ADVANCED ALGORITHMS & DATA STRUCTURES

Lecture-10
Dynamic programming

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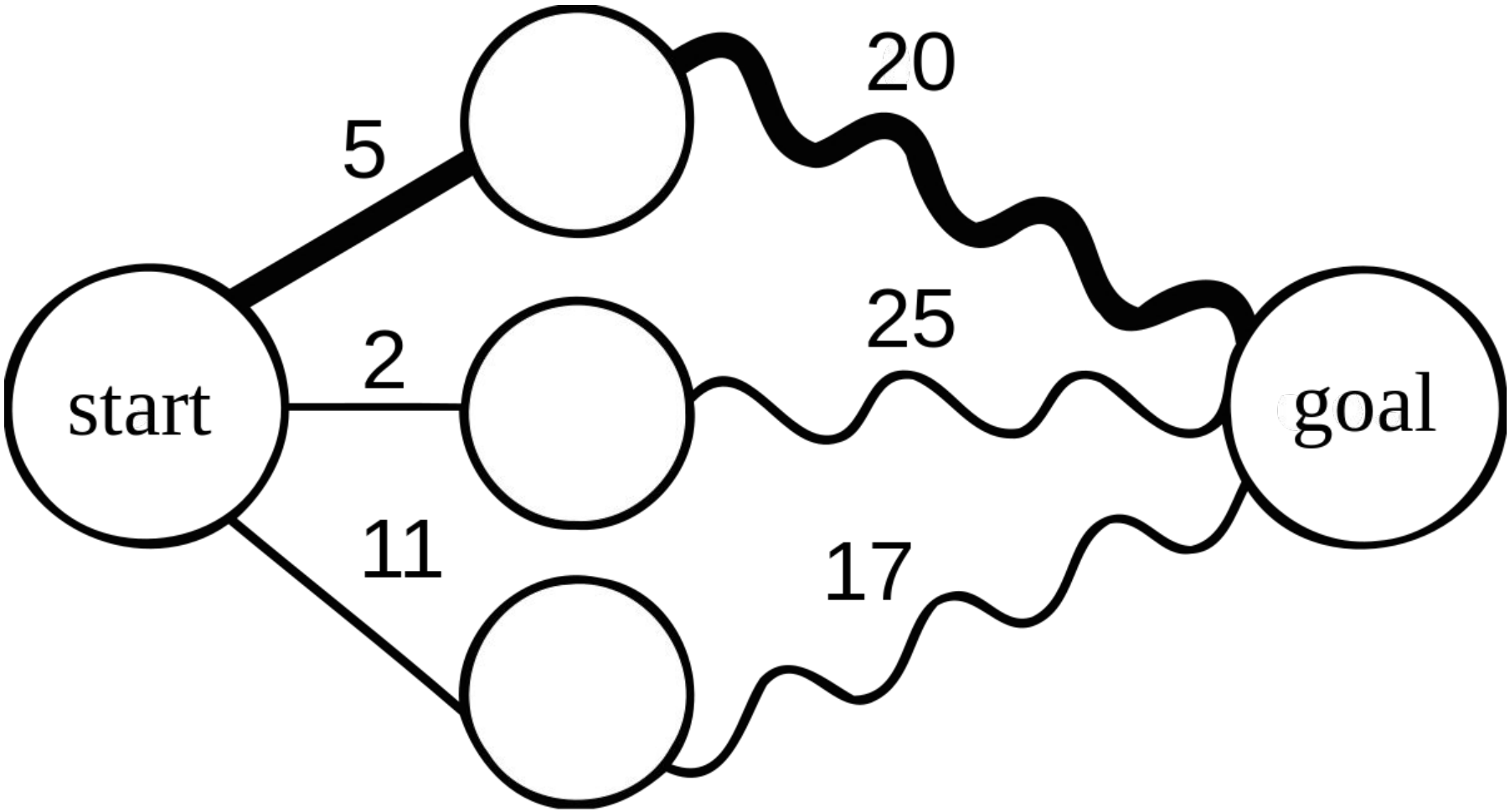
Dynamic Programming



Dynamic Programming

Dynamic Programming is mainly an optimization over plain recursion. Wherever we see a recursive solution that has repeated calls for same inputs, we can optimize it using Dynamic Programming.

The idea is to simply store the results of subproblems, so that we do not have to re-compute them when needed later. This simple optimization reduces time complexities from exponential to polynomial.

Dynamic Programming

For example, if we write simple recursive solution for Fibonacci Numbers, we get exponential time complexity and if we optimize it by storing solutions of subproblems, time complexity reduces to linear.

```
int fib(int n)
{
   if (n <= 1)
      return n;
   return fib(n-1) + fib(n-2);
}
```

Recursion : Exponential

```
f[0] = 0;
f[1] = 1;

for (i = 2; i <= n; i++)
{
    f[i] = f[i-1] + f[i-2];
}

return f[n];
```

Dynamic Programming : Linear

Dynamic Programming
1 Introduction

In this lecture we introduce *dynamic programming*, which is a high-level computational thinking concept rather than a concrete algorithm. Perhaps a more descriptive title for the lecture would be *sharing*, because dynamic programming is about sharing computation. We know that sharing of space is also crucial: binary decision diagrams in which subtrees are shared are (in practice) much more efficient than binary decision trees in which there is no sharing.

Dynamic Programming

In order to apply dynamic programming, we generally look for the following conditions:

1. The optimal solutions to a problem is composed of optimal solutions to subproblems, and
2. if there are several optimal solutions, we don't care which one we get.

2 Fibonacci Numbers

As a very simple example, we consider the computation of the Fibonacci numbers. They are defined by specifying, mathematically,

$$f0 = 0$$

$$f1 = 1$$

$$fn+2 = fn+1 + fn \ (n \geq 0)$$

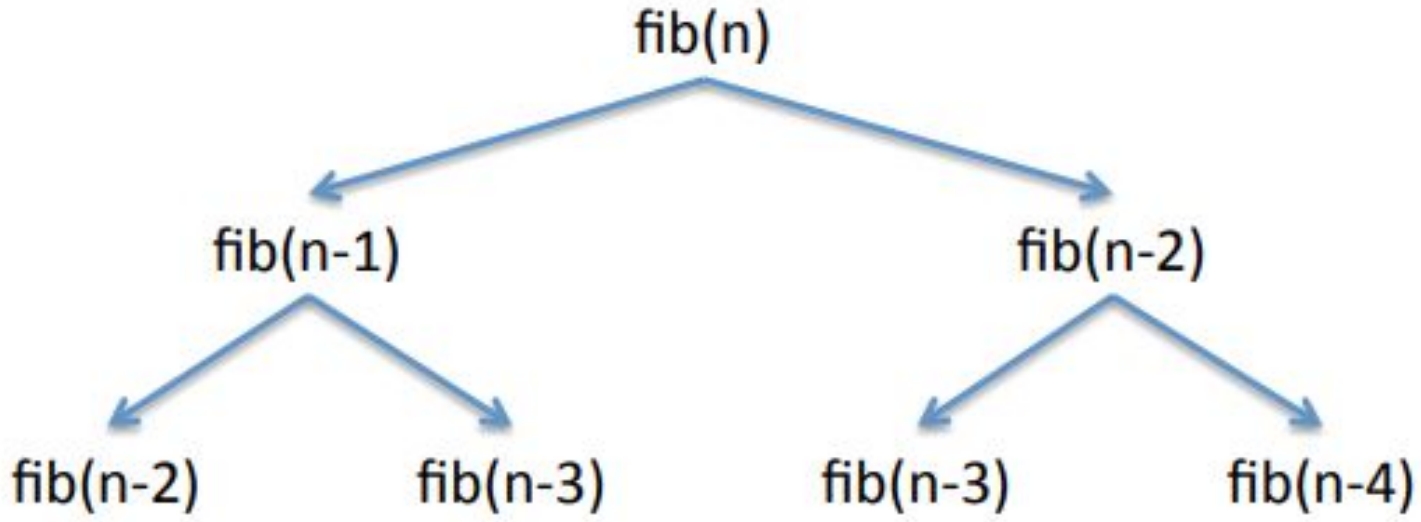
2 Fibonacci Numbers

A direct (and very inefficient) implementation is a recursive function

```
int fib0(int n) {
REQUIRES(n >= 0);
if (n == 0) return 0;
else if (n == 1) return 1;
else return fib0(n-1) + fib0(n-2);
}
```

When we draw the top part of a tree of the recursive calls that have to be made, we notice that values are computed multiple times.

2 Fibonacci Numbers



Before we move on to improve the efficiency of this program, did you notice that the program above is buggy in C, and the corresponding version would be questionable in C0? Think about it.

2 Fibonacci Numbers

The problem is that addition can overflow. In C, the result is undefined, and arbitrary behavior by the code would be acceptable. In C0 the result would be defined, but it would be the result of computing modulo 2^32 which could be clearly the wrong answer. We can fix this by explicitly checking for overflow.

2 Fibonacci Numbers

```
#include <limits.h>
int safe_plus(int x, int y) {
if ( (x > 0 && y > INT_MAX-x)
|| (x < 0 && y < INT_MIN-x) ) {
fprintf(stderr, "integer overflow\n");
abort();
} else {
return x+y;
}
}
int fib1(int n) {
REQUIRES(n >= 0);
if (n == 0) return 0;
else if (n == 1) return 1;
else return safe_plus(fib1(n-1),fib1(n-2));
}
```

3 Top-Down Dynamic Programming

In *top-down dynamic programming* we store the values as we compute them recursively. Then, if we need to compute a value we just reuse the value if we have computed it already. A characteristic pattern for top-down dynamic programming is a top-level function that allocates an array or similar structure to save computed results, and a recursive function that maintains this array.

3 Top-Down Dynamic Programming

```
int fib2_rec(int n, int* A) {
REQUIRES(n >= 0);
if (n == 0) return 0;
else if (n == 1) return 1;
else if (A[n] > 0) return A[n];
else {
int result = safe_plus(fib2_rec(n-1,A), fib2_rec(n-2,A));
A[n] = result; /* store A[n] == fib(n) */
return result;
}
}
int fib2(int n) {
REQUIRES(n >= 0);
int* A = calloc(n+1, sizeof(int));
if (A == NULL) { fprintf(stderr, "allocation failed\n");
abort(); }
/* calloc initializes the array with 0s */
int result = fib2_rec(n, A);
free(A);
return result;
}
```

3 Top-Down Dynamic Programming

We also call this programming technique *memoization*. We might be tempted to improve this function slightly, by looking up the second value:

```
int fib2_rec(int n, int* A) {
REQUIRES(n >= 0);
if (n == 0) return 0;
else if (n == 1) return 1;
else if (A[n] > 0) return A[n];
else {
int result = safe_plus(fib2_rec(n-1,A), A[n-2]);
A[n] = result; /* store A[n] == fib(n) */
return result;
}
}
```

This would be incorrect, but why?

4 Bottom-Up Dynamic Programming

Top-down dynamic programming retains the structure of the original (inefficient) recursive function. Bottom-up dynamic programming inverts the order and starts from the bottom of the recursion, building up the table of values. In bottom-up dynamic programming, recursion is often profitably replaced by iteration.

4 Bottom-Up Dynamic Programming

In our example, we would like to compute A[0], A[1], A[2], . . . in this order

```
int fib3(int n) {
REQUIRES(n >= 0);
int i;
int* A = calloc(n+1, sizeof(int));
if (A == NULL) { fprintf(stderr, "allocation failed\n");
abort(); }
A[0] = 0; A[1] = 1;
for (i = 2; i <= n; i++) {
/* loop invariant: 2 <= i && i <= n+1; */
/* loop invariant: A[i] = fib(i) for i in [0,i) */
A[i] = safe_plus(A[i-1], A[i-2]);
}
ASSERT(i == n+1);
int result = A[n];
free(A);
return result;
}
```

Program for Fibonacci numbers

The Fibonacci numbers are the numbers in the following integer sequence.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ……..

In mathematical terms, the sequence Fn of Fibonacci numbers is defined by the recurrence relation

Fn = Fn-1 + Fn-2

with seed values

F0 = 0 and F1 = 1.

Method 1 (Use recursion) vs Method 2 (Use Dynamic Programming)

```cpp
//Fibonacci Series using Recursion
#include<bits/stdc++.h>
using namespace std;

int fib(int n)
{
    if (n <= 1)
        return n;
    return fib(n-1) + fib(n-2);
}

int main ()
{
    int n = 9;
    cout << fib(n);
    getchar();
    return 0;
}

// This code is contributed
// by Akanksha Rai
```

Time Complexity: exponential 2^O(n)

```cpp
//Fibonacci Series using Dynamic Programming
#include<stdio.h>

int fib(int n)
{
  /* Declare an array to store Fibonacci numbers. */
  int f[n+2];   // 1 extra to handle case, n = 0
  int i;

  /* 0th and 1st number of the series are 0 and 1*/
  f[0] = 0;
  f[1] = 1;

  for (i = 2; i <= n; i++)
  {
      /* Add the previous 2 numbers in the series
         and store it */
      f[i] = f[i-1] + f[i-2];
  }

  return f[n];
}

int main ()
{
  int n = 9;
  printf("%d", fib(n));
  getchar();
  return 0;
}
```

Time Complexity: linear O(n)

Method 2
(Use Dynamic Programming)

```c
//Fibonacci Series using Dynamic Programming
#include<stdio.h>

int fib(int n)
{
  /* Declare an array to store Fibonacci numbers. */
  int f[n+2];   // 1 extra to handle case, n = 0
  int i;

  /* 0th and 1st number of the series are 0 and 1*/
  f[0] = 0;
  f[1] = 1;

  for (i = 2; i <= n; i++)
  {
      /* Add the previous 2 numbers in the series
         and store it */
      f[i] = f[i-1] + f[i-2];
  }

  return f[n];
}

int main ()
{
  int n = 9;
  printf("%d", fib(n));
  getchar();
  return 0;
}
```

Method 3 (Space Optimized Method 2)

```c
// Fibonacci Series using Space Optimized Method
#include<stdio.h>
int fib(int n)
{
  int a = 0, b = 1, c, i;
  if( n == 0)
    return a;
  for (i = 2; i <= n; i++)
  {
     c = a + b;
     a = b;
     b = c;
  }
  return b;
}

int main ()
{
  int n = 9;
  printf("%d", fib(n));
  getchar();
  return 0;
}
```

Time Complexity: linear O(n)
Extra Space: O(1)

Method 4 (Using power of the matrix {{1,1},{1,0}})

This another O(n) which relies on the fact that if we n times multiply the matrix M = {{1,1},{1,0}} to itself (in other words calculate power(M, n)), then we get the (n+1)th Fibonacci number as the element at row and column (0, 0) in the resultant matrix. The matrix representation gives the following closed expression for the Fibonacci numbers:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Time Complexity: O(n)
Extra Space: O(1)

Method 4 (Using power of the matrix {{1,1},{1,0}})

```
#include <stdio.h>

/* Helper function that multiplies 2 matrices F and M of size 2*2, and
  puts the multiplication result back to F[][] */
void multiply(int F[2][2], int M[2][2]);

/* Helper function that calculates F[][] raise to the power n and puts the
  result in F[][]
  Note that this function is designed only for fib() and won't work as general
  power function */
void power(int F[2][2], int n);

int fib(int n)
{
  int F[2][2] = {{1,1},{1,0}};
  if (n == 0)
      return 0;
  power(F, n-1);

  return F[0][0];
}
```

Method 4 (Using power of the matrix {{1,1},{1,0}})

```
void multiply(int F[2][2], int M[2][2])
{
  int x =  F[0][0]*M[0][0] + F[0][1]*M[1][0];
  int y =  F[0][0]*M[0][1] + F[0][1]*M[1][1];
  int z =  F[1][0]*M[0][0] + F[1][1]*M[1][0];
  int w =  F[1][0]*M[0][1] + F[1][1]*M[1][1];

  F[0][0] = x;
  F[0][1] = y;
  F[1][0] = z;
  F[1][1] = w;
}

void power(int F[2][2], int n)
{
  int i;
  int M[2][2] = {{1,1},{1,0}};

  // n - 1 times multiply the matrix to {{1,0},{0,1}}
  for (i = 2; i <= n; i++)
     multiply(F, M);
}
```

Method 4 (Using power of the matrix {{1,1},{1,0}})

```
/* Driver program to test above function */
int main()
{
  int n = 9;
  printf("%d", fib(n));
  getchar();
  return 0;
}
```

Time Complexity: O(n)
Extra Space: O(1)

Method 5 (Optimized Method 4)

The method 4 can be optimized to work in O(Logn) time complexity. We can do recursive multiplication to get power(M, n) in the prevous method (Similar to the optimization done in this post)

```
#include <stdio.h>

void multiply(int F[2][2], int M[2][2]);

void power(int F[2][2], int n);

/* function that returns nth
Fibonacci number */
int fib(int n)
{
  int F[2][2] = {{1,1},{1,0}};
  if (n == 0)
    return 0;
  power(F, n-1);
  return F[0][0];
}
```

```
/* Optimized version of power() in method 4 */
void power(int F[2][2], int n)
{
  if( n == 0 || n == 1)
     return;
  int M[2][2] = {{1,1},{1,0}};

  power(F, n/2);
  multiply(F, F);

  if (n%2 != 0)
     multiply(F, M);
}
```

Method 5 (Optimized Method 4)

```
void multiply(int F[2][2], int M[2][2])
{
  int x =  F[0][0]*M[0][0] + F[0][1]*M[1][0];
  int y =  F[0][0]*M[0][1] + F[0][1]*M[1][1];
  int z =  F[1][0]*M[0][0] + F[1][1]*M[1][0];
  int w =  F[1][0]*M[0][1] + F[1][1]*M[1][1];

  F[0][0] = x;
  F[0][1] = y;
  F[1][0] = z;
  F[1][1] = w;
}

/* Driver program to test above function */
int main()
{
  int n = 9;
  printf("%d", fib(9));
  getchar();
  return 0;
}
```

Time Complexity: **O(Logn)**
Extra Space: O(Logn) if we consider the function call stack size, otherwise O(1).

Method 6 (O(Log n) Time)

Below is one more interesting recurrence formula that can be used to find n'th Fibonacci Number in O(Log n) time.

If n is even then k = n/2:

$F(n) = [2*F(k-1) + F(k)]*F(k)$

If n is odd then k = (n + 1)/2

$F(n) = F(k)*F(k) + F(k-1)*F(k-1)$

Method 6 (O(Log n) Time)

```
// C++ Program to find n'th fibonacci Number in
// with O(Log n) arithmatic operations
#include <bits/stdc++.h>
using namespace std;

const int MAX = 1000;

// Create an array for memoization
int f[MAX] = {0};
```

```
// Returns n'th fuibonacci number using table f[]
int fib(int n)
{
    // Base cases
    if (n == 0)
        return 0;
    if (n == 1 || n == 2)
        return (f[n] = 1);

    // If fib(n) is already computed
    if (f[n])
        return f[n];

    int k = (n & 1)? (n+1)/2 : n/2;

    // Applyting above formula [Note value n&1 is 1
    // if n is odd, else 0.
    f[n] = (n & 1)? (fib(k)*fib(k) + fib(k-1)*fib(k-1))
           : (2*fib(k-1) + fib(k))*fib(k);

    return f[n];
}
```

```
/* Driver program to test above fur
int main()
{
    int n = 9;
    printf("%d ", fib(n));
    return 0;
}
```

Time complexity of this solution is O(Log n)

Method 7 Another approach:(Using formula)

In this method we directly implement the formula for nth term in the fibonacci series.

F_n = {[(√5 + 1)/2] ^ n} / √5

Reference:

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibFormula.html

```cpp
// C++ Program to find n'th fibonacci Number
#include<iostream>
#include<cmath>

int fib(int n) {
  double phi = (1 + sqrt(5)) / 2;
  return round(pow(phi, n) / sqrt(5));
}

// Driver Code
int main ()
{
  int n = 9;
  std::cout << fib(n) << std::endl;
  return 0;
}
//This code is contributed by Lokesh Mohanty.
```

Time Complexity: O(1)

Space Complexity: O(1)

5 BDD & ROBDD

In computer science, a **binary decision diagram** (**BDD**) or **branching program** is a data structure that is used to represent a Boolean function. On a more abstract level, BDDs can be considered as a compressed representation of sets or relations. Unlike other compressed representations, operations are performed directly on the compressed representation, i.e. without decompression. Other data structures used to represent Boolean functions include negation normal form (NNF), Zhegalkin polynomials, and propositional directed acyclic graphs (PDAG).

In popular usage, the term **BDD** almost always refers to **Reduced Ordered Binary Decision Diagram** (**ROBDD** in the literature, used when the ordering and reduction aspects need to be emphasized). The advantage of an ROBDD is that it is canonical (unique) for a particular function and variable order.[1] This property makes it useful in functional equivalence checking and other operations like functional technology mapping.
A path from the root node to the 1-terminal represents a (possibly partial) variable assignment for which the represented Boolean function is true. As the path descends to a low (or high) child from a node, then that node's variable is assigned to 0 (respectively 1).

https://en.wikipedia.org/wiki/Binary_decision_diagram

5 Implementing ROBDDs

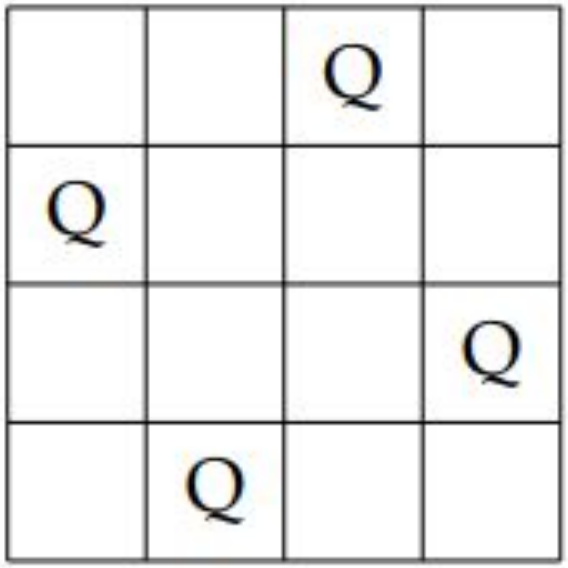
In the implementation of ROBDDs, dynamic programming plays a pervasive role. Binary decision diagrams (BDDs) satisfying two conditions:

- **Irredundancy:** The low and high successors of every node are distinct.
- **Uniqueness:** There are no two distinct nodes testing the same variable with the same successors.

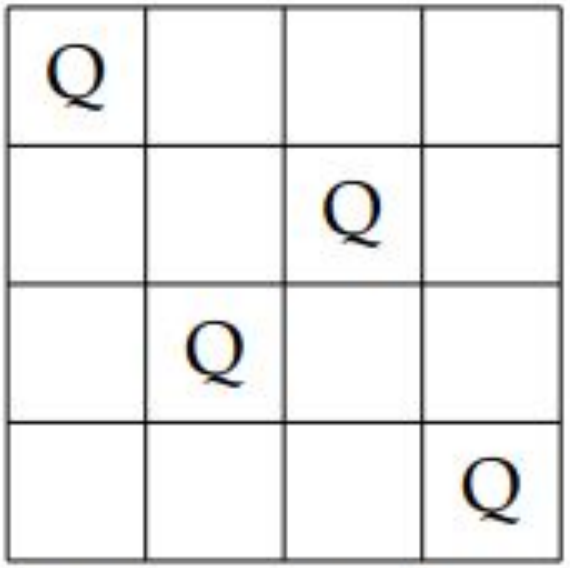
These two conditions guarantee canonicity of the representation of boolean functions and also make the data structure efficient in many common cases.

6 Encoding the n-Queens Problem

The n-queens problem is to fill an n ∗ n chessboard with n queens such that none attacks any other. Queens in chess can move horizontally, vertically, and diagonally. For example, the following are examples and counterexamples of solutions on a 4 ∗ 4 board.



Solution



Non-solution

6 Encoding the n-Queens Problem

We would like to encode n-queens problems as ROBDDS. The idea is to assign a boolean variable to each square, where a value of 1 means that the square is occupied by a queen, and a 0 means that the square is empty. We write these variables as xij for the square at column i and row j.

x_{03}	x_{13}	x_{23}	x_{33}
x_{02}	x_{12}	x_{22}	x_{32}
x_{01}	x_{11}	x_{21}	x_{31}
x_{00}	x_{10}	x_{20}	x_{30}

Now we need to generate constraints on these boolean variables such that a correct solution will be evaluated as true (1) and an incorrect situation will be evaluated as false (0).

6 Encoding the n-Queens Problem

For example, to encode that the column 0 has at least one queen on a $4 * 4$ board, we would write

$$x00 \vee x01 \vee x02 \vee x03$$

Similarly, to encode that the main diagonal has no more than one queen we might write

$$\begin{aligned} & (x_{00} \supset (\neg x_{11} \wedge \neg x_{22} \wedge \neg x_{33})) \\ \wedge \quad & (x_{11} \supset (\neg x_{00} \wedge \neg x_{22} \wedge \neg x_{33})) \\ \wedge \quad & (x_{22} \supset (\neg x_{00} \wedge \neg x_{11} \wedge \neg x_{33})) \\ \wedge \quad & (x_{33} \supset (\neg x_{00} \wedge \neg x_{11} \wedge \neg x_{22})) \end{aligned}$$

where b $\supset$ c (b implies c) is the same as (¬b) $\vee$ c in boolean logic. To see how this is programmed, we need to see the interface to the ROBDD package.

6 Encoding the n-Queens Problem

```
typedef struct bdd* bdd;
typedef int bdd_node;
bdd bdd_new(int k); /* k variables */
void bdd_free(bdd B);
int bdd_size(bdd B); /* total number of nodes */
bdd_node make(bdd B, int var, bdd_node low,
bdd_node high);
bdd_node apply(bdd B, int (*func)(int b1, int
b2), bdd_node u1, bdd_node u2);
int satcount(bdd B, bdd_node u);
void onesat(bdd B, bdd_node u);
void allsat(bdd B, bdd_node u);
```

6 Encoding the n-Queens Problem

The crucial functions here are make and apply.

make(B, x, u, v) takes a BDD B and a variable x and returns a node testing the variable x with low successor u and high successor v. Both u and v must be defined in B, and the result will be a node w also defined in B. We only use this to create variables and their negations, exploiting that the BDD nodes representing false and true are 0 and 1, respectively. The variable xij gets index i + j * n + 1, where 1 is added because the BDD library counts variables starting at 1. So we can obtain a BDD node representing just the BDD variable x33 on a 4 * 4 board with

```
bdd B = bdd_new(4*4);
x33 = make(B, 3+3*4+1, 0, 1);
```

where 0 means that the low successor of x33 will be 0 (false), and 1 means that the high successor of x will be 1 (true).

6 Encoding the n-Queens Problem

apply(B, op, u, v) takes two BDD nodes u and v and applies boolean operation op to them, returning a new node representation u op v. In the implementation this will be a function pointer, where the function implements the boolean operation on integers. It will be passed only 0 and 1 and must return either 0 or 1.

For example, the boolean expression

$$r = x00 \vee x01 \vee x02 \vee x03$$

could be represented as

6 Encoding the n-Queens Problem

```
bdd B = bdd_new(4*4);
int x00 = make(B, 1, 0, 1);
int x01 = make(B, 2, 0, 1);
int x02 = make(B, 3, 0, 1);
int x03 = make(B, 4, 0, 1);
int r = apply(B, &or, x00, x01);
r = apply(B, &or, r, x02);
r = apply(B, &or, r, x03);
```

where we have previously defined

```
int or(int b1, int b2) {
   return b1 | b2;
}
```

Now it is pretty straightforward to encode, in general, that each column has a queen. We assume B holds a BDD of n $*$ n variables.

6 Encoding the n-Queens Problem

```
r = 1;
/* each column has a queen */
for (i = 0; i < n; i++) {
u = 0; /* false */
for (j = 0; j < n; j++) {
x = make(B, i+j*n+1, 0, 1); /* x_ij */
u = apply(B, &or, u, x);
}
r = apply(B, &and, r, u);
}
```

The outer loop (i) goes through each column building up the result BDD for r, while the inner loop (j) goes through each row in the column i and builds up u.

Schematically we have

$$r = 1 \wedge u0 \wedge \cdots \wedge un{-}1$$

$$ui = 0 \vee xi0 \vee \cdots \vee xi(n{-}1)$$