Lecture 3
Two-Level Logic Minimization Algorithms

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ECE 303

Advanced Digital Design

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Outline

- CAD Tools for 2-level minimization
- Quine-McCluskey Method
- ESPRESSO Algorithm
- READING: Katz 2.4.1, 2.4.2, Dewey 4.5

Two-Level Simplification Approaches

Algebraic Simplification:

not an algorithm/systematic procedure

how do you know when the minimum realization has been found?

Computer-Aided Tools:

precise solutions require very long computation times, especially for functions with many inputs (>10)

heuristic methods employed —
"educated guesses" to reduce the amount of computation
good solutions not best solutions

Still Relevant to Learn Hand Methods:

insights into how the CAD programs work, and their strengths and weaknesses

ability to check the results, at least on small examples

Review of Karnaugh Map Method

Algorithm: Minimum Sum of Products Expression from a K-Map

Step 1: **Choose an element of ON-set not already covered by an implicant**

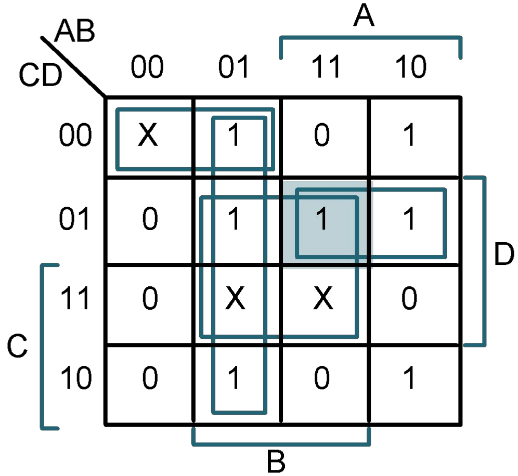
Step 2: **Find "maximal" groupings of 1's and X's adjacent to that element. Remember to consider top/bottom row, left/right column, and corner adjacencies. This forms *prime implicants* (always a power of 2 number of elements).**

Repeat Steps 1 and 2 to find all prime implicants

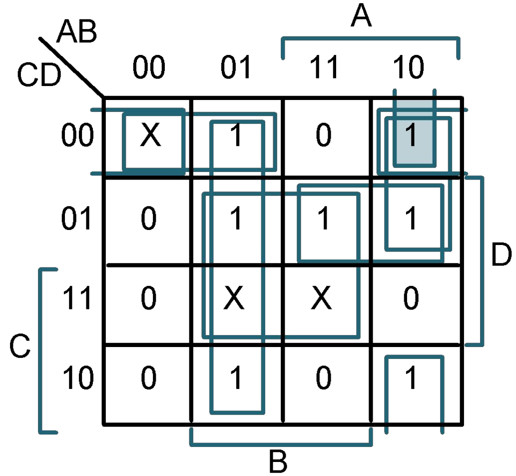
Step 3: **Revisit the 1's elements in the K-map. If covered by single prime implicant, it is *essential*, and participates in final cover. The 1's it covers do not need to be revisited**

Step 4: **If there remain 1's not covered by essential prime implicants, then select the smallest number of prime implicants that cover the remaining 1's**

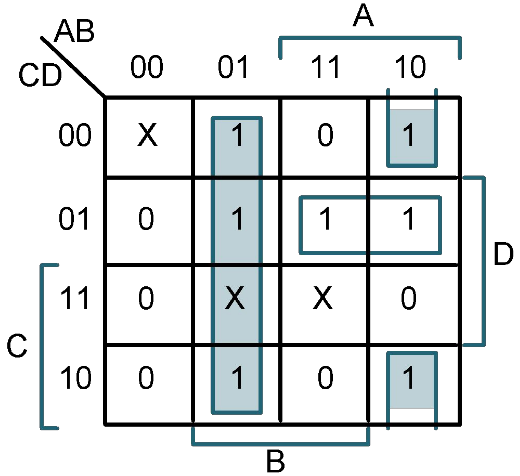
Example of Karnaugh Map Method



Primes around A B C' D



Primes around A B' C' D'



Essential Primes with Min Cover

Quine-McCluskey Method

Tabular method to systematically find all prime implicants

f(A,B,C,D) = Σm(4,5,6,8,9,10,13) + Σd(0,7,15)

Stage 1: Find all prime implicants

Step 1: Fill Column 1 with ON-set and DC-set minterm indices. Group by number of 1's.

Implication Table	
Column I	
0000	
0100	
1000	
0101	
0110	
1001	
1010	
0111	
1101	
1111	

Quine-McCluskey Method

Tabular method to systematically find all prime implicants

$f(A,B,C,D) = \Sigma\ m(4,5,6,8,9,10,13) + \Sigma\ d(0,7,15)$

Stage 1: Find all prime implicants

Step 1: Fill Column 1 with ON-set and DC-set minterm indices. Group by number of 1's.

Step 2: Apply Uniting Theorem— Compare elements of group w/ N 1's against those with N+1 1's. Differ by one bit implies adjacent. Eliminate variable and place in next column.

E.g., 0000 vs. 0100 yields 0-00
0000 vs. 1000 yields -000

When used in a combination, mark with a check. If cannot be combined, mark with a star. These are the prime implicants.

Repeat until no further combinations can be made.

Implication Table

Column I	Column II	
0000 ✓	0-00 -000	
0100 ✓ 1000 ✓	010- 01-0 100- 10-0	
0101 ✓ 0110 ✓ 1001 ✓ 1010 ✓	01-1 -101 011- 1-01	
0111 ✓ 1101 ✓	-111 11-1	
1111 ✓		

Quine Mcluskey Method

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$f(A,B,C,D) = \Sigma m(4,5,6,8,9,10,13) + \Sigma d(0,7,15)$

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E.g., 0000 vs. 0100 yields 0-00
0000 vs. 1000 yields -000

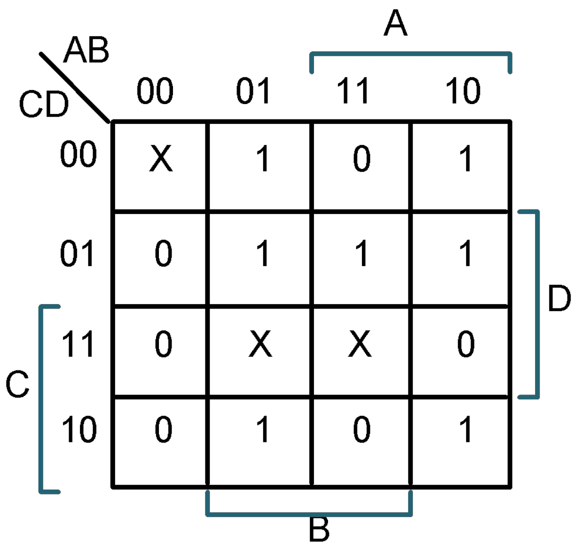
When used in a combination, mark with a check. If cannot be combined, mark with a star. These are the prime implicants.

Implication Table

Column I	Column II	Column III
0000 ¦	0-00 *	01-- *
	-000 *	
0100 ¦		-1-1 *
1000 ¦	010- ¦	
	01-0 ¦	
0101 ¦	100- *	
0110 ¦	10-0 *	
1001 ¦		
1010 ¦	01-1 ¦	
	-101 ¦	
0111 ¦	011- ¦	
1101 ¦	1-01 *	
1111 ¦	-111 ¦	
	11-1 ¦	

Repeat until no further combinations can be made.

Quine McCluskey Method (Contd)



Prime Implicants:

0-00 = A' C' D'

100- = A B' C'

1-01 = A C' D

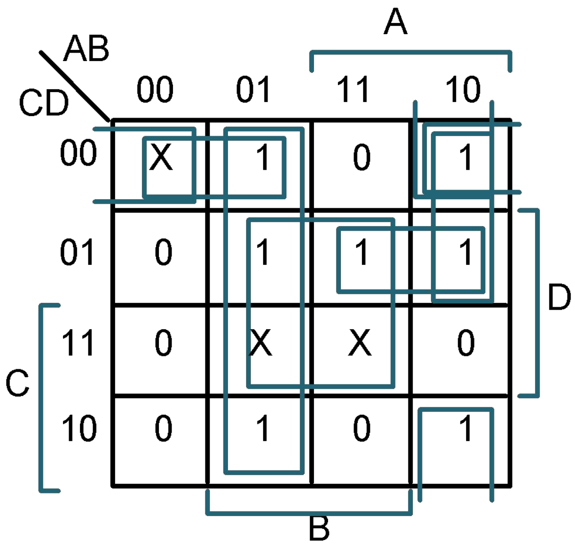
-1-1 = B D

-000 = B' C' D'

10-0 = A B' D'

01-- = A' B

Quine-McCluskey Method (Contd)



Prime Implicants:

0-00 = A' C' D'

-000 = B' C' D'

100- = A B' C'

10-0 = A B' D'

1-01 = A C' D

01-- = A' B

-1-1 = B D

Stage 2: find smallest set of prime implicants that cover the ON-set

recall that essential prime implicants must be in all covers

another tabular method– the prime implicant chart

Finding the Minimum Cover

- We have so far found all the prime implicants
- The second step of the Q-M procedure is to find the smallest set of prime implicants to cover the complete on-set of the function
- This is accomplished through the prime implicant chart
  - Columns are labeled with the minterm indices of the onset
  - Rows are labeled with the minterms covered by a given prime implicant
  - Example a prime implicant (-1-1) becomes minterms 0101, 0111, 1101, 1111, which are indices of minterms m5, m7, m13, m15

Prime Implicant Chart

	4	5	6	8	9	10	13
0,4(0-00)	X						
0,8(-000)				X			
8,9(100-)				X	X		
8,10(10-0)				X		X	
9,13(1-01)					X		X
4,5,6,7(01--)	X	X	X				
5,7,13,15(-1-1)		X					X

rows = prime implicants
columns = ON-set elements
place an "X" if ON-set element is covered by the prime implicant

Prime Implicant Chart

	4	5	6	8	9	10	13
0,4(0-00)	X						
0,8(-000)				X			
8,9(100-)				X	X		
8,10(10-0)				X			
9,13(1-01)					X		X
4,5,6,7(01--)	X	X					
5,7,13,15(-1-1)		X					X

If column has a single X, than the implicant associated with the row is essential. It must appear in minimum cover

Prime Implicant Chart (Contd)

	4	5	6	8	9	10	13
0,4(0-00)	X						
0,8(-000)				X			
8,9(100-)				X	X		
$\boxed{\text{8,10(10-0)}}$				X			
9,13(1-01)					X		X
$\boxed{\text{4,5,6,7(01--)}}$	X	X					
5,7,13,15(-1-1)		X					X

Eliminate all columns covered by essential primes

Prime Implicant Chart (Contd)

	4	5	6	8	9	10	13
0,4(0-00)	X						
0,8(-000)				X			
8,9(100-)				X	X		
$\boxed{\text{8,10(10-0)}}$				X		●	
$\boxed{\text{9,13(1-01)}}$					X		X
$\boxed{\text{4,5,6,7(01--)}}$	X	X	●				
5,7,13,15(-1-1)		X					X

Find minimum set of rows that cover the remaining columns

$$f = A\,B'\,D' + A\,C'\,D + A'\,B$$

Second Example of Q-M Method

Assume function $F(A,B,C,D) = \sum m(0, 1, 4, 5, 7, 12, 14, 15)$

Enumerate the minterms in order
of number of uncomplemented variables

Column I lists them
minterms with 0 : 0
minterms with 1: 1,4
minterms with 2: 5,12
minterms with 3: 7,14
minterms with 4: 15

Column II combines minterms that are
adjacent in one variable
example, 0,1 and 0.4 , etc.

Implication Table

Column I	Column II	
0(0000)	0,1 0,4	
1(0001) 4(0100)	1,5 4,5 4,12	
5(0101) 12(1100)	5,7 12,14	
7(0111) 14(1110)	7,15 14,15	
15(1111)		

Second Example (Contd)

Column III tries to combine adjacent terms in Column II

Example: 0,1 with 4,5 gives 0,1,4,5
0,4 with 1,5 gives 0,1,4,5
No other larger groups
End of procedure

FINAL PRIME IMPLICANTS
(0,1,4,5) representing -0-0 or A C
(4,12)
(5,7)
(12,14)
(7,15)
(14,15)

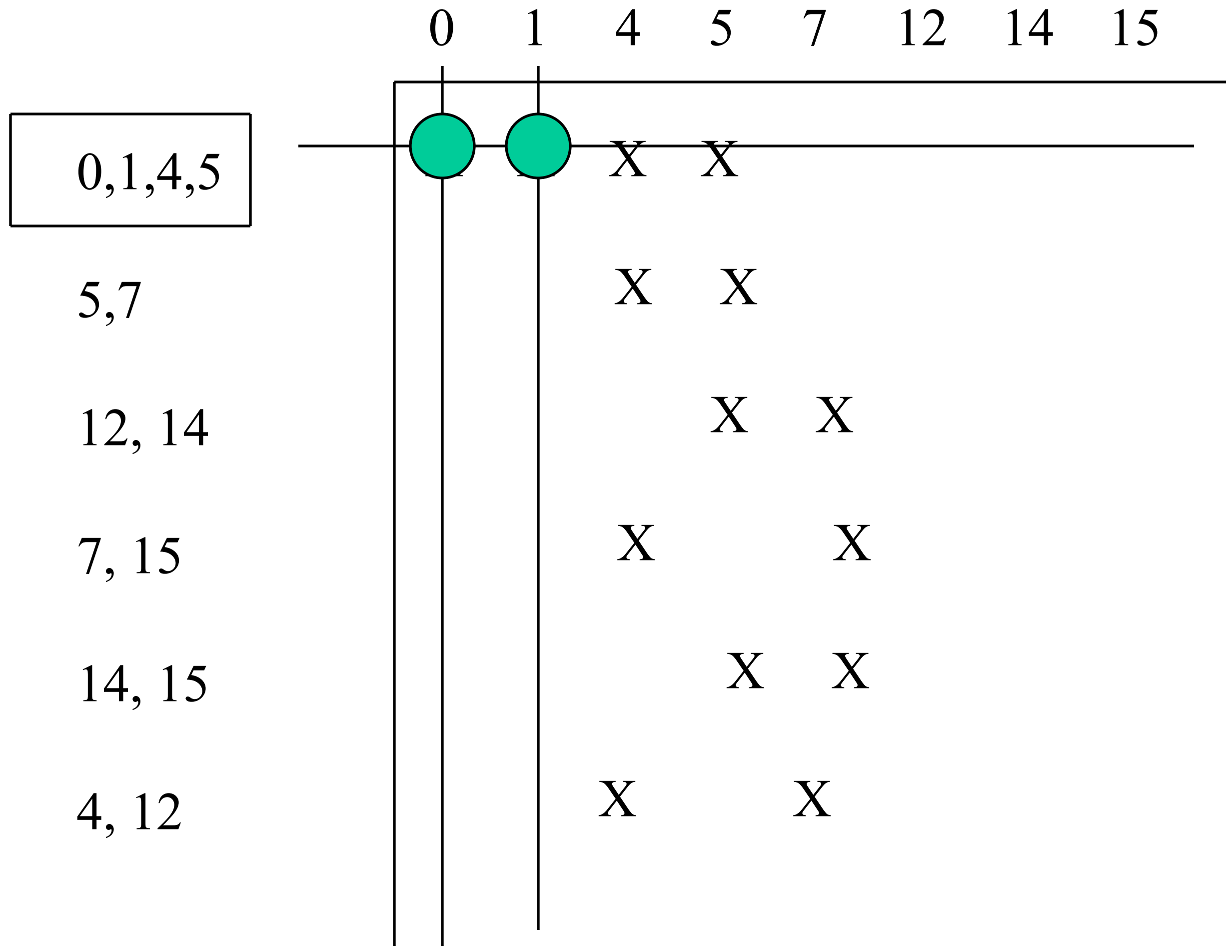
Implication Table

Column I	Column II	Column III
0(0000)	0,1	0,1,4,5
	0,4	0,4,1,5
1(0001)	1,5	
4(0100)	4,5	
	4,12	
5(0101)	5,7	
12(1100)	12,14	
7(0111)	7,15	
14(1110)	14,15	
15(1111)		

Prime Implicant Chart for Second Example

	0	1	4	5	7	12	14	15
0,1,4,5	X	X	X	X				
5,7			X	X				
12, 14				X	X			
7, 15			X		X			
14, 15				X	X			
4, 12			X		X			

Essential Primes for Example

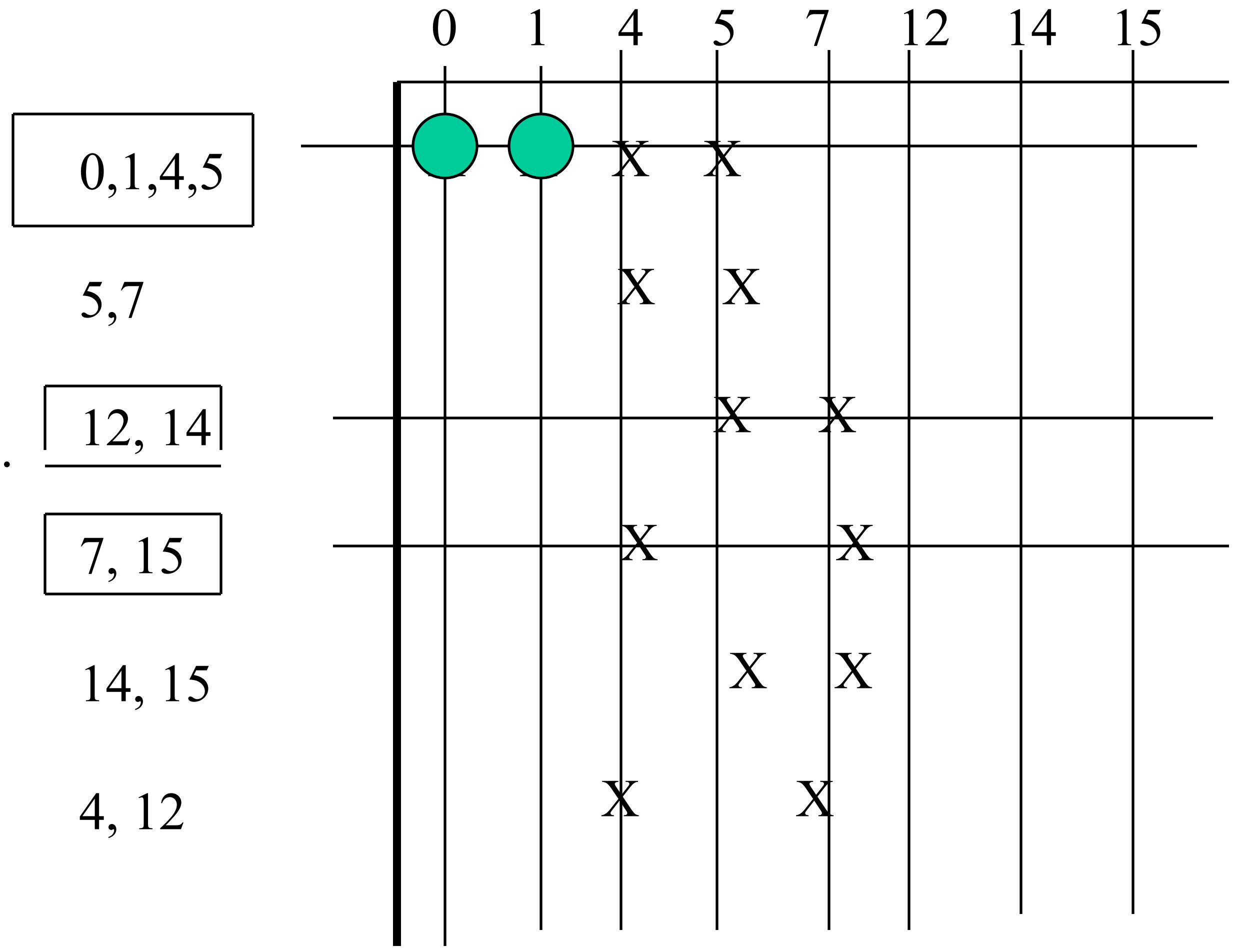


Delete Columns Covered by Essential Primes

	0	1	4	5	7	12	14	15
$\boxed{0,1,4,5}$	X	X	X	X				
5,7			X	X				
12, 14				X	X			
7, 15			X		X			
14, 15				X	X			
4, 12			X		X			

Resultant Minimum Cover

Several choices
of combinations
of prime implicants.



Resultant minimum function F = 0,1,4,5 + 7,15 + 12, 14

$= \underline{A\ C} + B\ C\ D + A\ B\ \underline{D}$

ESPRESSO Method

Problem with Quine-McCluskey: the number of prime implicants grows rapidly with the number of inputs

- **upper bound: $3^n/n$, where n is the number of inputs**
- **finding a minimum cover is NP-complete, i.e., a computational expensive process not likely to yield to any efficient algorithm**

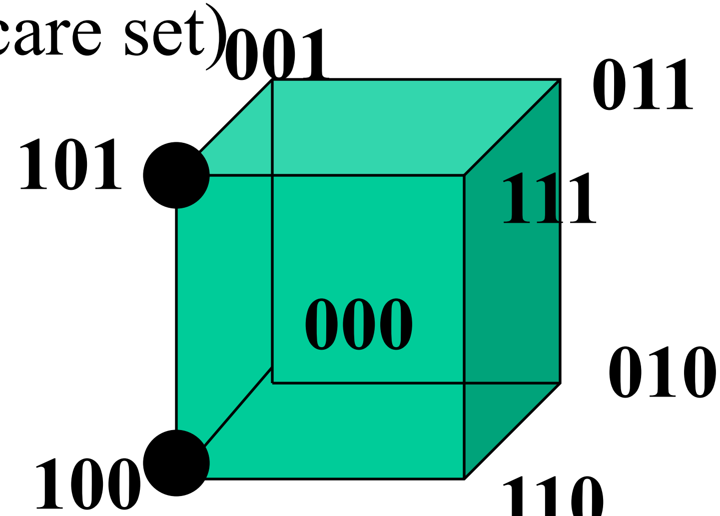
Espresso: trades solution speed for minimality of answer

- **don't generate *all* prime implicants (Quine-McCluskey Stage 1)**
- **judiciously select a subset of primes that still covers the ON-set**
- **operates in a fashion not unlike a human finding primes in a K-map**

Boolean Space

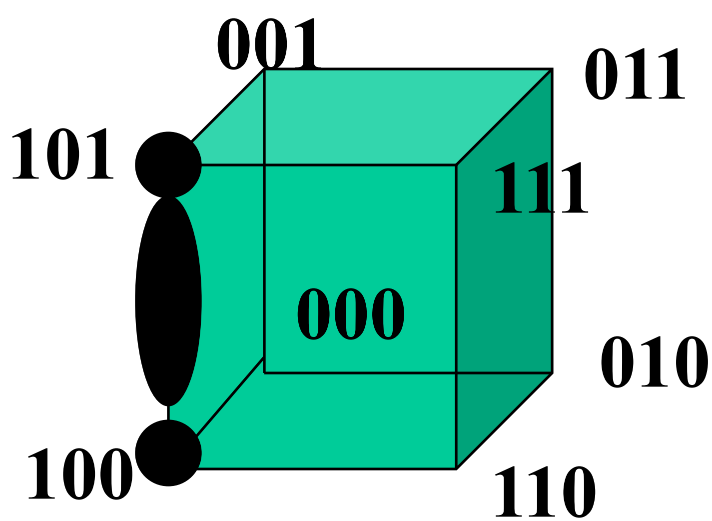
- The notion of redundancy can be formulated in Boolean space
- Every point in a Boolean space corresponds to an assignment of values (0 or 1) to variables.
- The on-set of a Boolean function is set of points (shown in black) where function is 1 (similarly for off-set and don't--care set)

Consider three Boolean variables x1, x2, x3



Boolean Space

- If g and h are two Boolean functions such that on-set of g is a subset of on-set of h, then we write
  - g $\underline{C}$ h
- Example g = x1 $\underline{x2}$ x3 and h = x1 $\underline{x2}$
- In general if f = p1 + p2 + ….pn, check if pi $\underline{C}$ p1 + p2 + …p I-1 + pn

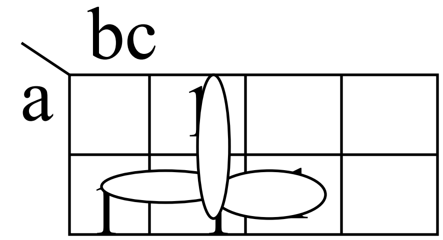


Redundancy in Boolean Space

- $x1\ \underline{x2}$ is said to cover $x1\ \underline{x2}\ x3$
- Thus redundancy can be identified by looking for inclusion or covering in the Boolean space
- While redundancy is easy to observe by looking at the product terms, it is not always the case
  - If $f = \underline{x2}\ \underline{x3} + x1\ \underline{x2} + x1\ x3$, then $x1\ \underline{x2}$ is redundant
- Situation is more complicated with multiple output functions
  - $f1 = p11 + p12 + \ldots + p1n$
  - $f2 = \ldots.$
  - $Fm = pm1 + pm2 + \ldots\ p\ mn$

Minimizing Two Level Functions

- Sometimes just finding an irredundant cover may not give minimal solution



- Example:
  - Fi = $\underline{b}$ c + $\underline{a}$ c + a $\underline{bc}$ (no cube is redundant)
- Can perform a reduction operation on some cubes
  - Fi = a $\underline{b}$ c + $\underline{a}$ c + a $\underline{bc}$ (add a literal a to $\underline{b}$ c)
- Now perform an expansion of some cubes
  - Fi = a $\underline{b}$ + $\underline{a}$ c + a $\underline{bc}$(remove literal c from a $\underline{b}$ c)
- Now perform irredundant cover
  - Fi = a $\underline{b}$ + $\underline{a}$ c (remove a $\underline{b\ c}$)
- At each step need to make sure that function remains same, I.e. Boolean equivalence

Espresso Algorithm

1. Expands implicants to their maximum size
   Implicants covered by an expanded implicant are removed from further consideration
   Quality of result depends on order of implicant expansion
   Heuristic methods used to determine order
   Step is called EXPAND

2. Irredundant cover (i.e., no proper subset is also a cover) is extracted from the expanded primes
   Just like the Quine-McCluskey Prime Implicant Chart
   Step is called IRREDUNDANT COVER

3. Solution usually pretty good, but sometimes can be improved
   Might exist another cover with fewer terms or fewer literals
   Shrink prime implicants to smallest size that still covers ON-set
   Step is called REDUCE

4. Repeat sequence REDUCE/EXPAND/IRREDUNDANT COVER to find alternative prime implicants
   Keep doing this as long as new covers improve on last solution

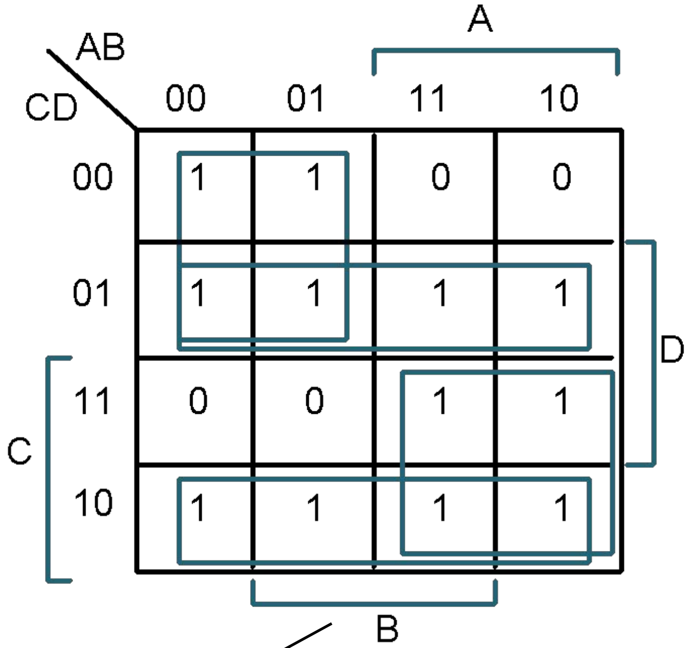
5. A number of optimizations are tried, e.g., identify and remove essential primes early in the process

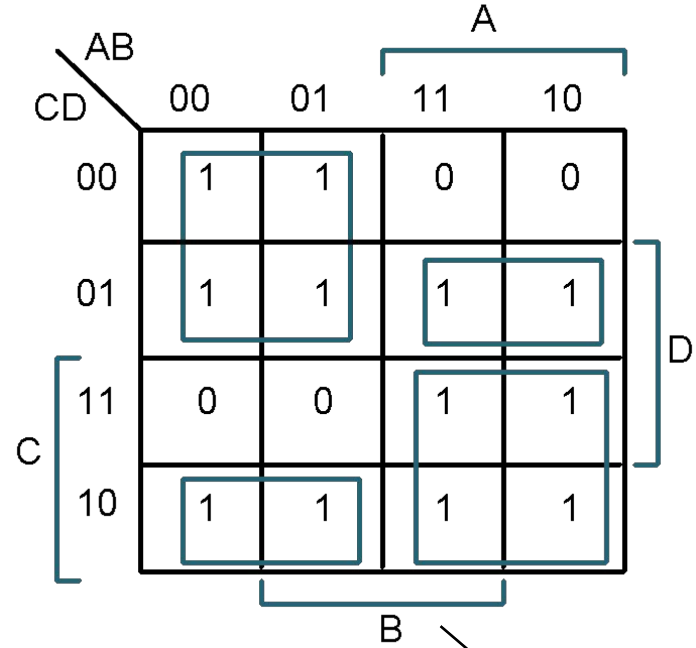
Details of ESPRESSO Algorithm

```
Procedure ESPRESSO ( F, D, R)  /* F is ON set, D is don't care, R OFF *
    R = COMPLEMENT(F+D);  /* Compute complement */
    F = EXPAND(F, R) ; /* Initial expansion */
    F = IRREDUNDANT(F,D);  /* Initial irredundant cover */
    F = ESSENTIAL(F,D) /* Detecting essential primes */
    F = F - E; /* Remove essential primes from F */
    D = D + E; /* Add  essential primes to D */
    WHILE Cost(F) keeps decreasing DO
      F = REDUCE(F,D);  /* Perform reduction, heuristic which cubes */
      F = EXPAND(F,R);  /* Perform expansion, heuristic which cubes */
      F = IRREDUNDANT(F,D);  /* Perform irredundant cover */
    ENDWHILE;
    F = F + E;
    RETURN F;
END Procedure;
```

Need for Iterations in ESPRESSO

Espresso: Why Iterate on Reduce, Irredundant Cover, Expand?





Initial Set of Primes found by Steps1 and 2 of the Espresso Method

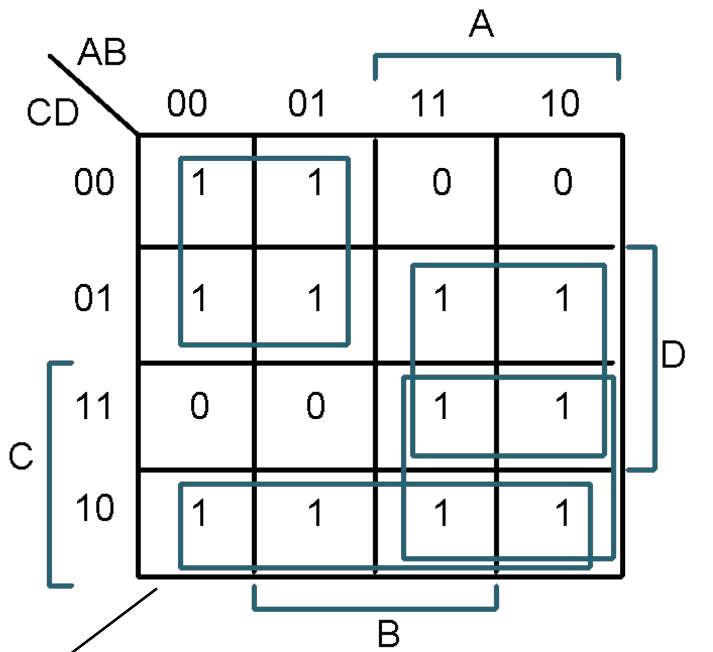
4 primes, irredundant cover, but not a minimal cover!

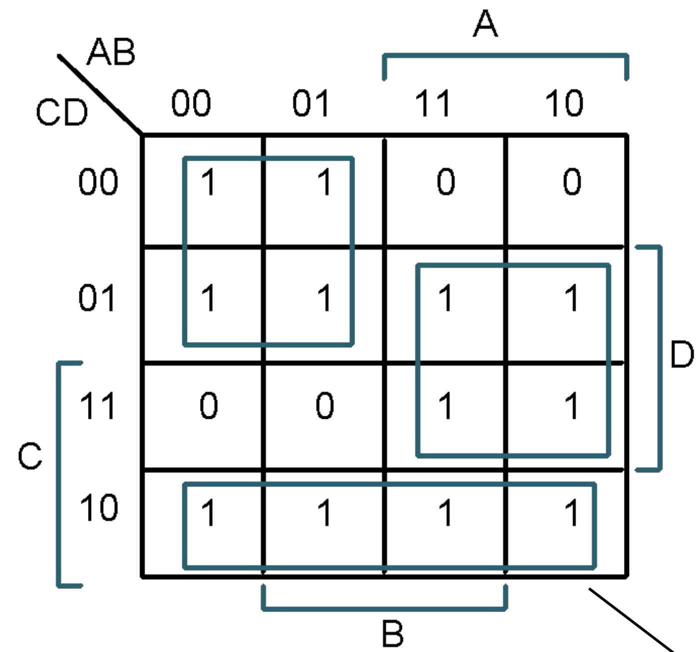
Result of REDUCE: Shrink primes while still covering the ON-set

Choice of order in which to perform shrink is important

ESPRESSO Example

Espresso Iteration (Continued)





Second EXPAND generates a different set of prime implicants

IRREDUNDANT COVER found by final step of espresso

Only three prime implicants!

Example of ESPRESSO Input/Output

ƒ(A,B,C,D) = □m(4,5,6,8,9,10,13) + d(0,7,15)

Espresso Input

```
.i 4            -- # inputs
.o 1            -- # outputs
.ilb a b c d    -- input names
.ob f           -- output name
.p 10           -- number of product terms
0100   1        -- A'BC'D'
0101   1        -- A'BC'D
0110   1        -- A'BCD'
1000   1        -- AB'C'D'
1001   1        -- AB'C'D
1010   1        -- AB'CD'
1101   1        -- ABC'D
0000   -        -- A'B'C'D' don't care
0111   -        -- A'BCD don't care
1111   -        -- ABCD don't care
.e              -- end of list
```

Espresso Output

```
.i 4
.o 1
.ilb a b c d
.ob f
.p 3
1-01   1
10-0   1
01--   1
.e
```

ƒ = A C' D + A B' D' + A' B

Two-Level Logic Design Approach

Primitive logic building blocks

INVERTER, AND, OR, NAND, NOR, XOR, XNOR

Canonical Forms

Sum of Products, Products of Sums

Incompletely specified functions/don't cares

Logic Minimization

Goal: two-level logic realizations with fewest gates and fewest number of gate inputs

Obtained via Laws and Theorems of Boolean Algebra

or Boolean Cubes and the Uniting Theorem

or K-map Methods up to 6 variables

or Quine-McCluskey Algorithm

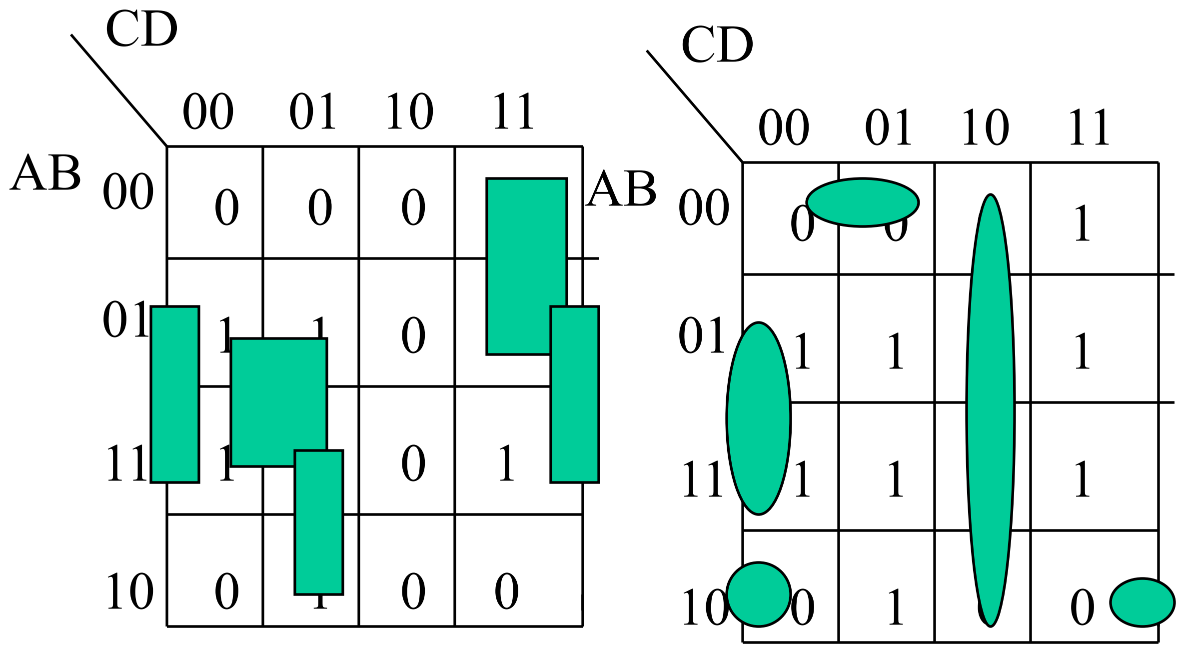
or Espresso CAD Tool

SOP and POS Two-Level Logic Forms

- We have looked at two-level logic expressions
- Sum of products form
  - $F = a\ b\ \underline{c} + \underline{b}\ c\ \underline{d} + a\ b\ \underline{d} + \underline{a}\ c$
  - This lists the ON sets of the functions, minterms that have the value 1
- Product of sums form (another equivalent form)
  - $\overline{F} = (\ \underline{a} + \underline{b} + c\ ) . (\ b + \underline{c} + d\ ) . (\ \underline{a} + \underline{b} + d\ ) . (\ a + \underline{c})$
  - This lists the OFF sets of the functions, maxterms that have the value 0
- Relationship between forms
  - minimal POS form of F = $\overline{\text{minimal SOP form of } \overline{F}}$
  - minimal SOP form of F = $\overline{\text{minimal POS form of } \overline{F}}$

SOP and POS Forms

AB \ CD	00	01	10	11
00	0	0	0	1
01	1	1	0	1
11	1	1	0	1
10	0	1	0	0



SOP form

POS form

F = E m(2,4,5,6,8,9,10,13)

F = II M(0,1,3,7,11,15)

F= B $\underline{C}$ + B $\underline{D}$ + A $\underline{C}$ D + $\underline{A}$ C $\underline{D}$

F =($\underline{C}$ + $\underline{D}$)($\underline{A}$+B+D)(A+B+C)

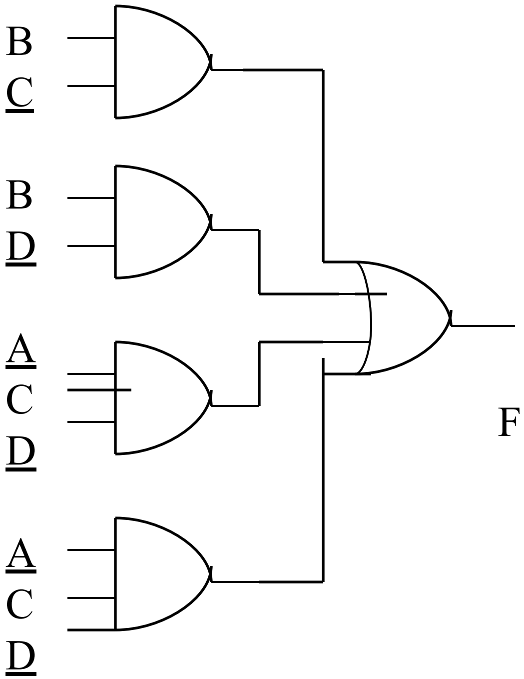
Product of Sums Minimization

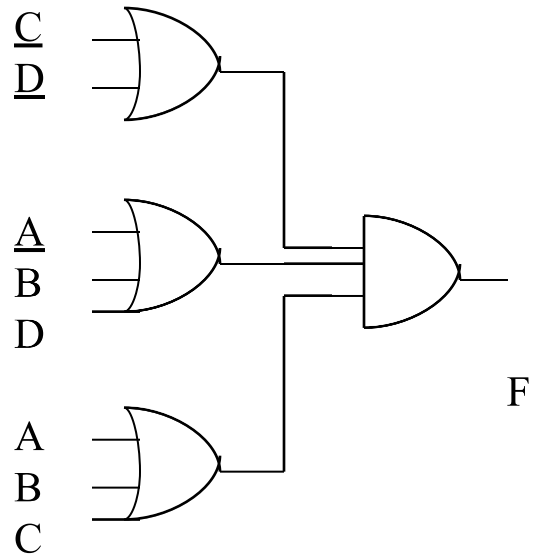
- For a given function shown as a K-map, in an SOP realization one groups the 1s
- Example: $F = B\,\underline{C} + B\,\underline{D} + A\,\underline{C}\,D + \underline{A}\,C\,\underline{D}$
- For the same function in a K-map, in a POS realization one groups the 0s
- Example: $\overline{F(A,B,C,D)} = (C.D) + (A.\underline{B}.\underline{D}) + (\underline{A}.\underline{B}.\underline{C})$
- With De Morgan's theorem

$$F = (\underline{C} + \underline{D}) \,.\, (\underline{A} + B + D) \,.\, (A + B + C)$$

- Can generalize Quine McCluskey and ESPRESSO techniques for POS forms as well

Two Level Logic Forms





Summary

- CAD Tools for 2-level minimization
- Quine-McCluskey Method
- ESPRESSO Algorithm
- NEXT LECTURE: Combinational Logic Implementation Technologies
- READING: Katz 4.1, 4.2, Dewey 5.2, 5.3, 5.4, 5.5 5.6, 5.7, 6.2