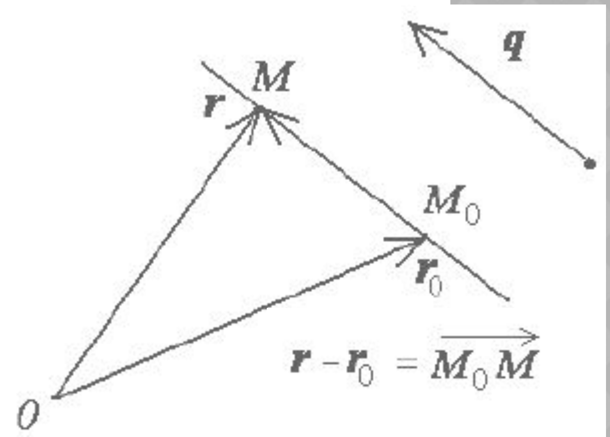
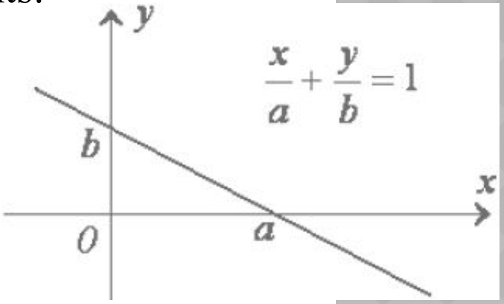
LECTURE 10: LINE IN THE PLANE

- **10.1. Equations of Lines**
- A *direction vector* of a straight line is a vector parallel to the line.
- A point M_0 and a direction vector $\boldsymbol{q}$ determine the straight line L.
- Let M be an arbitrary point on the line.
- $\vec{r} - \vec{r}_0$ − the difference between the radius-vectors of M and M_0 is a vector in the line:
- $$\boldsymbol{r} - \boldsymbol{r}_0 \parallel \boldsymbol{q}\,.$$
- Two parallel vectors are proportional: $\boldsymbol{r} - \boldsymbol{r}_0 = t\boldsymbol{q}$ (10.1)
- This vector equality is called the *vector equation of the line*.
- An arbitrary number t is said to be a *parameter*.
- Assume: a rectangular Cartesian coordinate system is chosen.
- Then $\boldsymbol{r}$, $\boldsymbol{r}_0$ and $\boldsymbol{q}$ are represented by their coordinates:
- $$\boldsymbol{r} - \boldsymbol{r}_0 = \{x - x_0, y - y_0\},$$
- $$\boldsymbol{q} = \{q_x, q_y\}.$$
- where x and y are running coordinates of a point on the line.
- Then (10.1) can be written in the coordinate form as the system
- of linear equations:
- $$\begin{cases} x = x_0 + q_x t \\ y = y_0 + q_x t \end{cases} \quad (10.2)$$
- which is called the *parametric equation* of a line.
- Solving system (10.2) by elimination of t, we obtain the *canonical equations of a line*:
- $$\frac{x - x_0}{q_x} = \frac{y - y_0}{q_y} \quad (10.3)$$



- If $M_0\,(x_0, y_0)$ and $M_1(x_1, y_1)$ are two given points on a line,
- then the vector $\boldsymbol{q} = \{x_1 - x_0, y_1 - y_0\}$ joining these points
- serves as a direction vector of the line.
- Therefore, we get an equation of a line passing through two given points:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}.$$



- Sometimes we express a straight-line equation in the *x, y*–plane as
- (1($\frac{x}{a} + \frac{y}{b} = 1.$
- In this case, $y = 0$ implies $x = a$, and $x = 0$ implies $y = b$.
- Equation (10.4) is called an *equation of a line in the intercept form*.
- A line on the *x,y*–plane may be also given by the *equation in the slope intercept form:*
- $y = kx + b,$
- where b is the y-intercept of a graph of the line, and k is the slope of the line.
- If $M_0\,(x_0, y_0)$ is a point on the line, i.e, $y_0 = kx_0 + b$, then the *point–slope equation:*
- $y - y_0 = k(x - x_0)$.

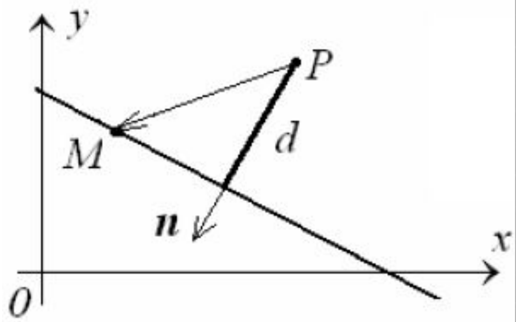
- On the *x, y*–plane, a line can be also described by the linear equation
- $$Ax + By + C = 0 \,. \qquad (10.5)$$
- If $M_0\ (x_0, y_0)$ is a point on the line then
- $$Ax_0 + By_0 + C = 0 \,. \qquad (10.6)$$
- Subtracting identity (10.6) from equation (10.5), we obtain
- the *equation of a line passing through the point* $M_0\ (x_0, y_0)$:
- $$A(x - x_0) + B(y - y_0) = 0 \,. \qquad (10.6a)$$
- The expression on the left hand side has a form of the scalar product of the vectors
- $\boldsymbol{n} = \{A, B\}$ and $\boldsymbol{r} - \boldsymbol{r}_0 = \{x - x_0, y - y_0\}$:
- $$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0 \,.$$
- Therefore, the coefficients *A* and *B* can be interpreted geometrically as the coordinates of a vector in the *x, y*–plane, being perpendicular to the line.

- **10.2. Angle between two lines**
- The *angle between two lines* is the angle between direction vectors of the lines.
- If $\boldsymbol{p} = \{p_x, p_y\}$ and $\boldsymbol{q} = \{q_x, q_y\}$ are direction vectors of lines, then:

$$\cos\theta = \frac{p \cdot q}{|p| \cdot |q|} = \frac{p_x q_x + p_y q_y}{\sqrt{p_x^{\,2} + p_y^{\,2}}\sqrt{q_x^{\,2} + q_y^{\,2}}}$$

- If lines are perpendicular to each other then their direction vectors are also perpendicular
- => scalar product of the direction vectors is equal to zero:
- $$\boldsymbol{p} \cdot \boldsymbol{q} = p_x q_x + p_y q_y = 0.$$
- If two lines are parallel then their direction vectors are proportional: $\boldsymbol{p} = c\boldsymbol{q}$,
- where *c* is a number.
- In the coordinate form, this condition looks like $\dfrac{p_x}{q_x} = \dfrac{p_y}{q_y}$

- If two lines in the x, y–plane are given by the equations in the slope intercept form
- $y = k_1 x + b_1$ and $y = k_2 x + b_2$,
- and θ is the angle between the lines, then $\tan\theta = \dfrac{k_2 - k_1}{1 + k_1 k_2}$.
- The lines are parallel, if $k_1 = k_2$.
- The lines are perpendicular, if $k_1 k_2 = -1$.



- **10.3. Distance From a Point to a Line**
- Consider a line in the x, y–plane.
- Let $\boldsymbol{n}$ be a normal vector to the line and $M(x_0, y_0)$ be any point on the line.
- Then *the distance d* from a point P not on the line is equal to the absolute value of the projection of the vector PM on $\boldsymbol{n}$:

$$d = |\mathrm{Proj}_{\boldsymbol{n}} \vec{PM}| = \left| \frac{\vec{PM} \cdot \boldsymbol{n}}{|\boldsymbol{n}|} \right|$$

- In particular, if the line is given by the equation $Ax + By + C = 0$,
- and the coordinates of the point P are x_1 and y_1, that is,
- $\boldsymbol{n} = \{A, B\}$ and $PM = \{x_1 - x_0, y_1 - y_0\}$,
- then *the distance from P to the line* is calculated according to the following formula:

$$d = \frac{|A(x_1 - x_0) + B(y_1 - y_0)|}{\sqrt{A^2 + B^2}}.$$

- Since $M(x_0, y_0)$ is a point on the line, $Ax_0 + By_0 + C = 0$.
- Therefore, we obtain $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.