Solution Methods for Bilevel Optimization

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Overview

- Definition of a bilevel problem and its general form
- Optimality (KKT-type) conditions
- Reformulation of a general bilevel problem
- Iterative (descent direction) methods
- Numerical results

Stackelberg Game (Bilevel problem)

- Players: the Leader and the Follower
- The Leader is first to make a decision
- Follower reacts optimally to Leader’s decision
- The payoff for the Leader depends on the follower’s reaction

Example

- Taxation of a factory
- Leader – government
- Objectives: maximize profit and minimize pollution
- Follower – factory owner
- Objectives: maximize profit

General structure of a Bilevel problem

$$
\begin{aligned}
&\min_{x \in X} F(x, y) \\
&\text{subject to } G(x, y) \leq 0 \\
&\qquad\qquad\quad H(x, y) = 0 \\
&\qquad\qquad\quad \min_{y \in Y} f(x, y) \\
&\qquad\qquad\quad \text{subject to } g(x, y) \leq 0 \\
&\qquad\qquad\qquad\qquad\quad\; h(x, y) = 0
\end{aligned}
$$

Important Sets

- $$\min_{x \in X} F(x,y)$$
  $$\text{subject to } G(x,y) \le 0$$
  $$H(x,y) = 0$$
  $$\min_{y \in Y} f(x,y)$$
  $$\text{subject to } g(x,y) \le 0$$
  $$h(x,y) = 0$$

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General linear Bilevel problem

n $F(x, y)$

bject to $G(x, y) \leq 0$

$$H(x, y) = 0$$

$$\min_{y \in Y} f(x, y)$$

subject to $g(x, y) \leq 0$

$$h(x, y) = 0$$

Solution methods

- Vertex enumeration in the context of Simplex method
- Kuhn-Tucker approach
- Penalty approach
- Extract gradient information from a lower objective function to compute directional derivatives of an upper objective function

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Concept of KKT conditions

$$\min_{x \in X} F(x,y)$$

$$\text{subject to } G(x,y) \le 0$$

$$H(x,y) = 0$$

$$\min_{y \in Y} f(x,y)$$

$$\text{subject to } g(x,y) \le 0$$

$$h(x,y) = 0$$

Value function reformulation

$$
\begin{aligned}
&\min_{x \in X} F(x,y) \\
&\text{subject to } G(x,y) \leq 0 \\
&\qquad\qquad\quad H(x,y) = 0 \\
&\qquad\qquad\quad \min_{y \in Y} f(x,y) \\
&\qquad\qquad\quad \text{subject to } g(x,y) \leq 0 \\
&\qquad\qquad\qquad\qquad\quad\;\; h(x,y) = 0
\end{aligned}
$$

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KKT for value function reformulation

$$
\begin{aligned}
\min_{x \in X}\ & F(x,y) \\
\text{subject to}\ & G(x,y) \le 0 \\
& H(x,y) = 0 \\
& \min_{y \in Y}\ f(x,y) \\
& \text{subject to}\ g(x,y) \le 0 \\
& \qquad\qquad\quad h(x,y) = 0
\end{aligned}
$$

$$
\begin{aligned}
{}_{x \in X}\ & \\
\text{subject to}\ & G(x,y) \le 0 \\
& H(x,y) = 0 \\
& \min_{y \in Y}\ f(x,y) \\
& \text{subject to}\ g(x,y) \le 0 \\
& \qquad\qquad\quad h(x,y) = 0
\end{aligned}
$$

Assumptions

$$\min_{x \in X} F(x, y)$$

$$\text{subject to } G(x, y) \le 0$$

$$H(x, y) = 0$$

$$\min_{y \in Y} f(x, y)$$

$$\text{subject to } g(x, y) \le 0$$

KKT-type optimality conditions for B

$$
\begin{aligned}
\nabla_x F(\bar{x}, \bar{y}) + r\nabla_x f(\bar{x}, \bar{y}) - r\sum_{s=1}^{n+1} \eta_s \nabla_x f(\bar{x}, y_s)& \\
+\nabla_x g(\bar{x}, \bar{y})^T u - r\sum_{s=1}^{n+1} \eta_s \nabla_x g(\bar{x}, y_s)^T u_s& \\
+\nabla_x h(\bar{x}, \bar{y})^T v - r\sum_{s=1}^{n+1} \eta_s \nabla_x h(\bar{x}, y_s)^T v_s& \\
+\nabla G(\bar{x})^T u' + \nabla H(\bar{x})^T v' &= 0, \\
\nabla_y F(\bar{x}, \bar{y}) + r\nabla_y f(\bar{x}, \bar{y}) + \nabla_y g(\bar{x}, \bar{y})^T u + \nabla_y h(\bar{x}, \bar{y})^T v &= 0, \\
\nabla_y f(\bar{x}, y_s) + \nabla_y g(\bar{x}, y_s)^T u_s + \nabla_y h(\bar{x}, y_s)^T v_s &= 0, \\
u \geq 0, u^T g(\bar{x}, \bar{y}) &= 0, \\
u' \geq 0, u'^T G(\bar{x}) &= 0, \\
u_s \geq 0, u_s^T g(\bar{x}, y_s) &= 0.
\end{aligned}
$$

Further Assumptions (for simpler version)

$$\min_{x \in X} F(x,y)$$

$$\text{subject to } G(x,y) \leq 0$$

$$H(x,y) = 0$$

$$\min_{y \in Y} f(x,y)$$

$$\text{subject to } g(x,y) \leq 0$$

$$h(x,y) = 0$$

Simpler version of KKT-type conditions

$$\min_{x \in X} F(x,y)$$

subject to $G(x,y) \leq 0$

$$H(x,y) = 0$$

$$\min_{y \in Y} f(x,y)$$

subject to $g(x,y) \leq 0$

$$h(x,y) = 0$$

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NCP-Functions

$$\min_{x \in X} F(x,y)$$

$$\text{subject to } G(x,y) \leq 0$$

$$H(x,y) = 0$$

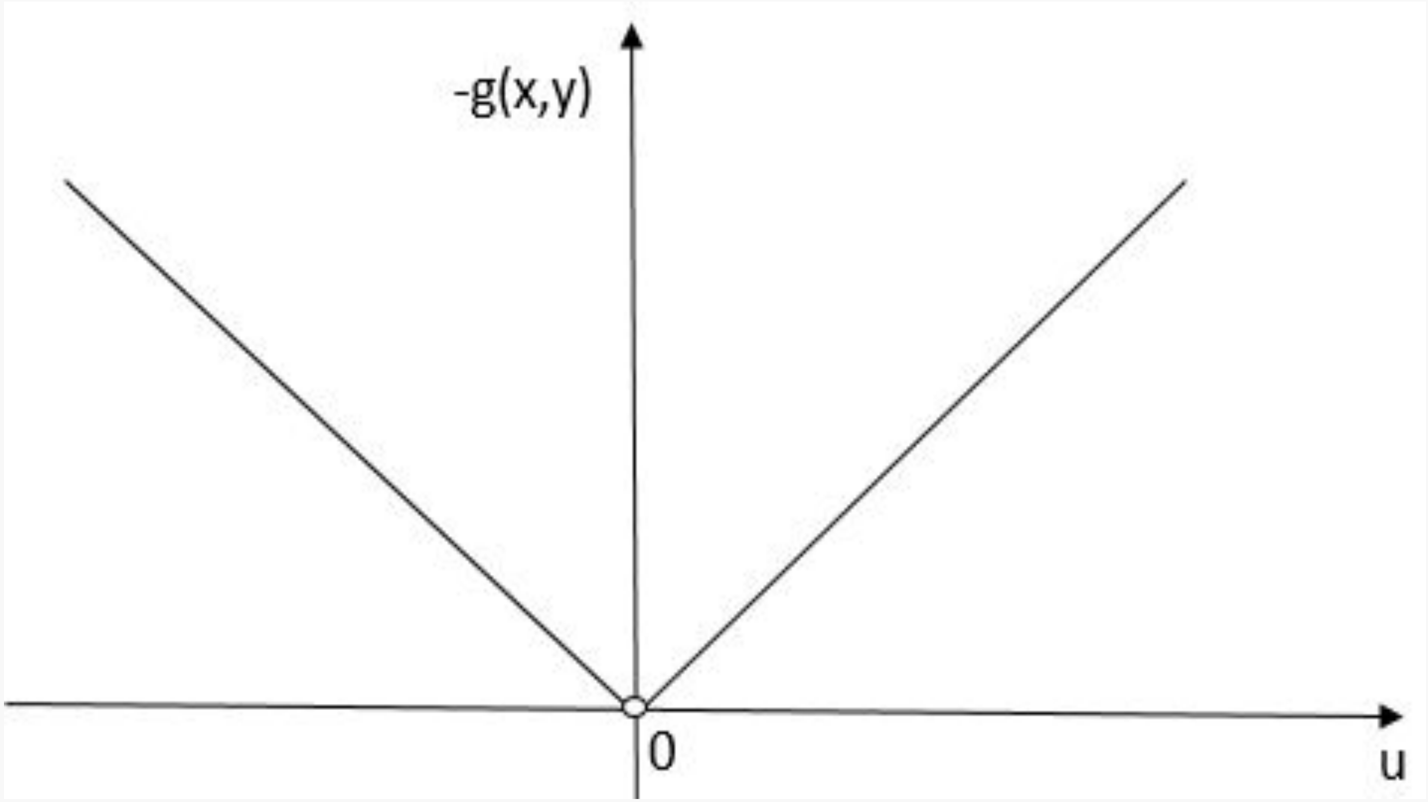
$$\min_{y \in Y} f(x,y)$$

$$\text{subject to } g(x,y) \leq 0$$

$$h(x,y) = 0$$

Problems with differentiability

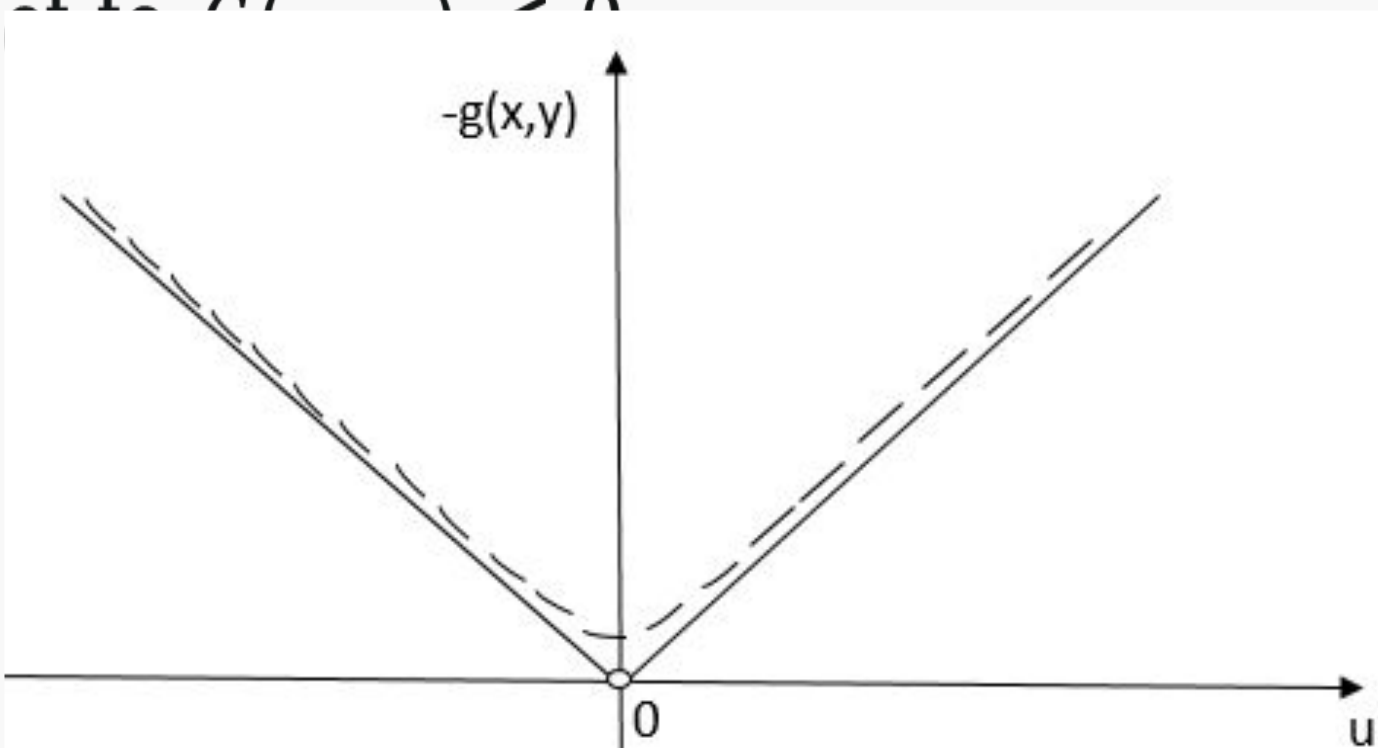
- Fischer-Burmeister is not differentiable at 0



$$\text{subject to } G(x,y) \leq 0$$
$$H(x,y) = 0$$
$$\min_{y \in Y} f(x,y)$$
$$\text{subject to } g(x,y) \leq 0$$

$$\min_{x \in X} F(x,y)$$

subje



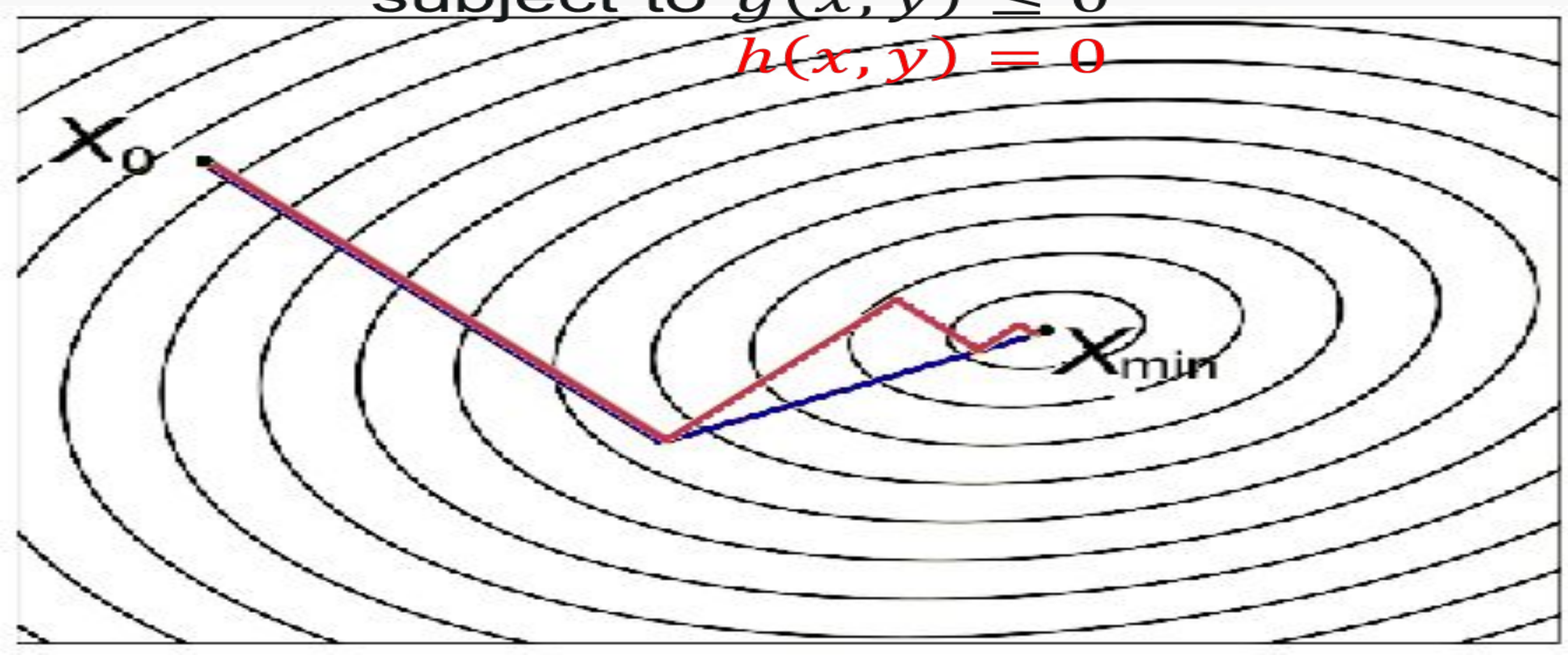
Simpler version with perturbed Fischer-Burmeister NCP functions

$$\begin{aligned}
&\min_{x \in X} F(x,y) \\
&\text{subject to } G(x,y) \leq 0 \\
&\qquad\qquad\quad H(x,y) = 0 \\
&\qquad\qquad\quad \min_{y \in Y} f(x,y) \\
&\qquad\qquad\quad \text{subject to } g(x,y) \leq 0 \\
&\qquad\qquad\qquad\qquad\qquad h(x,y) = 0
\end{aligned}$$

$$\begin{aligned}
&\min_{x \in X} F(x,y) \\
&\text{subject to } G(x,y) \leq 0 \\
&\qquad\qquad\quad H(x,y) = 0 \\
&\qquad\qquad\quad \min_{y \in Y} f(x,y) \\
&\qquad\qquad\quad \text{subject to } g(x,y) \leq 0 \\
&\qquad\qquad\qquad\qquad\qquad h(x,y) = 0
\end{aligned}$$

Iterative methods

$$\min_{x \in X} F(x,y)$$
$$\text{subject to } G(x,y) \leq 0$$
$$H(x,y) = 0$$
$$\min_{y \in Y} f(x,y)$$
$$\text{subject to } g(x,y) \leq 0$$
$$h(x,y) = 0$$



Newton method

$$\min_{x \in X} F(x,y)$$

$$\text{subject to } G(x,y) \leq 0$$

$$H(x,y) = 0$$

$$\boxed{\min_{y \in Y} f(x,y)}$$

$$\text{subject to } g(x,y) \leq 0$$

$$h(x,y) = 0$$

Pseudo inverse

$$\min_{x \in X} F(x, y)$$
$$\text{subject to } G(x, y) \leq 0$$
$$H(x, y) = 0$$
$$\min_{y \in Y} f(x, y)$$
$$\text{subject to } g(x, y) \leq 0$$

$$\min_{x \in X} F(x, y)$$
$$\text{subject to } G(x, y) \leq 0$$
$$H(x, y) = 0$$
$$\min_{y \in Y} f(x, y)$$
$$\text{subject to } g(x, y) \leq 0$$
$$h(x, y) = 0$$

$$\min_{x \in X} F(x, y)$$
$$\text{subject to } G(x, y) \leq 0$$
$$H(x, y) = 0$$
$$\min_{y \in Y} f(x, y)$$
$$\text{subject to } g(x, y) \leq 0$$
$$h(x, y) = 0$$

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Gauss-Newton method

$$\min_{x \in X} F(x,y)$$

$$\text{subject to } G(x,y) \le 0$$

$$H(x,y) = 0$$

$$\min_{y \in Y} f(x,y)$$

$$\boxed{\text{subject to } g(x,y) \le 0}$$

$$h(x,y) = 0$$

Singular Value Decomposition (SVD)

$$\min_{x \in X} F(x, y)$$

$$\text{subject to } G(x, y) \leq 0$$

$$H(x, y) = 0$$

$$\min_{y \in Y} f(x, y)$$

$$\text{subject to } g(x, y) \leq 0$$

$$h(x, y) = 0$$

SVD for wrong direction

$$\min F(x, y)$$

$$\begin{pmatrix} \sigma_1 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & 0 & \sigma_n \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \sigma_2 & 0 & \dots & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & \vdots & \dots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 0 & \sigma_n & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 & 0 \\ 0 & \sigma_2^2 & 0 & \dots & \vdots & 0 \\ \vdots & 0 & \ddots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & \ddots & 0 & \vdots \\ 0 & \vdots & \dots & 0 & \sigma_n^2 & 0 \\ 0 & 0 & \dots & \dots & 0 & 0 \end{pmatrix}.$$

$$\min_{y \in Y} f(x, y)$$

$$\text{subject to } g(x, y) \leq 0$$

SVD for right direction

$$\min_{x \in X} F(x, y)$$

$$\begin{pmatrix} \sigma_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \sigma_2 & 0 & \dots & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & \vdots & \dots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 0 & \sigma_n & 0 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & 0 & \sigma_n \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & 0 \\ 0 & \vdots & \dots & 0 & \sigma_n^2 \end{pmatrix}.$$

subject to $g(x, y) \leq 0$

$h(x, y) = 0$

Levenberg-Marquardt method

$$\min_{x \in X} F(x, y)$$

subject to $G(x, y) \leq 0$

$$H(x, y) = 0$$

$$\min_{y \in Y} f(x, y)$$

subject to $g(x, y) \leq 0$

$$h(x, y) = 0$$

Numerical results

Name	multiplier	B(average)	iterations	exitflag	message
Bard1	10^(-3)	9.966087	6	-2	No solution found (regular)
Dempe9	10^(-9)	9.702413	21	1	Equation solved
Hend58	10^(-2)	3.929357	5	-2	No solution found (regular)
Shim97	10^(-9)	5.794647	10	1	Equation solved
Clarke90	10^(-10)	1.248215	12	1	Equation solved
Shimi81P1	10^(-2)	1.044174	23	-2	No solution found (regular)
BIPA1	10^(-1)	9.685432	32	-2	No solution found (regular)
BIPA2	10^(-4)	6.614896	19	1	Equation solved
BIPA3	10^(-1)	2.591180	159	-2	No solution found (ineffecti
BIPA4	10^(-3)	7.637497	7	-2	No solution found (regular)

Plans for further work

$$
\begin{aligned}
&\min_{x \in X} F(x,y) \\
&\text{subject to } G(x,y) \le 0 \\
&\qquad\qquad\quad {\color{red} H(x,y) = 0} \\
&\qquad\qquad\quad \min_{y \in Y} f(x,y) \\
&\qquad\qquad\quad \text{subject to } g(x,y) \le 0 \\
&\qquad\qquad\qquad\qquad\qquad {\color{red} h(x,y) = 0}
\end{aligned}
$$

Plans for further work

6. Construct the own code for Levenberg-Marquardt method in the context of solving bilevel problems within defined reformulation.

7. Search for good starting point techniques for our problem. 8. Do the numerical calculations for the harder reformulation defined .

9. Code Newton method with pseudo-inverse.

10. Solve the problem assuming strict complementarity

11. Look at other solution methods.

Thank you!

Questions?

References

$$\min_{x \in X} F(x,y)$$

$$\text{subject to } G(x,y) \le 0$$

$$H(x,y) = 0$$

$$\min_{y \in Y} f(x,y)$$

$$\text{subject to } g(x,y) \le 0$$

$$h(x,y) = 0$$

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References

$$
\begin{aligned}
&\min_{x\in X} F(x,y)\\
&\text{subject to } G(x,y) \le 0\\
&\qquad\qquad\quad H(x,y) = 0\\
&\qquad\qquad\quad \min_{y\in Y} f(x,y)\\
&\qquad\qquad\quad \text{subject to } g(x,y) \le 0\\
&\qquad\qquad\qquad\qquad\qquad h(x,y) = 0
\end{aligned}
$$