7

VECTORS AND THE GEOMETRY OF SPACE

A line in the *xy*-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given.

- The equation of the line can then be written using the point-slope form.

Equations of Lines and Planes

In this section, we will learn how to:

Define three-dimensional lines and planes using vectors.

EQUATIONS OF LINES

A line *L* in three-dimensional (3-D) space is determined when we know:

- A point $P_0(x_0, y_0, z_0)$ on *L*
- The direction of *L*

EQUATIONS OF LINES

In three dimensions, the direction of a line is conveniently described by a vector.

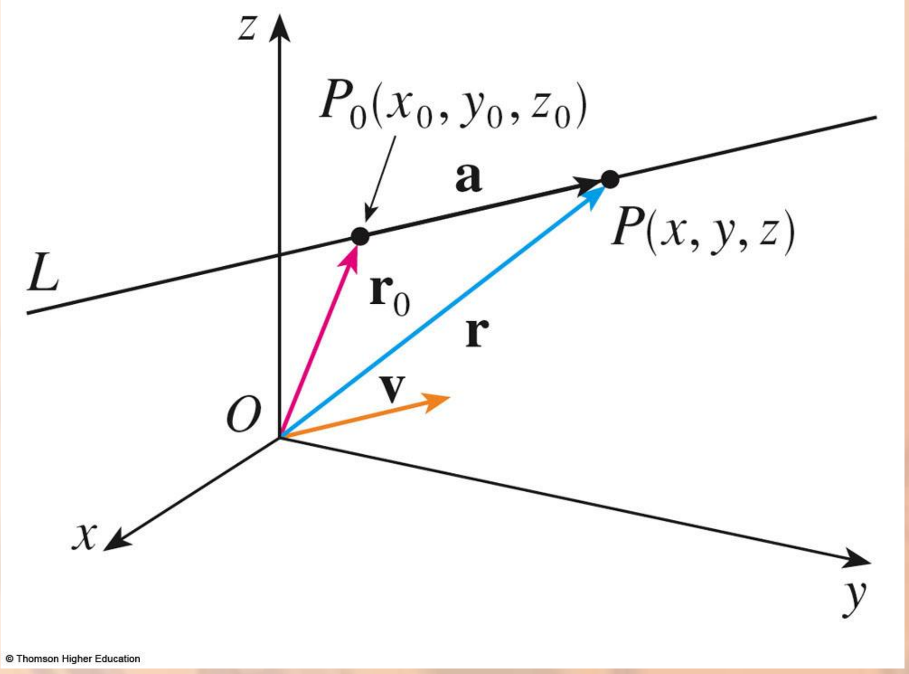
So, we let **v** be a vector parallel to *L*.

- Let *P*(*x*, *y*, *z*) be an arbitrary point on *L.*
- Let $\mathbf{r}_0$ and **r** be the position vectors of P_0 and *P.* That is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$.

EQUATIONS OF LINES

If **a** is the vector with representation $\overrightarrow{P_0P}$, then the Triangle Law for vector addition gives:

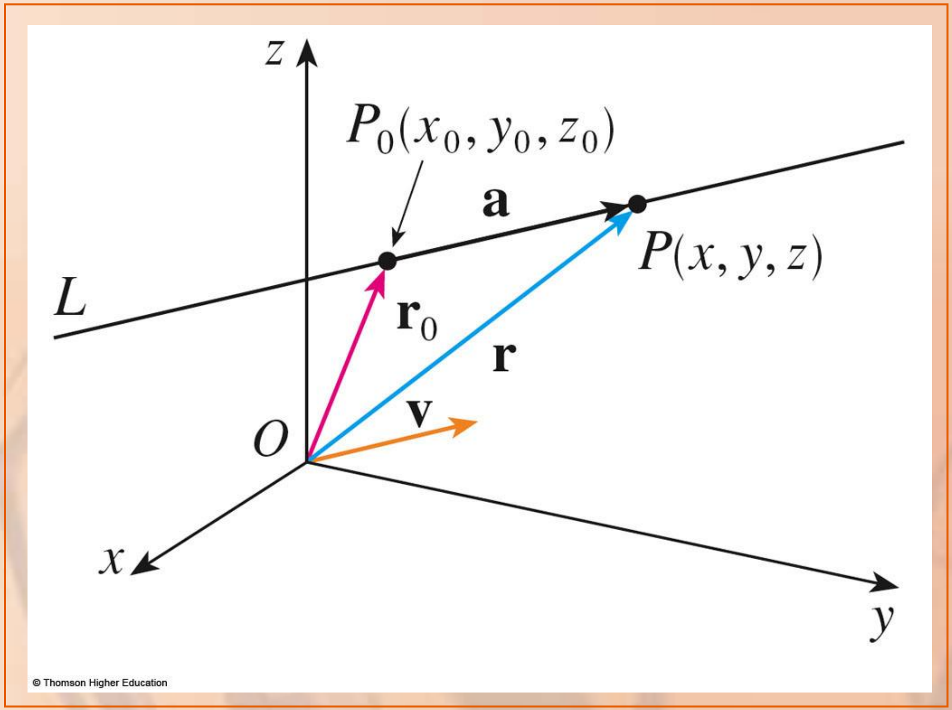
$$\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$$



EQUATIONS OF LINES

However, since **a** and **v** are parallel vectors, there is a scalar *t* such that

$$\mathbf{a} = t\mathbf{v}$$



Thus,

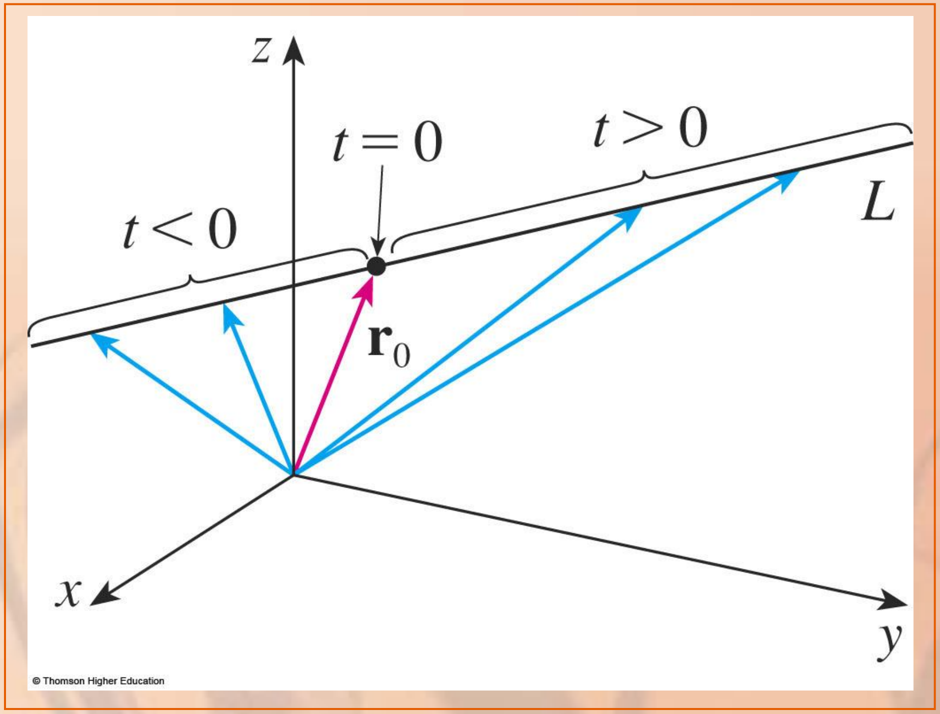
$$\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v}$$

- This is a vector equation of *L*.

VECTOR EQUATION

Each value of the parameter *t* gives the position vector **r** of a point on *L*.

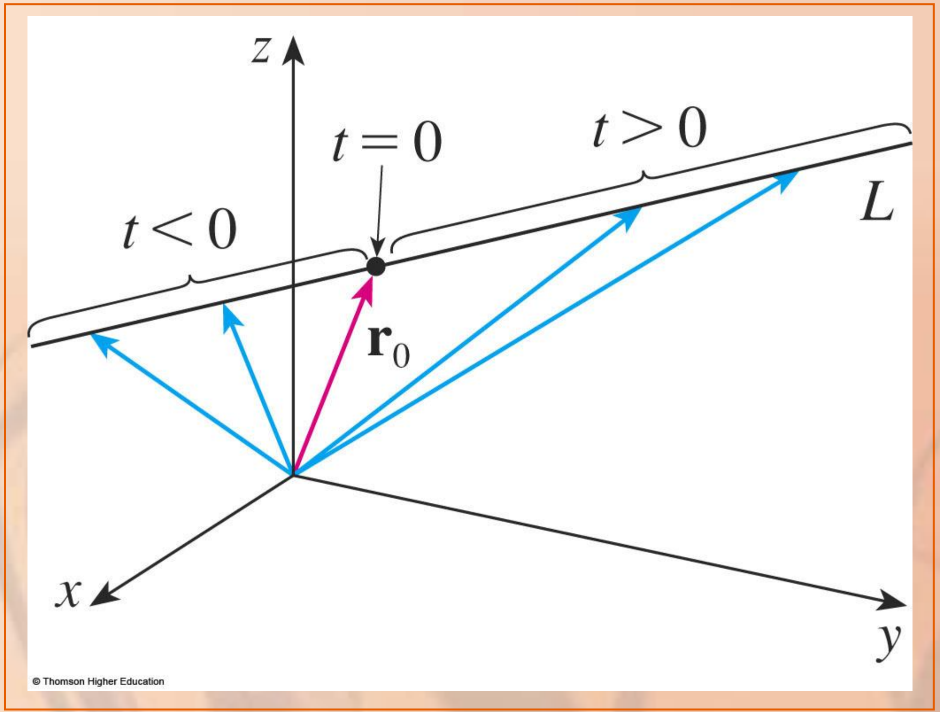
- That is, as *t* varies, the line is traced out by the tip of the vector **r**.



VECTOR EQUATION

Positive values of *t* correspond to points on *L* that lie on one side of P_0.

Negative values correspond to points that lie on the other side.



VECTOR EQUATION

If the vector **v** that gives the direction of the line *L* is written in component form as $\mathbf{v} = \langle a, b, c \rangle$, then we have:

$$t\mathbf{v} = \langle ta, tb, tc \rangle$$

VECTOR EQUATION

We can also write:

$$\mathbf{r} = \langle x, y, z \rangle \quad \text{and} \quad \mathbf{r}_0 = \langle x_0, y_0, \mathbf{z}_0 \rangle$$

- So, vector Equation 1 becomes:

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Two vectors are equal if and only if corresponding components are equal.

Hence, we have the following three scalar equations.

SCALAR EQUATIONS OF A LINE **Equations 2**

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

- Where, $t \in \mathbb{R}$

PARAMETRIC EQUATIONS

These equations are called parametric equations of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$.

- Each value of the parameter t gives a point (x, y, z) on L.

a. Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector **i** + 4 **j** – 2 **k**.

b. Find two other points on the line.

Here, $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\,\mathbf{i} + \mathbf{j} + 3\,\mathbf{k}$

and $\mathbf{v} = \mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k}$

- So, vector Equation 1 becomes:

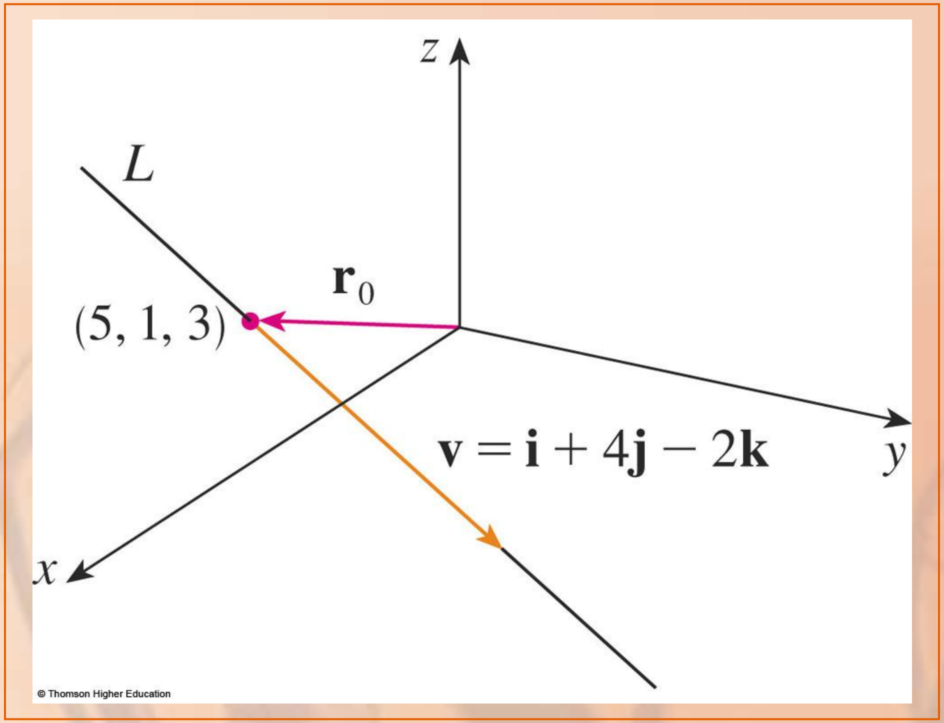
$$\mathbf{r} = (5\,\mathbf{i} + \mathbf{j} + 3\,\mathbf{k}) + t(\mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k})$$

or

$$\mathbf{r} = (5 + t)\,\mathbf{i} + (1 + 4t)\,\mathbf{j} + (3 - 2t)\,\mathbf{k}$$

Parametric equations are:

$$x = 5 + t \qquad y = 1 + 4t \qquad z = 3 - 2t$$



Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$.

So, (6, 5, 1) is a point on the line.

- Similarly, $t = -1$ gives the point (4, –3, 5).

The vector equation and parametric equations of a line are not unique.

- If we change the point or the parameter or choose a different parallel vector, then the equations change.

For instance, if, instead of (5, 1, 3), we choose the point (6, 5, 1) in Example 1, the parametric equations of the line become:

$$x = 6 + t \qquad y = 5 + 4t \qquad z = 1 - 2t$$

Alternatively, if we stay with the point (5, 1, 3) but choose the parallel vector 2 **i** + 8 **j** – 4 **k**, we arrive at:

$$x = 5 + 2t \qquad y = 1 + 8t \qquad z = 3 - 4t$$

DIRECTION NUMBERS

In general, if a vector **v** = $\langle a, b, c \rangle$ is used to describe the direction of a line *L*, then the numbers *a*, *b*, and *c* are called direction numbers of *L*.

DIRECTION NUMBERS

Any vector parallel to **v** could also be used.

Thus, we see that any three numbers proportional to *a*, *b*, and *c* could also be used as a set of direction numbers for *L*.

Equations 3

Another way of describing a line *L* is to eliminate the parameter *t* from Equations 2.

- If none of *a*, *b*, or *c* is 0, we can solve each of these equations for *t*, equate the results, and obtain the following equations.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called symmetric equations of *L*.

SYMMETRIC EQUATIONS

Notice that the numbers *a*, *b*, and *c* that appear in the denominators of Equations 3 are direction numbers of *L*.

- That is, they are components of a vector parallel to *L*.

SYMMETRIC EQUATIONS

If one of *a*, *b*, or *c* is 0, we can still eliminate *t*.

For instance, if $a = 0$, we could write the equations of *L* as:

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- This means that *L* lies in the vertical plane $x = x_0$.

a. Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.

b. At what point does this line intersect the xy-plane?

We are not explicitly given a vector parallel to the line.

However, observe that the vector **v** with representation $\overrightarrow{AB}$ is parallel to the line and

$$\mathbf{v} = \langle 3 - 2, -1 - 4, 1 - (-3) \rangle = \langle 1, -5, 4 \rangle$$

Thus, direction numbers are:

$$a = 1,\ b = -5,\ c = 4$$

Taking the point (2, 4, –3) as P_0,

we see that:

- Parametric Equations 2 are:

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t$$

- Symmetric Equations 3 are:

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}$$

The line intersects the *xy*-plane when $z = 0$.

So, we put $z = 0$ in the symmetric equations and obtain:

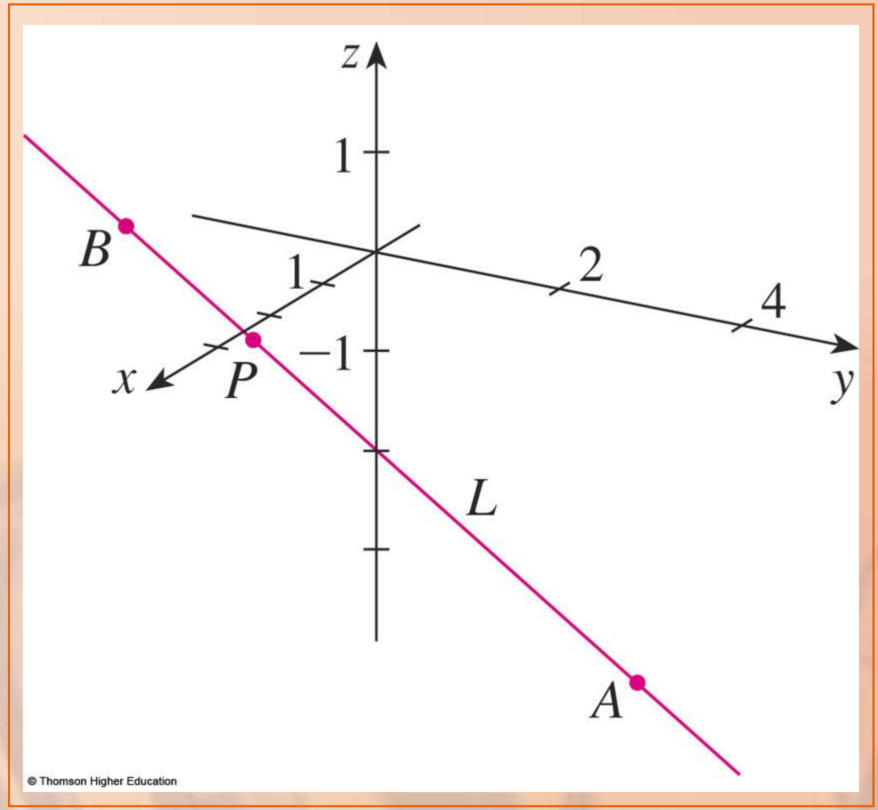
$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{3}{4}$$

- This gives $x = \frac{11}{4}$ and $y = \frac{1}{4}$.

 Example 2 b

The line intersects the *xy*-plane at the point

$$\left(\frac{11}{4},\frac{1}{4},0\right)$$



In general, the procedure of Example 2 shows that direction numbers of the line *L* through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are: $x_1 - x_0$ $y_1 - y_0$ $z_1 - z_0$

- So, symmetric equations of *L* are:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

EQUATIONS OF LINE SEGMENTS

Often, we need a description, not of an entire line, but of just a line segment.

- How, for instance, could we describe the line segment *AB* in Example 2?

EQUATIONS OF LINE SEGMENTS

If we put $t = 0$ in the parametric equations in Example 2 a, we get the point (2, 4, –3).

If we put $t = 1$, we get (3, –1, 1).

So, the line segment *AB* is described by either:

- The parametric equations

  $$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t$$

  where $0 \leq t \leq 1$

- The corresponding vector equation

  $$\mathbf{r}(t) = \langle 2 + t, 4 - 5t, -3 + 4t \rangle$$

  where $0 \leq t \leq 1$

In general, we know from Equation 1 that the vector equation of a line through the (tip of the) vector $\mathbf{r}_0$ in the direction of a vector $\mathbf{v}$ is:

$$\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v}$$

EQUATIONS OF LINE SEGMENTS

If the line also passes through (the tip of) $\mathbf{r}_1$, then we can take $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$.

So, its vector equation is:

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1$$

- The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the parameter interval $0 \le t \le 1$.

EQUATIONS OF LINE SEGMENTS **Equation 4**

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1$$

where $0 \le t \le 1$

Show that the lines L_1 and L_2 with parametric equations

$$x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$

$$x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are skew lines.

- That is, they do not intersect and are not parallel, and therefore do not lie in the same plane.

The lines are not parallel because the corresponding vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel.

- Their components are not proportional.

If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$1 + t = 2s$$

$$-2 + 3t = 3 + s$$

$$4 - t = -3 + 4s$$

However, if we solve the first two equations, we get:

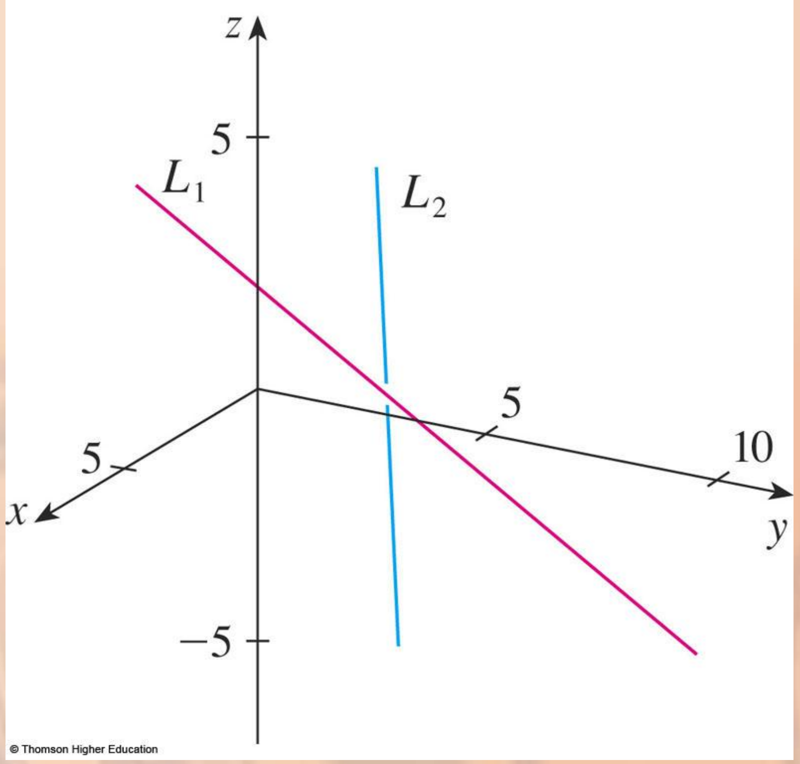
$$t = \frac{11}{5} \quad \text{and} \quad s = \frac{8}{5}$$

- These values don't satisfy the third equation.

Thus, there are no values of *t* and *s* that satisfy the three equations.

- So, L_1 and L_2 do not intersect.

Hence, L_1 and L_2 are skew lines.



PLANES

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe.

- A single vector parallel to a plane is not enough to convey the 'direction' of the plane.

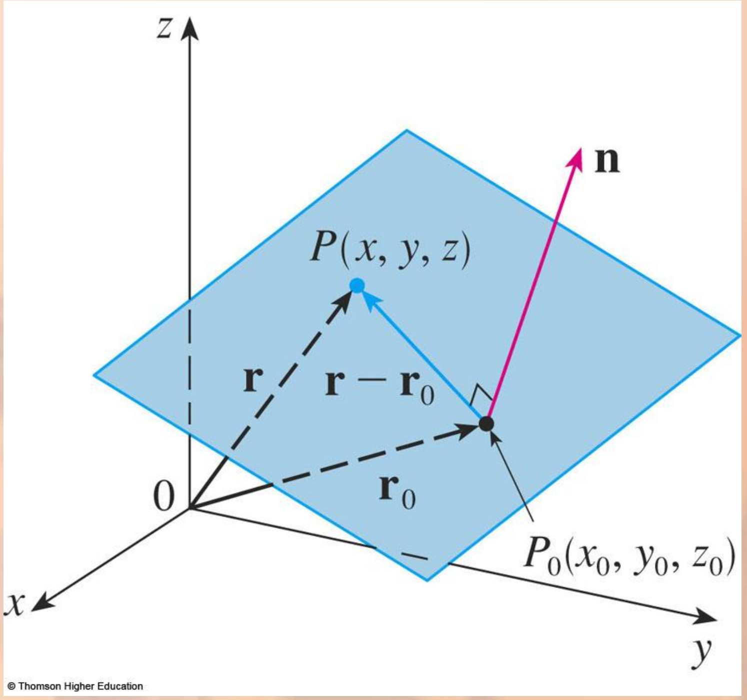
PLANES

However, a vector perpendicular to the plane does completely specify its direction.

PLANES

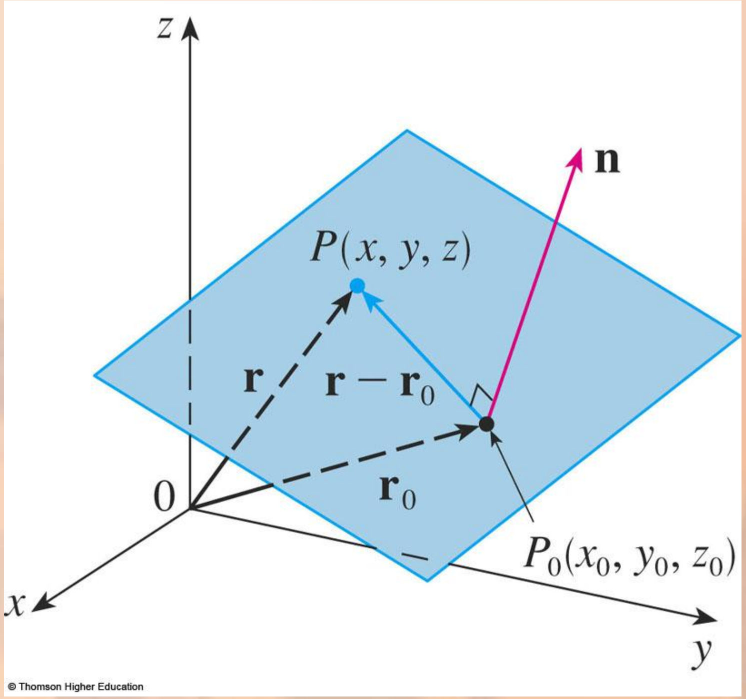
Thus, a plane in space is determined by:

- A point $P_0(x_0, y_0, z_0)$ in the plane
- A vector **n** that is orthogonal to the plane



NORMAL VECTOR

This orthogonal vector **n** is called a normal vector.

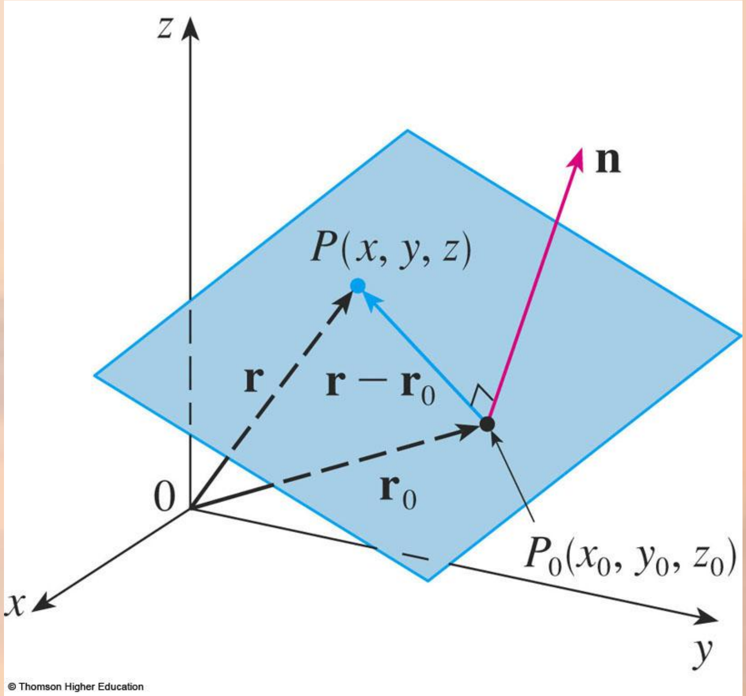


PLANES

Let $P(x, y, z)$ be an arbitrary point in the plane.

Let $\mathbf{r}_0$ and $\mathbf{r}_1$ be the position vectors of P_0 and P.

- Then, the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$



PLANES

The normal vector **n** is orthogonal to every vector in the given plane.

In particular, **n** is orthogonal to $\mathbf{r} - \mathbf{r}_0$.

Thus, we have:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

That can also be written as:

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

VECTOR EQUATION

Either Equation 5 or Equation 6 is called a vector equation of the plane.

EQUATIONS OF PLANES

To obtain a scalar equation for the plane, we write:

$$\mathbf{n} = \langle a, b, c \rangle$$

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$$

Then, the vector Equation 5 becomes:

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

That can also be written as:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- This equation is the scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$.

Find an equation of the plane through the point (2, 4, –1) with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$.

Find the intercepts and sketch the plane.

In Equation 7, putting

$$a = 2,\ b = 3,\ c = 4,\ x_0 = 2,\ y_0 = 4,\ z_0 = -1,$$

we see that an equation of the plane is:

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

$$2x + 3y + 4z = 12$$

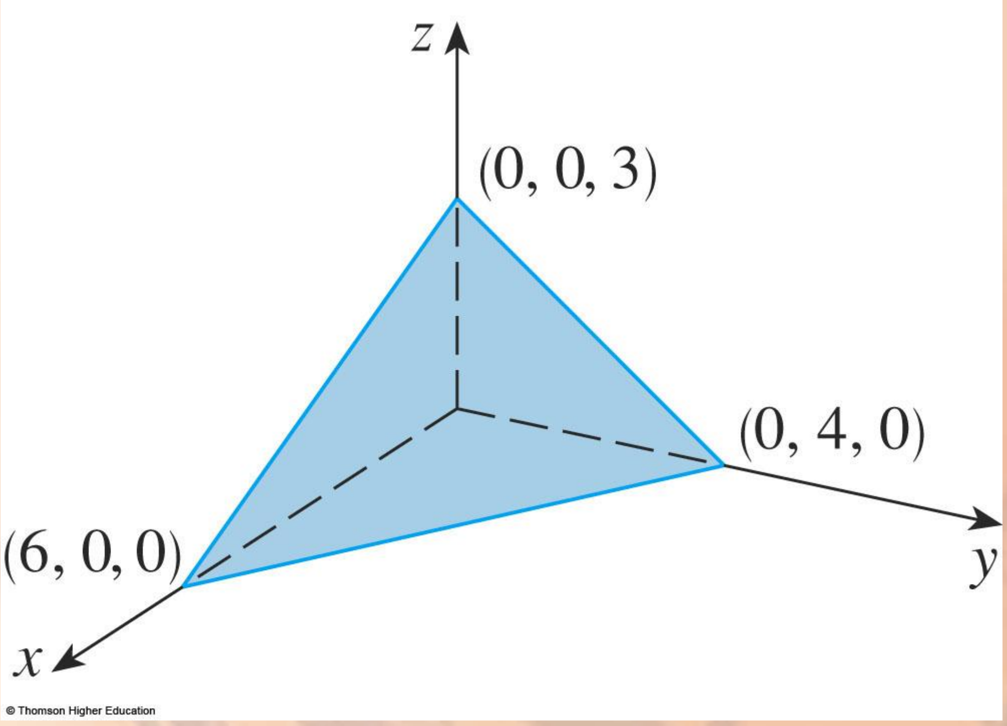
To find the x-intercept, we set $y = z = 0$ in the equation, and obtain $x = 6$.

Similarly, the y-intercept is 4 and the z-intercept is 3.

EQUATIONS OF PLANES

Example 4

This enables us to sketch the portion of the plane that lies in the first octant.



By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as follows.

Equation 8

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$

- This is called a linear equation in *x*, y, and z.

LINEAR EQUATION

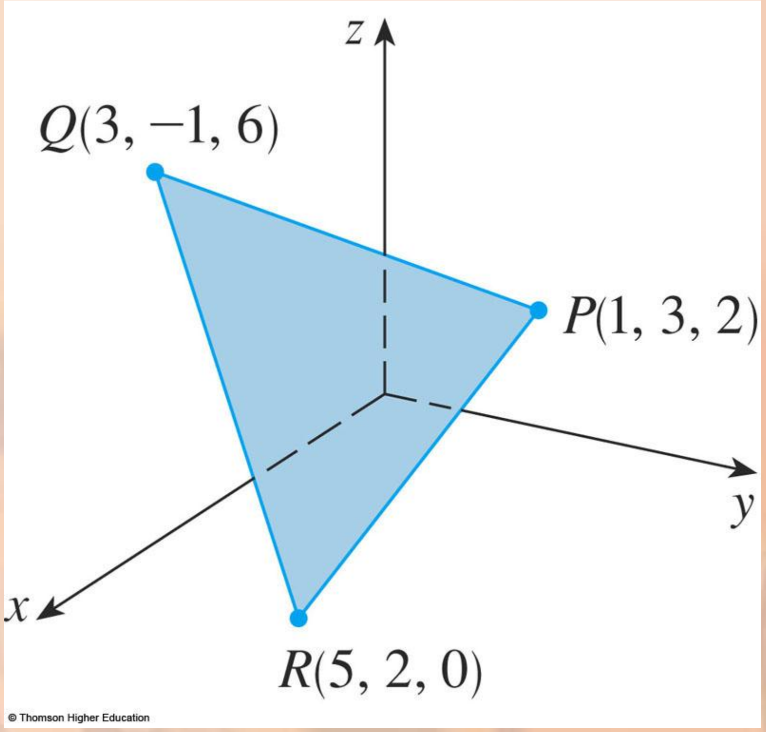
Conversely, it can be shown that, if *a*, *b*, and *c* are not all 0, then the linear Equation 8 represents a plane with normal vector $\langle a, b, c \rangle$.

- See Exercise 77.

EQUATIONS OF PLANES **Example 5**

Find an equation of the plane that passes through the points

$P(1, 3, 2)$, $Q(3, -1, 6)$, $R(5, 2, 0)$



The vectors **a** and **b** corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are:

$$\mathbf{a} = \langle 2, -4, 4 \rangle \qquad \mathbf{b} = \langle 4, -1, -2 \rangle$$

Since both **a** and **b** lie in the plane, their cross product **a** x **b** is orthogonal to the plane and can be taken as the normal vector.

Thus,

$$\mathbf{n} = \mathbf{a} \times \mathbf{b}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= 12\,\mathbf{i} + 20\,\mathbf{j} + 14\,\mathbf{k}$$

With the point $P(1, 2, 3)$ and the normal vector **n**, an equation of the plane is:

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

Find the point at which the line with parametric equations

$$x = 2 + 3t \qquad y = -4t \qquad z = 5 + t$$

intersects the plane

$$4x + 5y - 2z = 18$$

We substitute the expressions for x, y, and z from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

That simplifies to $-10t = 20$.

Hence, $t = -2$.

- Therefore, the point of intersection occurs when the parameter value is $t = -2$.

Then,

$$x = 2 + 3(-2) = -4$$

$$y = -4(-2) = 8$$

$$z = 5 - 2 = 3$$

- So, the point of intersection is (–4, 8, 3).

PARALLEL PLANES

Two planes are parallel if their normal vectors are parallel.

PARALLEL PLANES

For instance, the planes

$$x + 2y - 3z = 4 \text{ and } 2x + 4y - 6z = 3$$

are parallel because:

- Their normal vectors are

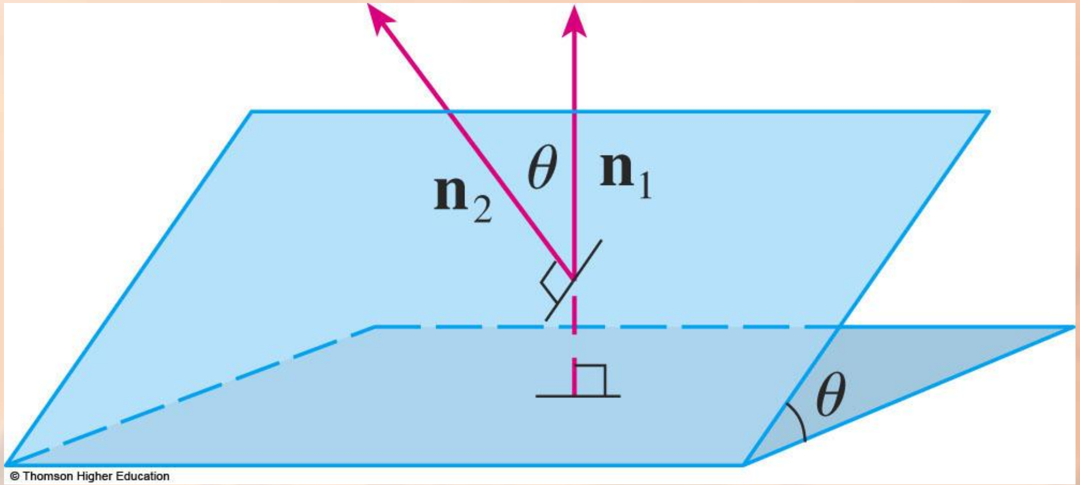
  $$\mathbf{n}_1 = \langle 1, 2, -3 \rangle \text{ and } \mathbf{n}_2 = \langle 2, 4, -6 \rangle$$

  and $\mathbf{n}_2 = 2\mathbf{n}_1$.

NONPARALLEL PLANES

If two planes are not parallel, then

- They intersect in a straight line.
- The angle between the two planes is defined as the acute angle between their normal vectors.



a. Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$

b. Find symmetric equations for the line of intersection L of these two planes.

The normal vectors of these planes are:

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle \qquad \mathbf{n}_2 = \langle 1, -2, 3 \rangle$$

So, if θ is the angle between the planes, Corollary 6 in Section 12.3 gives:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{1(1)+1(-2)+1(3)}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^{\circ}$$

We first need to find a point on L.

- For instance, we can find the point where the line intersects the xy-plane by setting $z = 0$ in the equations of both planes.
- This gives the equations
  $$x + y = 1 \text{ and } x - 2y = 1$$
  whose solution is $x = 1$, $y = 0$.
- So, the point (1, 0, 0) lies on L.

As L lies in both planes, it is perpendicular to both the normal vectors.

- Thus, a vector **v** parallel to L is given by the cross product

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

So, the symmetric equations of *L* can be written as:

$$\frac{x-1}{5}=\frac{y}{-2}=\frac{z}{-3}$$

NOTE

A linear equation in *x*, *y*, and *z* represents a plane.

Also, two nonparallel planes intersect in a line.

- It follows that two linear equations can represent a line.

NOTE

The points (x, y, z) that satisfy both

$$a_1x + b_1y + c_1z + d_1 = 0$$

and $$a_2x + b_2y + c_2z + d_2 = 0$$

lie on both of these planes.

- So, the pair of linear equations represents the line of intersection of the planes (if they are not parallel).

NOTE

For instance, in Example 7, the line L was given as the line of intersection of the planes

$$x + y + z = 1 \text{ and } x - 2y + 3z = 1$$

NOTE

The symmetric equations that we found for *L* could be written as:

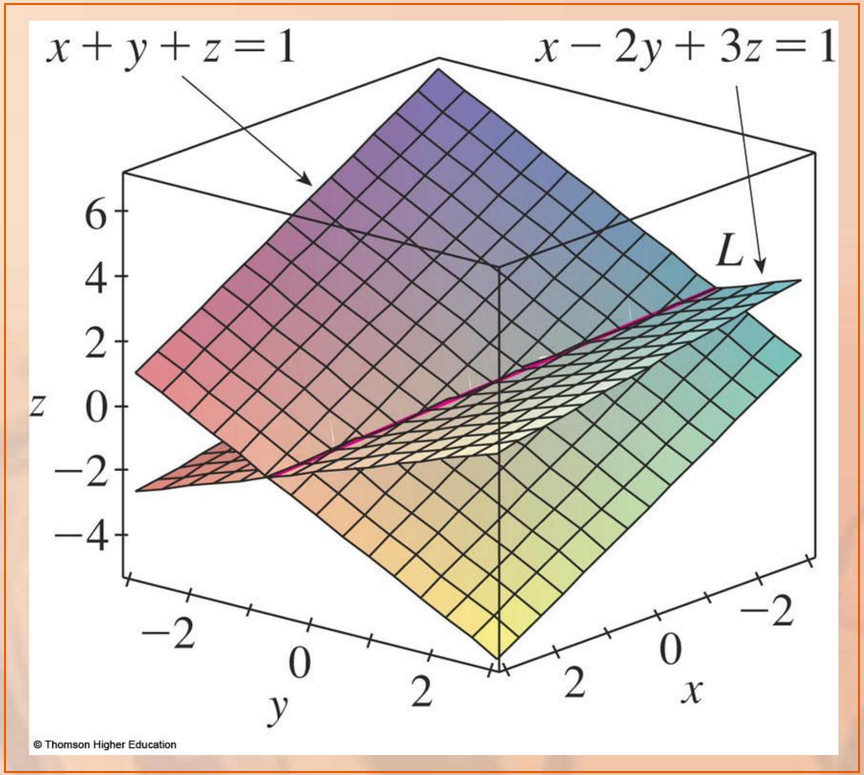
$$\frac{x-1}{5} = \frac{y}{-2} \quad \text{and} \quad \frac{y}{-2} = \frac{z}{-3}$$

This is again a pair of linear equations.

NOTE

They exhibit *L* as the line of intersection of the planes

$$(x - 1)/5 = y/(-2) \text{ and } y/(-2) = z/(-3)$$



NOTE

In general, when we write the equations of a line in the symmetric form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

we can regard the line as the line of intersection of the two planes

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \quad \text{and} \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

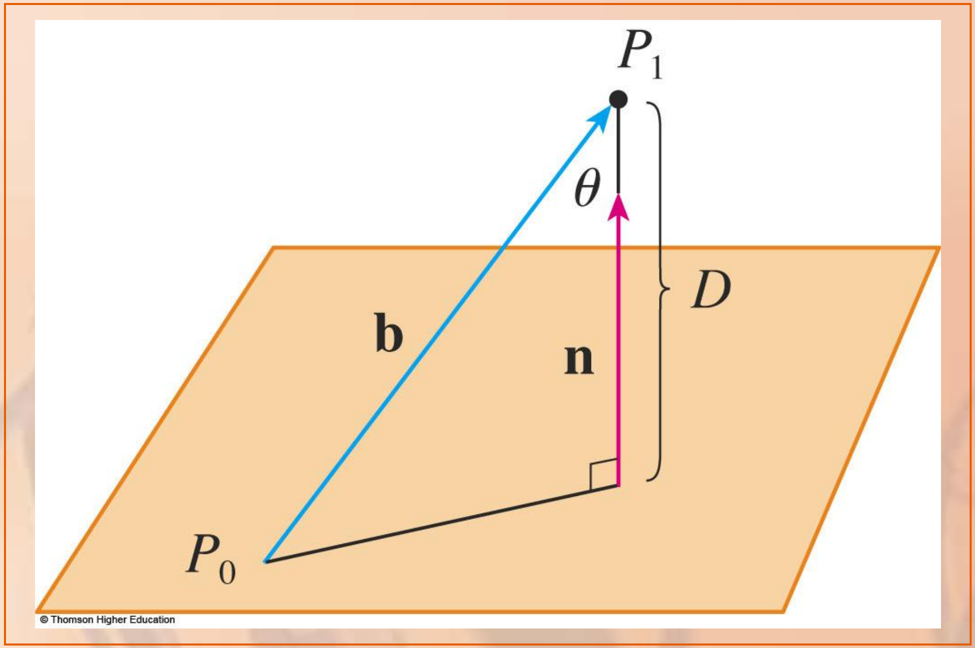
Let $P_0(x_0, y_0, z_0)$ be any point in the plane.

Let **b** be the vector corresponding to $\overrightarrow{P_0P_1}$.

- Then,

  $\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

You can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of **b** onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$.



Thus,

$$D = \left|\text{comp}_{\text{n}}\, b\right|$$

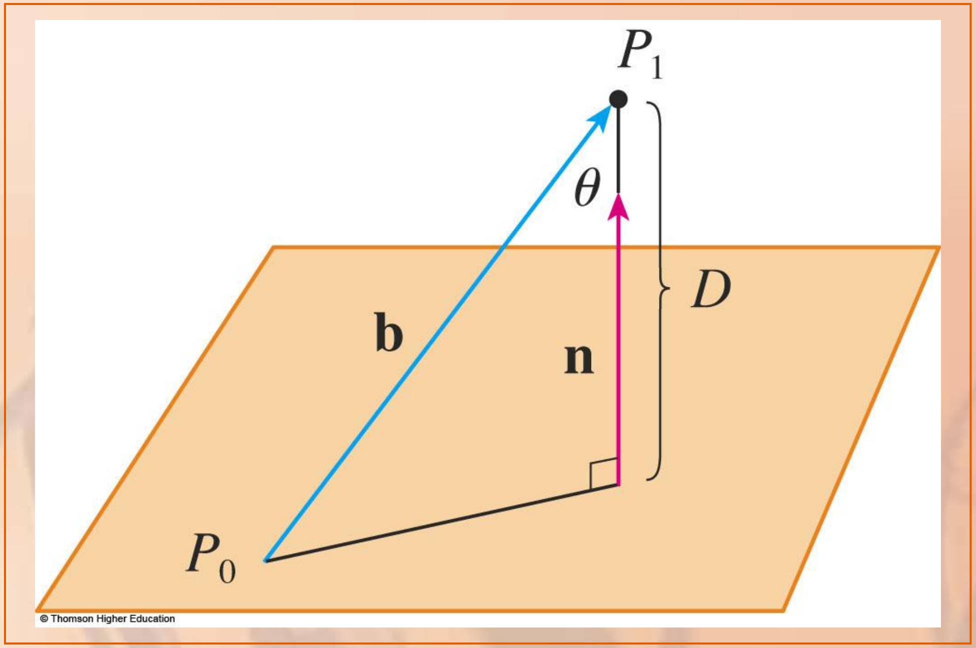
$$= \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|}$$

$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

Since P_0 lies in the plane, its coordinates satisfy the equation of the plane.

- Thus, we have $ax_0 + by_0 + cz_0 + d = 0$.



 E. g. 8—Formula 9

Hence, the formula for *D* can be written as:

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Find the distance between the parallel planes

$$10x + 2y - 2z = 5 \text{ and } 5x + y - z = 1$$

First, we note that the planes are parallel because their normal vectors

$$\langle 10, 2, -2 \rangle \text{ and } \langle 5, 1, -1 \rangle$$

are parallel.

To find the distance D between the planes, we choose any point on one plane and calculate its distance to the other plane.

- In particular, if we put $y = z = 0$ in the equation of the first plane, we get $10x = 5$.
- So, (½, 0, 0) is a point in this plane.

By Formula 9, the distance between (½, 0, 0) and the plane $5x + y - z - 1 = 0$ is:

$$D = \frac{\left|5\left(\frac{1}{2}\right) + 1(0) - 1(0) - 1\right|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

- So, the distance between the planes is $\sqrt{3}/6$.

In Example 3, we showed that the lines

L_1: $x = 1 + t$ $y = -2 + 3t$ $z = 4 - t$

L_2: $x = 2s$ $y = 3 + s$ $z = -3 + 4s$

are skew.

Find the distance between them.

Since the two lines L_1 and L_2 are skew, they can be viewed as lying on two parallel planes P_1 and P_2.

- The distance between L_1 and L_2 is the same as the distance between P_1 and P_2.
- This can be computed as in Example 9.

The common normal vector to both planes must be orthogonal to both

$$\mathbf{v}_1 = \langle 1, 3, -1 \rangle \quad (\text{direction of } L_1)$$

$$\mathbf{v}_2 = \langle 2, 1, 4 \rangle \quad (\text{direction of } L_2)$$

So, a normal vector is:

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix}$$

$$= 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$$

If we put $s = 0$ in the equations of L_2, we get the point (0, 3, –3) on L_2.

- So, an equation for P_2 is:

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0$$

or

$$13x - 6y - 5z + 3 = 0$$

If we now set $t = 0$ in the equations for L_1, we get the point (1, –2, 4) on P_1.

So, the distance between L_1 and L_2 is the same as the distance from (1, –2, 4) to $13x - 6y - 5z + 3 = 0$.

By Formula 9, this distance is:

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}}$$

$$= \frac{8}{\sqrt{230}} \approx 0.53$$