AUTOMATICS and AUTOMATIC CONTROL

LECTURE 2

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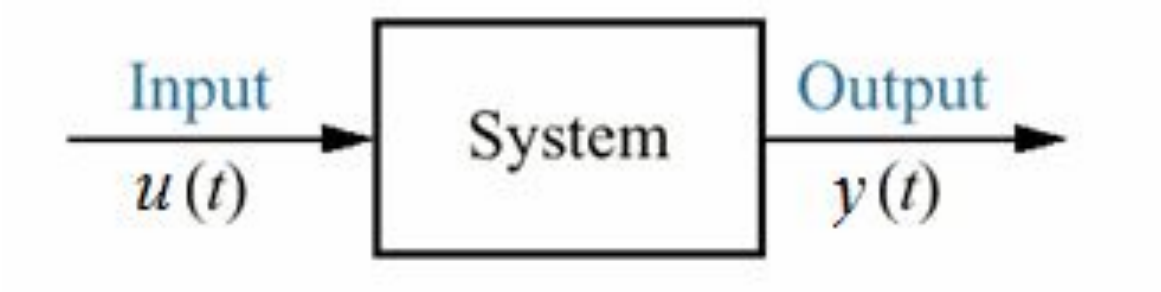
SYSTEM MODELS

In order to describe relationship between output signals of the control system and the system inputs the mathematical model is used



A matchematical model which is used to describe control system, usually is the starting point for analysies, as well as for synthesis of a control system

SYSTEM MODELS

In order to describe properties of dynamic systems (control systems), differential equations are often used as a basic models.

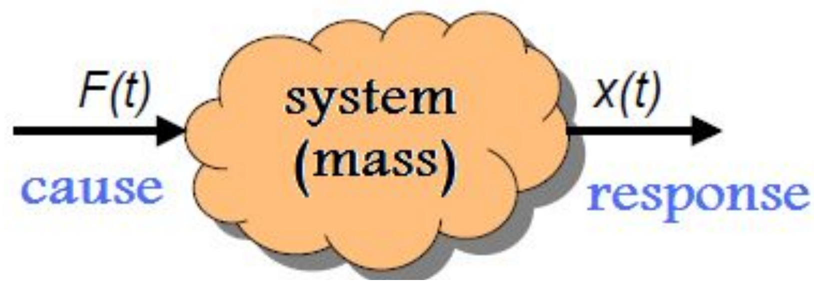
System models in form of differential equations can be obtained by using physical laws that govern the basic phenomena

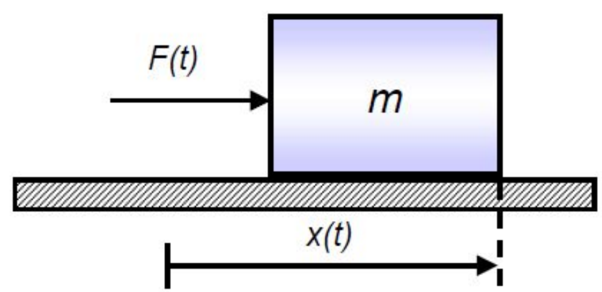
$$F = m \cdot a$$

$$F(t) = m\frac{d^2x(t)}{dt^2}$$

$$m\frac{d^2x(t)}{dt^2} - F(t) = 0$$

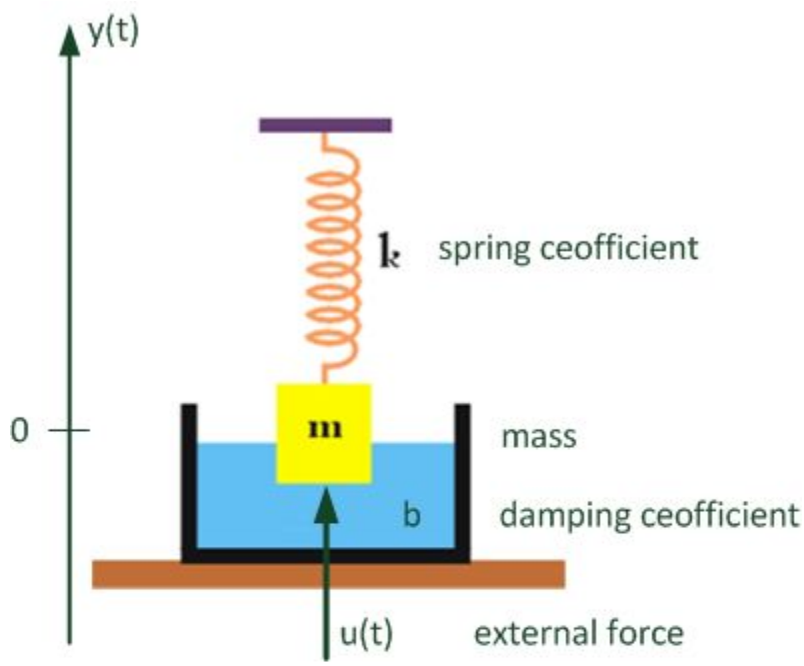
$$\boxed{f(\ddot{x}, F) = 0}$$





SYSTEM MODELS

Example1: **damped harmonic oscillator**

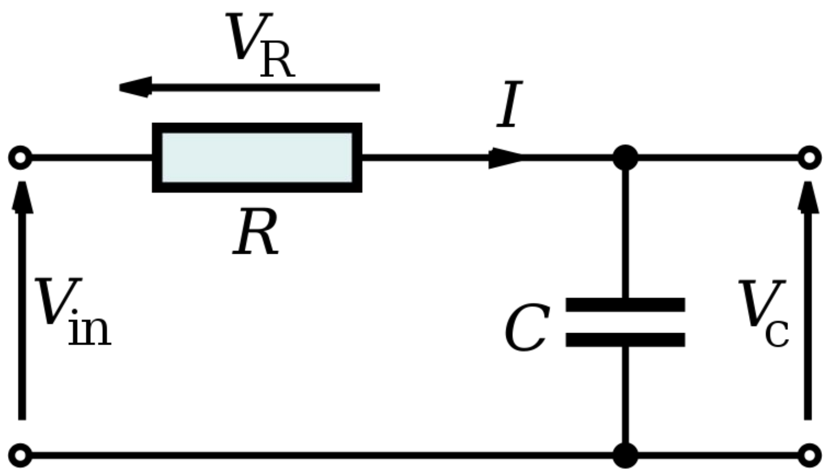


$$m\frac{d^2y}{dt^2}+b\frac{dy}{dt}+ky=u$$

Mass *m* attached to a fixed location by a spring *k*. Mass moves in a damping environment *b*

SYSTEM MODELS

Example 2: RC circuit



From II-nd **Kirchhoff's law** we can obtain:

SYSTEM MODELS

Classification of dynamical system models:

I.

1. **Linear** my” + by’ + ky = u
2. **Nonlinear** my” + by’ + f(y)y = u k=f(y)

II.

1. **with lumped parameters** (spring mass lumped in one point)
2. **with distributed parameters** (spring mass distributed towards the direction y axis)

III.

1. **Stationary** (parameters are constant *m=const.*)
2. **Nonstationary** (parameters change in time *m=f(t)*)

STATE SPACE MODELS

The state space model is an anather way (kind of an ordered way) of writing the diiferential equetions of the system.

General differential equetion:

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = b_0 u(t)$$

State space model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t)$$

STATE SPACE MODELS

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix},$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_m(t) \end{bmatrix}, \quad \mathbf{g}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_r(t) \end{bmatrix}$$

x(t) **- state variables**, are the smallest possible subset of system variables that can represent the entire state of the system at any given time. The minimum number of state variables required to represent a given system, is usually equal to the order of the system's defining differential equation

u(t) **- input variables**

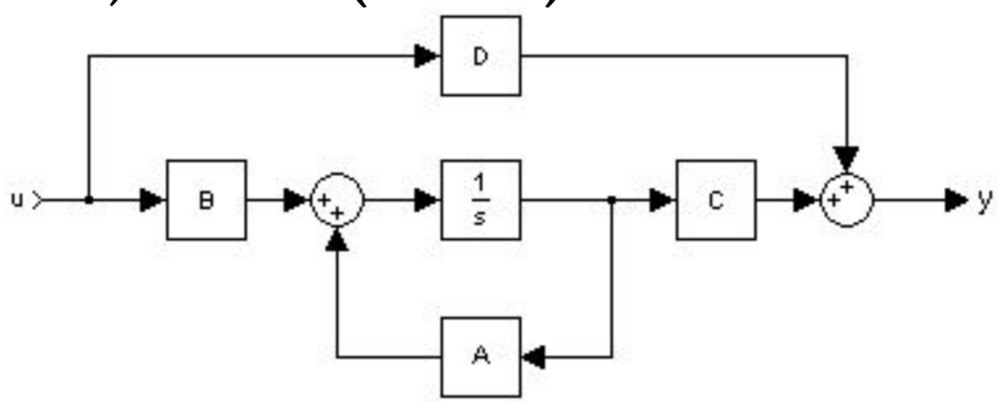
STATE SPACE MODELS

If vector functions f and g are linear:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Where *A* is called the state matrix (*n* x *n*), *B* the input matrix (*n* x *r*), C(t) the output matrix (*m* x *n*), and *D* the feedthrough (or direct transmition) matrix (*m* x *r*).



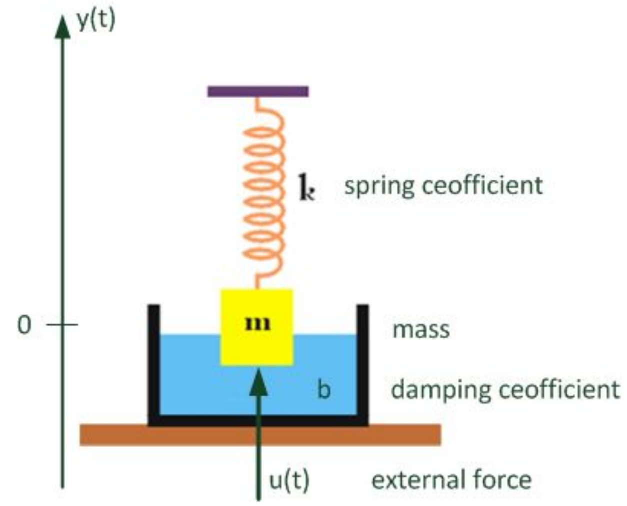
STATE SPACE MODELS

Example 1 : damped harmonic oscillator

$$m\ddot{y}(t) = u(t) - b\,\dot{y}(t) - k\,y(t)$$

Where:

- $\mathbf{y}(t)$ is position; $\dot{y}(t)$ is velocity; $\ddot{y}(t)$ is acceleration
- $u(t)$ is an applied force
- *b* is the damping coefficient
- *k* is the spring constant
- *m* is the mass of the object



The state equation would then become:

$$\begin{bmatrix}\dot{\mathbf{x}}_1(t)\\ \dot{\mathbf{x}}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k2}{m} & -\frac{k1}{m}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1(t)\\ \mathbf{x}_2(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}\mathbf{u}(t)$$

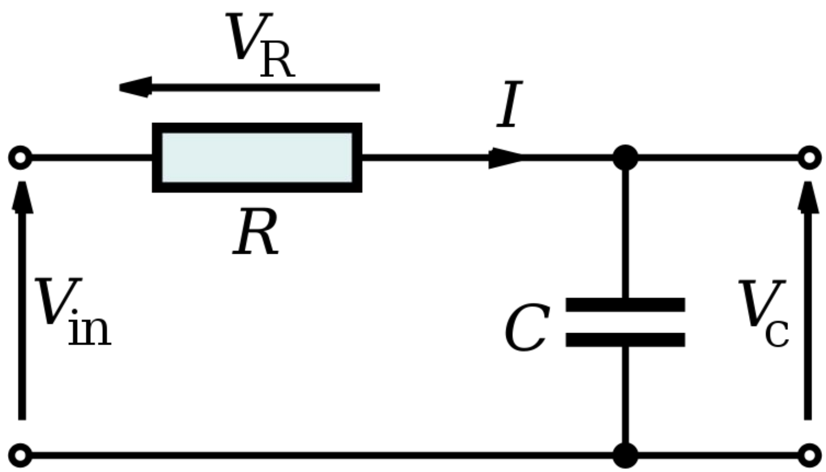
$$\mathbf{y}(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}_1(t)\\ \mathbf{x}_2(t)\end{bmatrix}$$

where:

- $x_1(t)$ represents **the position of the object**
- $x_2(t) = \dot{x}_1(t)$ **is the velocity of the object**
- the output is the position of the object

STATE SPACE MODELS

Exercise 2: RC circuit



LAPLACE TRANSFORM

The **Laplace transform** is a linear operation of a function *f*(*t*) with a real argument *t* ($t \geq 0$) that transforms it to a function *F*(*s*) with a complex argument *s*.

The Laplace transform of a function *f*(*t*), defined for all real numbers $t \geq 0$ (positive numbers), is the function *F*(*s*), defined by:

$$F(s) = \int_0^\infty f(t) \cdot e^{-st} dt$$

where: *s* is a complex number:

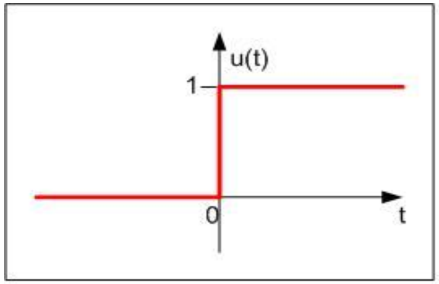
$$s = a + jb$$

with real numbers *a*, *b* *and* imaginary unit $j = \sqrt{-1}$

LAPLACE TRANSFORM

Example 1: **Heaviside step function** or **unit step function**

f(t) = ***1****(t)*



$$F(s) = \int_0^{\infty} 1e^{-st}dt$$

$$F(s) = -\frac{1}{s}\left[e^{-st}\right]_0^{\infty}$$

$$F(s) = \frac{1}{s}$$

Exercise 2: Exponential function

$$x(t) = e^{-at}$$

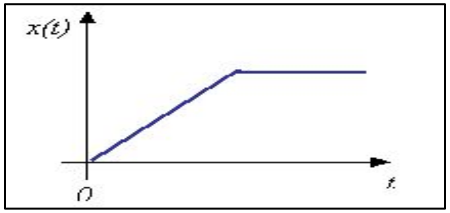
LAPLACE TRANSFORM

It is often convenient to use the differentiation property of the Laplace transform to find the transform of a function's derivative. This can be derived from the basic expression for a Laplace transform as follows:

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_{0^-}^{\infty} e^{-st} f(t)\,dt \\ &= \left[\frac{f(t)e^{-st}}{-s}\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{e^{-st}}{-s} f'(t)\,dt \quad \text{(by parts)} \\ &= \left[-\frac{f(0)}{-s}\right] + \frac{1}{s}\mathcal{L}\{f'(t)\},\end{aligned}$$

Exercise 1: Ramp function

f(t) = t



LAPLACE TRANSFORM

Laplace transforms of typical functions (TABLE OF TRANSFORMS):

f(t)	F(s)
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$

LAPLACE TRANSFORM

Laplace transforms of typical functions:

f(t)	F(s)
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(wt)$	$\frac{w}{s^2+w^2}$
$\cos(wt)$	$\frac{s}{s^2+w^2}$

LAPLACE TRANSFORM

Laplace transform properties:

1. **Linearity:**

$$f_1(t) + f_2(t) \text{ becomes } F_1(s) + F_2(s)$$
$$f_1(t) - f_2(t) \text{ becomes } F_1(s) - F_2(s)$$

The multiplication of a function by a constant, becomes the multiplication of the Laplace transform of the function by the same constant

$$a \cdot f(t) \text{ becomes } a \cdot F(s)$$

LAPLACE TRANSFORM

Laplace transform properties:

2. **Differentiation in time domain:**

$$\frac{d^n f(t)}{dt^n}$$

becomes

$$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s^0 f^{(n-1)} f(0)$$

$$= s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$$

LAPLACE TRANSFORM

Laplace transform properties:

3. **Integration in time domain:**

$$\int_0^\infty f(t) \text{ becomes } \frac{1}{s}F(s)$$

4. **Time shift:**

$$f(t-t_0) \quad \text{becomes} \quad e^{-t_o s}F(s)$$

LAPLACE TRANSFORM

Laplace transform properties:

5. **Initial value theorem:**

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t)$$

6. **Final value theorem:**

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$$

The final value theorem is useful because it gives the long-term behaviour without having to perform difficult algebra (solving differential equations, etc.)

INVERSE LAPLACE TRANSFORM

The Inverse Laplace Transform is defined by:

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{ts}\,ds$$

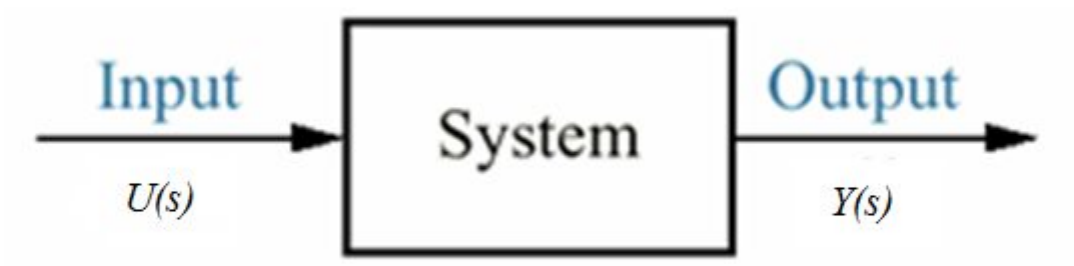
If the algebraic equation is solved in s, we can find the solution of the differential equation using Inverse Laplace transform.

The most common procedure is to break the function F(s) in fractions, calculate the inverse transforms in each, using the transform table and add the analytical expressions for each of them to find the function f(t) **(partial fractions decomposition or expansion**)

TRANSFER FUNCTION

Transfer function is defined as the ratio of the Laplace transform of the output signal to the Laplace transform of the input signal under the assumption that all initial conditions are zero)

$$G(s) = \frac{Y(s)}{U(s)}$$



TRANSFER FUNCTION

From general differential equetion:

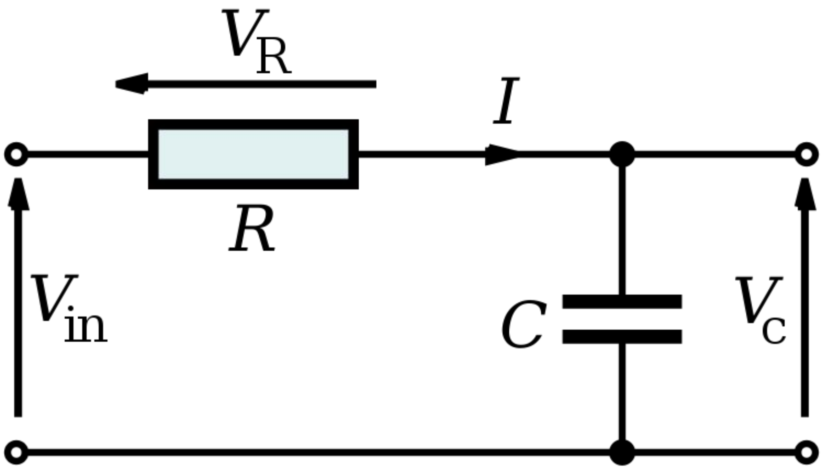
$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \ldots + a_1 y'(t) + a_0 y(t) = b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \ldots + b_1 u'(t) + b_0 u(t)$$

we can obtain transfer function :

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

TRANSFER FUNCTION

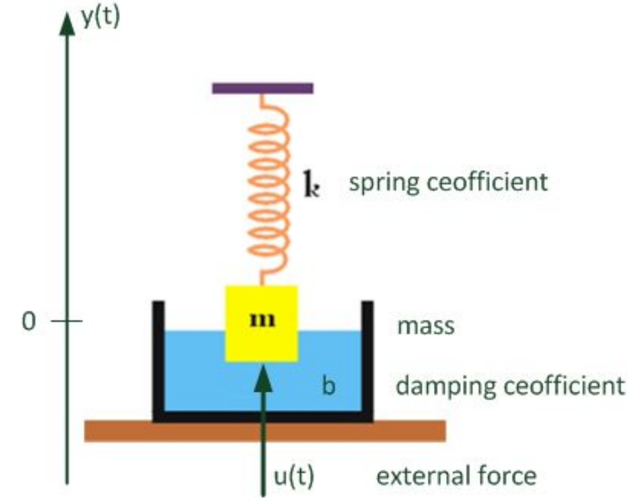
Example1 : RC circuit



TRANSFER FUNCTION

Example2 : damped harmonic oscillator

$$m\ddot{y}(t) = u(t) - b\,\dot{y}(t) - k\,y(t)$$



TRANSFER FUNCTION

If we obtain the roots in the numerator and denominator of transfer function:

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

we can change the standard form of transfer function to the form of ZERO-POLE

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

z1, z2,... z3: zeros, numbers of s which makes the transfer function zero
p1, p2,... p3: poles, numbers of s which make the transfer function infinity
K is the gain

Zeros and poles could be complex numbers

TRANSFER FUNCTION AND STATE SPACE

We can transform State space model to transfer function by performing following operations:

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$

$$Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s)$$

Since the transfer function was previously defined as the ratio of the Laplace transform of the output to the Laplace transform of the input when the initial conditions were zero, we set x(0) in the previous equation to be zero. We operate and substitute and then we have:

$$Y(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\right]U(s)$$

,I – unit matrix

and finally:

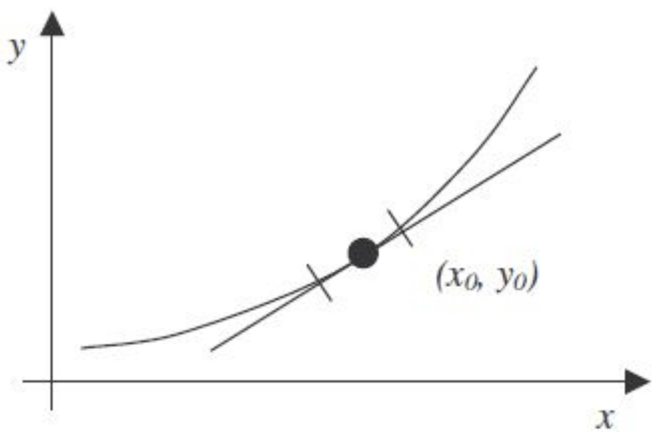
$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

MODEL LINEARIZATION

Control systems often works with small changes of input and output quantities around some given steady state value. For this small changes the system can be described in an approximate way by linear equation (differential equation)

- Static linearization

$$f(x,y)=0$$



Taylor series expansion in steady state point:

$$f(x,y)=f(x_0,y_0)+\left.\frac{\partial f(x,y)}{\partial x}\right|_{\substack{x=x_0\\y=y_0}}(x-x_0)+\left.\frac{\partial f(x,y)}{\partial y}\right|_{\substack{x=x_0\\y=y_0}}(y-y_0)$$

because $f(x_0,y_0)=0$:

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{\substack{x=x_0\\y=y_0}}(x-x_0)+\left.\frac{\partial f(x,y)}{\partial y}\right|_{\substack{x=x_0\\y=y_0}}(y-y_0)=0 \quad \Rightarrow \quad y=y_0-\left.\frac{\frac{\partial f(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y}}\right|_{\substack{x=x_0\\y=y_0}}(x-x_0)$$

MODEL LINEARIZATION

- Static linearization example: $f(x,y) = x - \sqrt{x \cdot y} - 1 = 0$

$$(x_0, y_0) = (2, 0.5)$$

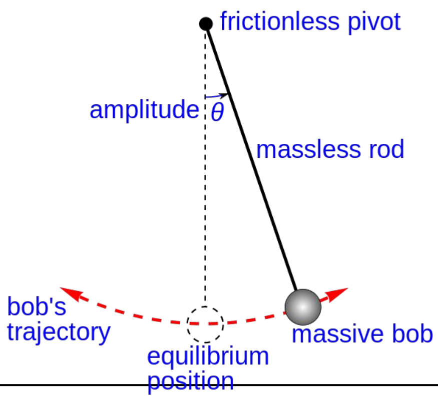
$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{\substack{x=x_0 \\ y=y_0}} = 1 - \frac{y_0}{2\sqrt{x_0 \cdot y_0}} = \frac{3}{4}$$

$$\left.\frac{\partial f(x,y)}{\partial y}\right|_{\substack{x=x_0 \\ y=y_0}} = -\frac{x_0}{2\sqrt{x_0 \cdot y_0}} = -1$$

$$y = 0.5 + 0.75 \cdot (x - 2) = 0.75 \cdot x - 1$$

- Static linearization exercise:

Pendulum T=m*g*l*sin(x) (To,xo) = (0,0)



MODEL LINEARIZATION

- Linearization of differential equation (dynamic linearization)

$$f\left(x,\dot{x},\dots,x^{(m)},y,\dot{y},\dots,y^{(n)}\right)=0$$

Taylor series expansion in steady state point $S_0=(x_0,0,\dots,0,y_0,0,\dots,0)$
:

$$f\left(x,\dot{x},\dots,x^{(m)},y,\dot{y},\dots,y^{(n)}\right)=f\left(x_0,0,\dots,0,y_0,0,\dots,0\right)+$$

$$+\left.\frac{\partial f}{\partial x}\right|_{S_0}\cdot(x-x_0)+\left.\frac{\partial f}{\partial \dot{x}}\right|_{S_0}\cdot(\dot{x}-0)+\dots+\left.\frac{\partial f}{\partial x^{(m)}}\right|_{S_0}\cdot\left(x^{(m)}-0\right)+$$

$$+\left.\frac{\partial f}{\partial y}\right|_{S_0}\cdot(y-y_0)+\left.\frac{\partial f}{\partial \dot{y}}\right|_{S_0}\cdot(\dot{y}-0)+\dots+\left.\frac{\partial f}{\partial y^{(n)}}\right|_{S_0}\cdot\left(y^{(n)}-0\right)$$

because $f(x_0,0,\dots,0,y_0,0,\dots,0)=0$:

$$\left.\frac{\partial f}{\partial x}\right|_{S_0}\cdot(x-x_0)+\left.\frac{\partial f}{\partial \dot{x}}\right|_{S_0}\cdot\dot{x}+\dots+\left.\frac{\partial f}{\partial x^{(m)}}\right|_{S_0}\cdot x^{(m)}+$$

$$+\left.\frac{\partial f}{\partial y}\right|_{S_0}\cdot(y-y_0)+\left.\frac{\partial f}{\partial \dot{y}}\right|_{S_0}\cdot\dot{y}+\dots+\left.\frac{\partial f}{\partial y^{(n)}}\right|_{S_0}\cdot y^{(n)}=0$$

MODEL LINEARIZATION

- Linearization of differential equation (dynamic linearization)

example:

DC generator equation

$$f(u,y) = L\dot{y} + (c_1 u + R)y - c_2 u_w = 0$$

$$\left.\frac{\partial f(u,y)}{\partial \dot{y}}\right|_0 = L \qquad \left.\frac{\partial f(u,y)}{\partial y}\right|_0 = c_1 u_0 + R \qquad \left.\frac{\partial f(u,y)}{\partial u}\right|_0 = c_1 y_0$$

$$L\dot{y} + (c_1 u_0 + R)(y - y_0) + c_1 y_0 (u - u_0) = 0$$

THANK YOU