and ***modeling*** are discussed throughout the chapter, in particular in Secs. 1.1, 1.3, 1.5 (***population dynamics***, etc.), and 1.6 (***trajectories***).

Picard's ***existence*** and ***uniqueness theorems*** are explained in Sec. 1.7 (and *Picard's iteration* in Problem Set 1.7).

Numeric methods for first-order ODEs can be studied in Secs. 21.1 and 21.2 immediately after this chapter, as indicated in the chapter opening.