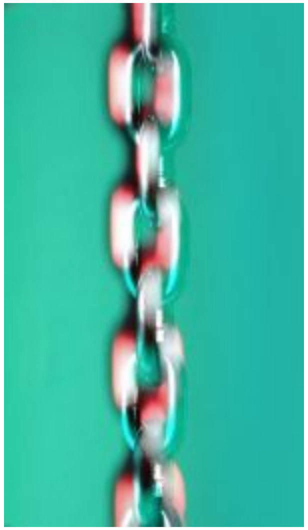
Learning Objectives for Section 11.4 The Chain Rule

- The student will be able to form the composition of two functions.
- The student will be able to apply the general power rule.
- The student will be able to apply the chain rule.



Composite Functions

Definition: A function m is a **composite** of functions f and g if

$$m(x) = f[g(x)]$$

The domain of m is the set of all numbers x such that x is in the domain of g and $g(x)$ is in the domain of f.

General Power Rule

We have already made extensive use of the power rule:

$$\frac{d}{dx}x^n = nx^{n-1}$$

Now we want to generalize this rule so that we can differentiate composite functions of the form $[u(x)]^n$, where $u(x)$ is a differentiable function. Is the power rule still valid if we replace x with a function $u(x)$?

Example

Let $u(x) = 2x^2$ and $f(x) = [u(x)]^3 = 8x^6$. Which of the following is $f'(x)$?

(a) $3[u(x)]^2$ (b) $3[u'(x)]^2$ (c) $3[u(x)]^2 u'(x)$

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We know that $f'(x) = 48x^5$.

(a) $3[u(x)]^2 = 3(2x^2)^2 = 3(4x^4) = 12\,x^4$. This is not correct.

(b) $3[u'(x)]^2 = 3(4x)^2 = 3(16x^2) = 48x^2$. This is not correct.

(c) $3[u(x)]^2\, u'(x) = 3[2x^2]^2(4x) = 3(4x^4)(4x) = 48x^5$. This is the correct choice.

Generalized Power Rule

What we have seen is an example of the **generalized power rule:** If u is a function of x, then

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

For example, $$\frac{d}{dx}(x^2+3x+5)^3 = 3(x^2+3x+5)^2(2x+3)$$

Here u is x^2+3x+5 and $\frac{du}{\mathrm{dx}} = 2x+3$

Chain Rule

We have used the generalized power rule to find derivatives of composite functions of the form $f(g(x))$ where $f(u) = u^n$ is a power function. But what if f is not a power function? It is a more general rule, the **chain rule**, that enables us to compute the derivatives of many composite functions of the form f(g(x)).

Chain Rule: If $y = f(u)$ and $u = g(x)$ define the composite function $y = f(u) = f[g(x)]$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \quad \text{provided } \frac{dy}{du} \text{ and } \frac{du}{dx} \text{ exist.}$$

Generalized Derivative Rules

1. $$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x)$$

   If $y = u^n$, then
   $y' = nu^{n-1} \cdot du/dx$

2. $$\frac{d}{dx}\ln[f(x)] = \frac{1}{f(x)} \cdot f'(x)$$

   If $y = \ln u$, then
   $y' = 1/u \cdot du/dx$

3. $$\frac{d}{dx}e^{f(x)} = e^{f(x)} f'(x)$$

   If $y = e^u$, then
   $y' = e^u \cdot du/dx$

Examples for the Power Rule

Chain rule terms are marked:

$$y = x^5,\ y' = 5x^4$$

$$y = (2x)^5,\ y' = 5(2x)^4\boxed{(2)} = 160x^4$$

$$y = (2x^3)^5,\ y' = 5(2x^3)^4\boxed{(6x^2)} = 480x^{14}$$

$$y = (2x+1)^5,\ y' = 5(2x+1)^4\boxed{(2)} = 10(2x+1)^4$$

$$y = (e^x)^5,\ y' = 5(e^x)^4\boxed{(e^x)} = 5e^{5x}$$

$$y = (\ln x)^5,\ y' = 5(\ln x)^4\boxed{(1/x)}$$

Examples for Exponential Derivatives

$$\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$$

$$y = e^{3x},\ y' = e^{3x}\ (3) = 3e^{3x}$$

$$y = e^{3x+1},\ y' = e^{3x+1}\ (3) = 3e^{3x+1}$$

$$y = e^{4x^2-3x+5},\ y' = e^{4x^2-3x+5}\ (8x-3)$$

$$y = e^{\ln x} = x,\ y' = e^{\ln x}\ (\frac{1}{x}) = \frac{x}{x} = 1$$

Examples for Logarithmic Derivatives

$$\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$$

$$y = \ln(4x), \qquad y' = \frac{1}{4x}\cdot 4 = \frac{1}{x}$$

$$y = \ln(4x+1), \qquad y' = \frac{1}{4x+1}\cdot 4 = \frac{4}{4x+1}$$

$$y = \ln(x^2), \qquad y' = \frac{1}{x^2}\cdot(2x) = \frac{2}{x}$$

$$y = \ln(x^2+2x-4), \qquad y' = \frac{1}{x^2+2x-4}\cdot(2x+2) = \frac{2x+2}{x^2+2x-4}$$