Discrete Mathematics

Probability-1

Adil M. Khan

Professor of Computer Science

Innopolis University

"Information: The Negative Reciprocal Value of Probability!"
- Claude Shannon -

Probability---Introduction

- One of the most important disciplines in Computer Science (CS).
- Algorithm Design and Game Theory
- Information Theory
- Signal Processing
- Cryptography

Probability---Introduction---Cont.

But it is also *probably* the least well understood

- Human intuition and Random events

Goal: To try our best to teach you how to easily and confidently solve problems involving probability

- "*What is the probability that* … ?"

Probability

❑ Contents

- Basic definitions and an elementary 4-step process
- Counting

- Conditional probability and the concept of independence
- Random Variable
- Expected value and Standard Deviation

Probability

❑ Let's Make a Deal

- **The famous game show** (you might have seen this problem in your books)

- Participant is given a choice of three doors. Behind one door is a car, behind the others, useless stuff. The participant picks a door (**say door 1**). The host, who knows what is behind the doors, opens another door (**say door 3**) which has the useless stuff. He then asks the participant if he would like to switch (**pick door 2**)?

  Is it to participant's advantage to switch or not?

Probability

❑ Precise Description

- The car is <u>equally likely</u> to be hidden behind the three doors.

Equally likely events are events that have the same likelihood of occurring. For example. each numeral on a die is equally likely to occur when the die is tossed.

Probability

❑ Precise Description

- The car is equally likely to be hidden behind the three doors.
- The player is equally likely to pick each of the doors.
- After the player picks a door, the host must open a different door (with the useless thing behind it) and offer the player to switch.
  - ❖ When a host has a choice of which door to pick, he is equally likely to pick each of them.

Now here comes the question:

"What is the probability that a player who switches wins the car?"

Probability

❑ Solving Problems Involving Probability

- ❖ Model the situation mathematically
- ❖ Solve the resulting mathematical problem

Probability

❑ Solving Problems Involving Probability

Step 1: Finding the *sample space*

❖ Set of all possible outcomes of a random process

To say that a process is random means that when it takes place, one outcome from a set of outcomes is sure to occur, but it is impossible to predict with certainty which outcome that will be.

For example: tossing a coin, choosing winners in state lotteries.

Probability

❑ Solving Problems Involving Probability

Step 1: Finding the *sample space*

- Set of all possible outcomes of a <u>random process</u>

The set of all possible outcomes that can result from a random process is is called a sample space.

Probability

❑ Solving Problems Involving Probability

Step 1: Finding the *sample space*

- Set of all possible outcomes of a random process

To find this, we must understand the quantities involve in the random process

Probability

❑ Solving Problems Involving Probability

Step 1: Finding the *sample space*

- Set of all possible outcomes of a random process

To find this, we must understand the quantities involve in the random process

Quantities in the above problem:

- The door concealing the car
- The door initially chosen by the player
- The door that host opens to reveal the useless thing

Probability

- **Finding the Sample Space**

Every possible value of these quantities is called an outcome.

And (as said earlier) the set of all possible outcomes is called the sample space

Probability

❑ Finding the Sample Space

Every possible value of these quantities is called an outcome.

And (as said earlier) the set of all possible outcomes is called the sample space

A *tree structure* (Possibility tree) is a useful tool for keeping track of all outcomes

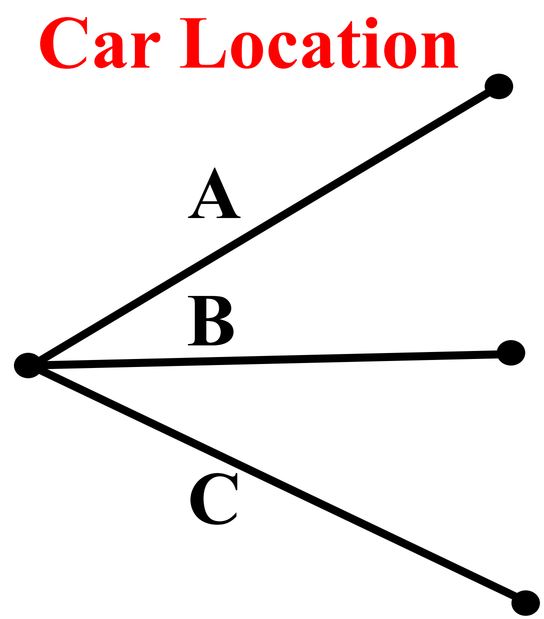
❖ When the number of possible outcomes is not too large

Probability

❑ Possibility Tree

The first quantity in our example is the door concealing the car

Represent this as a root of tree with three branches (three doors)



Probability

- **Possibility Tree --- Cont.**

1. The car can be at any of these three locations

Probability

❑ Possibility Tree --- Cont.

1. The car can be at any of these three locations

2. For each possible location of the car, the player can choose any of the three doors

Probability

❑ Possibility Tree --- Cont.

1. The car can be at any of these three locations

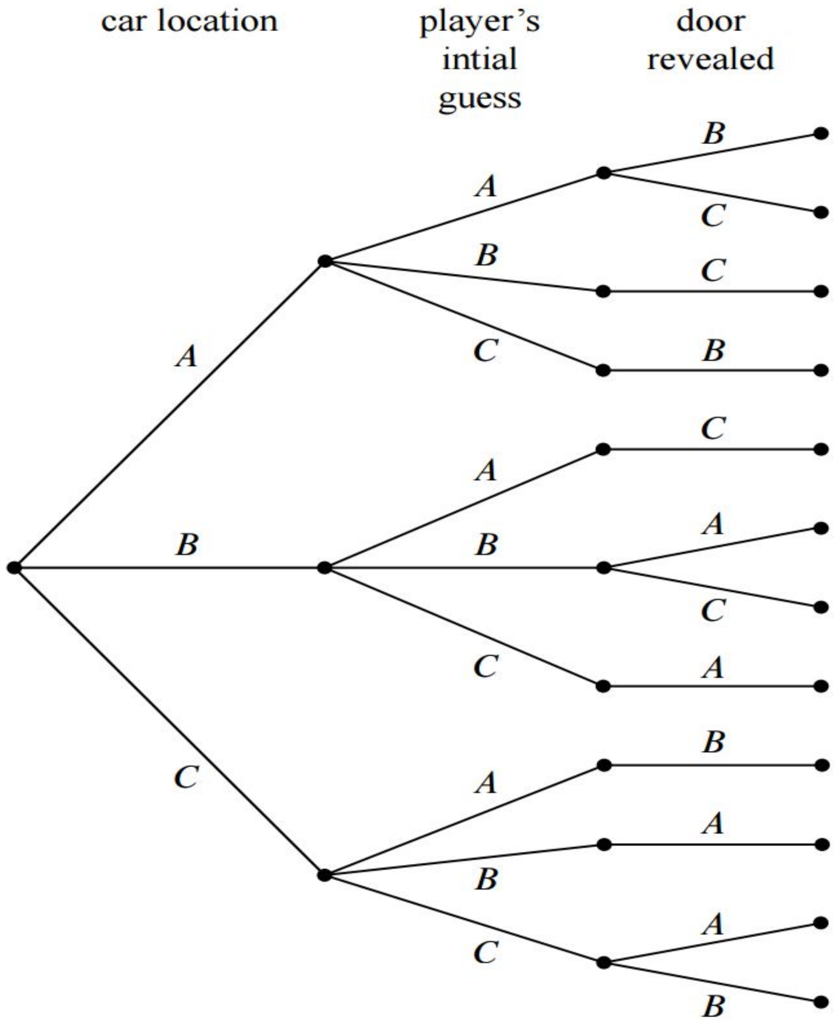
2. For each possible location of the car, the player can choose any of the three doors

3. Then the final possibility is regarding the host opening a door to reveal the useless thing

- Overall tree turns out to be

Probability

❑ Possibility Tree --- Cont.



Probability

- **Finding The Sample Space**

  The leaves of the possibility tree represent the outcomes of a random process

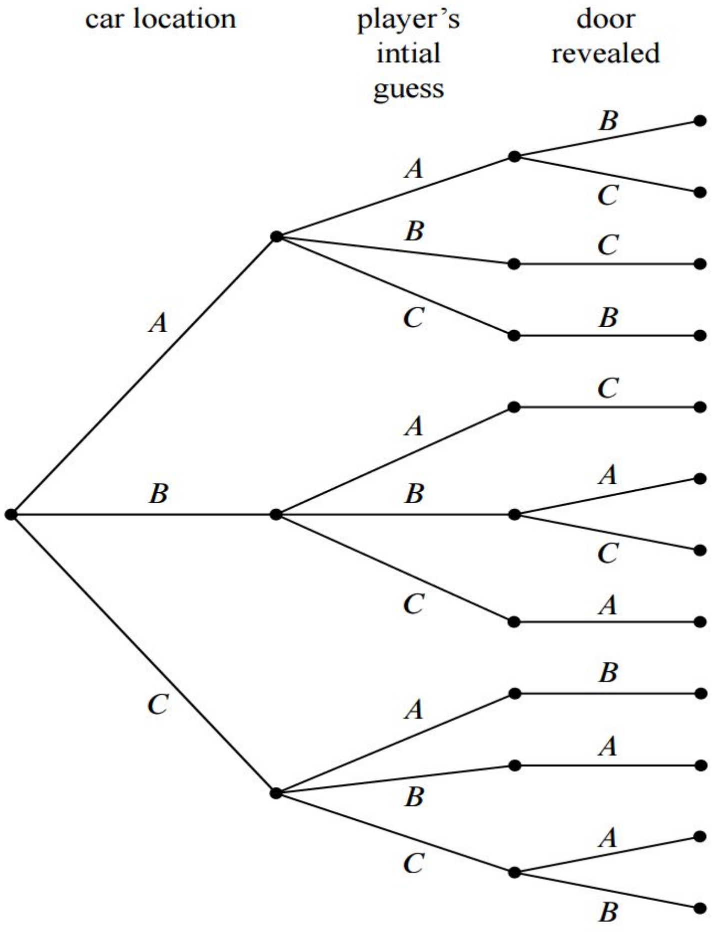
  The set of all leaves represent the sample space

Probability

- **Finding The Sample Space**

In our example, if we represent the leaves as a sequence of "labels" of intermediate nodes including the leaf node then,

$$S = \begin{Bmatrix} (A, A, B), (A, A, C), (A \\ (B, B, C), (B, C, A), (\end{Bmatrix}$$



Probability

- ## Solving Problems Involving Probability

Step 2: Defining the Events of Interest:

Probability

❑ Solving Problems Involving Probability

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Remember, we want to answer the questions of type:

- "What is the probability that … ?"

Probability

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Replacing the "…" with some specific event. For example,

Probability

❑ Solving Problems Involving Probability

Step 2: Defining the Events of Interest:

Remember, we want to answer the questions of type:

- "What is the probability that … ?"

Replacing the "…" with some specific event. For example,

- "What is the probability that the car is behind door C?"

Doing this reduces S to some specific outcomes, called *event of interest*.

Probability

❑ Event of Interest

For the event,

- "What is the probability that the car is behind door C?"

The set of possible outcomes reduces to

$$\{(C;\ A;\ B);\ (C;\ B;\ A);\ (C;\ C;\ A);\ (C;\ C;\ B)\}$$

Probability

❑ Event of Interest

For the event,

- "What is the probability that the car is behind door C?"

The set of possible outcomes reduces to

$$\{(C;\ A;\ B);\ (C;\ B;\ A);\ (C;\ C;\ A);\ (C;\ C;\ B)\}$$

- Simply speaking, **an event is a subset of S**

Probability

❑ Solving Problems Involving Probability

Coming back to our example

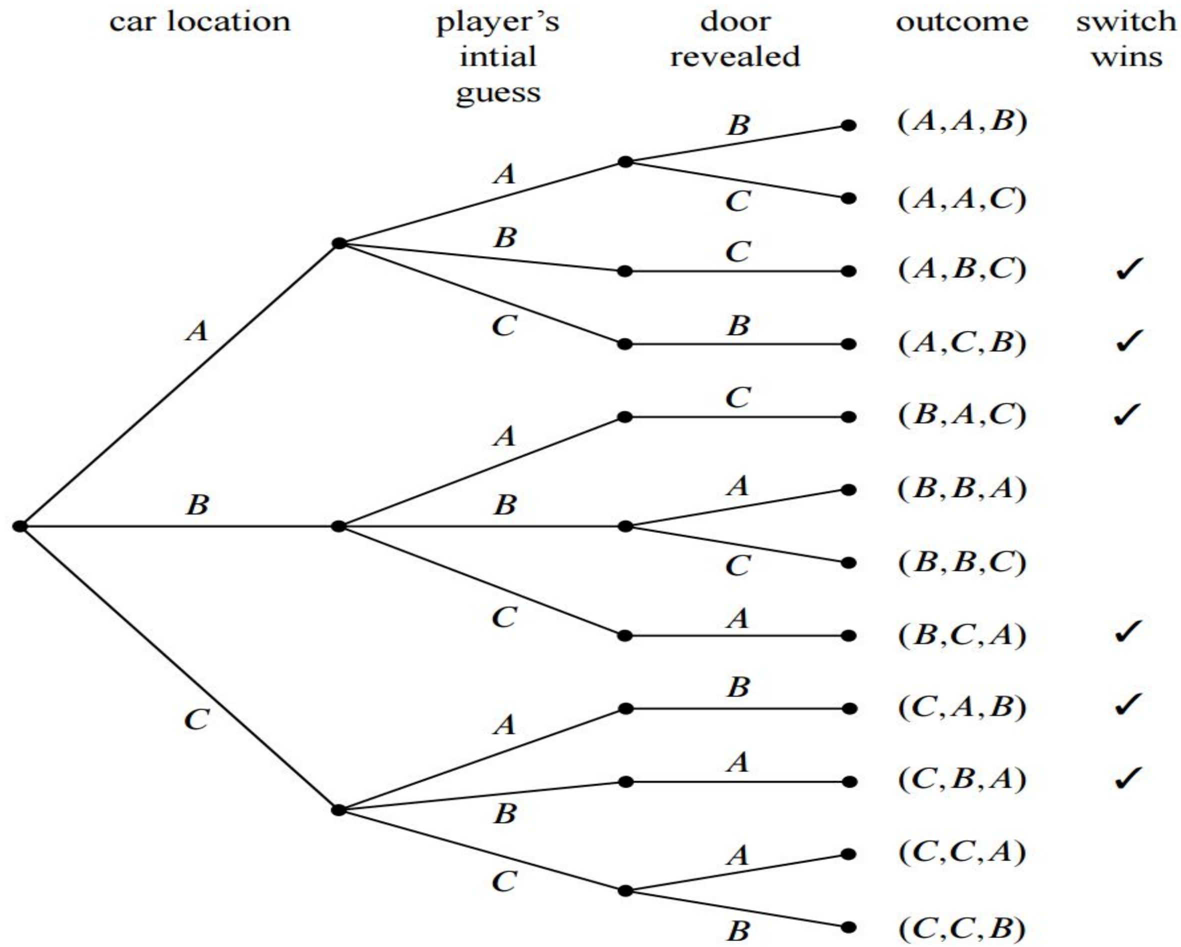
We want to know:

"What is the probability that the player will win by switching?"

This event can be represented as the following set

Probability

❑ Solving Problems Involving Probability---Cont.



Notice: Half of the outcomes are checked. Does this mean that the player wins by switching in half of all outcomes?

Probability

- **Solving Problems Involving Probability---Cont.**

Step 3: Determining Outcome Probability

1. **Assign Edge Probabilities**

2. **Compute Outcome Probabilities**

Probability

❑ Equally likely probability formula

$$P(E) = \frac{the\ number\ of\ outcomes\ in\ E}{the\ total\ number\ of\ outcomes\ in\ S}$$

$$\boldsymbol{P(E) = \frac{|E|}{|S|}}$$

E: the equally likely event
S: the sample space

Probability

- **Solving Problems Involving Probability---Cont.**

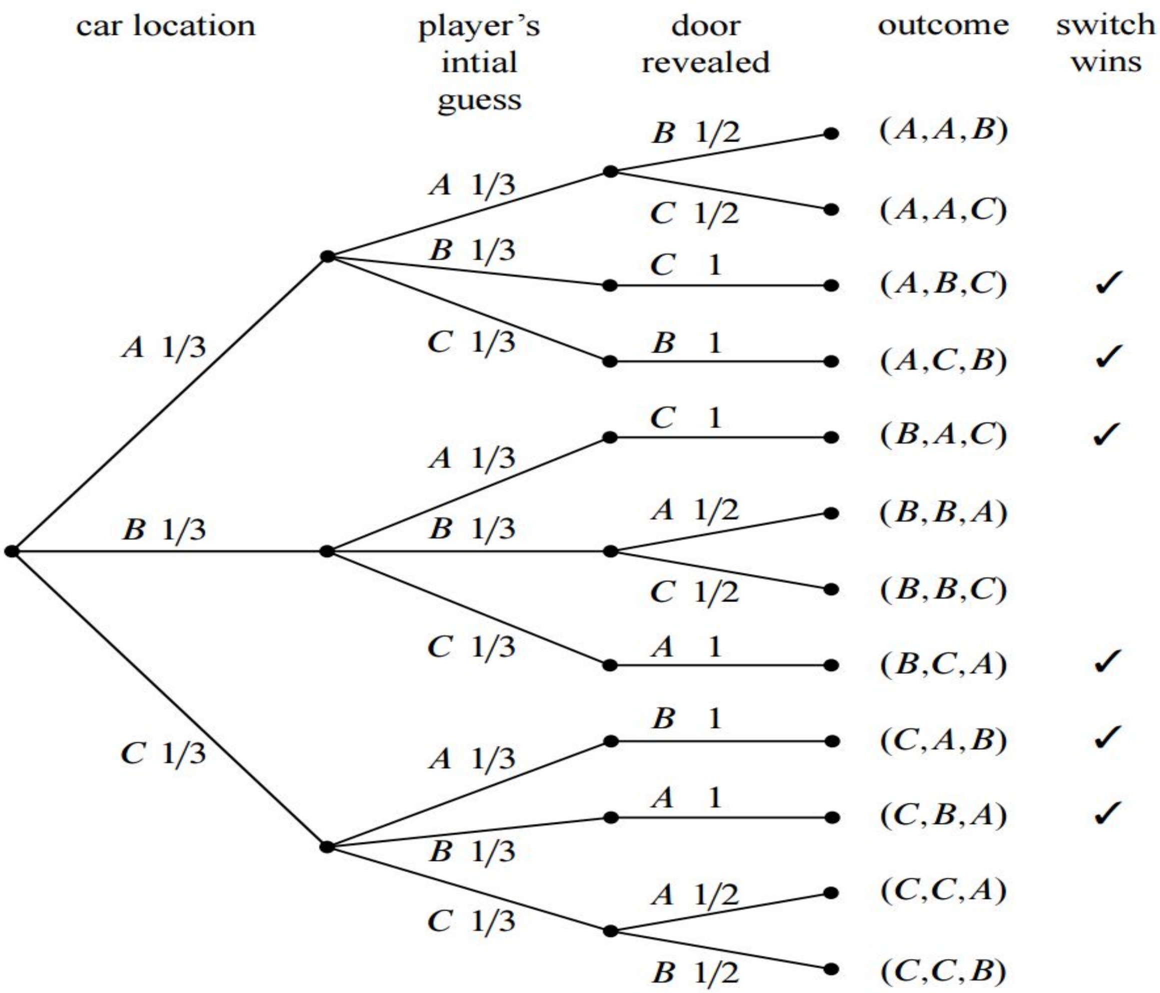
Step 3: Determining Outcome Probability

1. **Assign Edge Probabilities**

2. **Compute Outcome Probabilities**

Probability

❑ Edge Probabilities



To understand, let's analyze the path leading to the leaf node (A, A, B)!

Probability

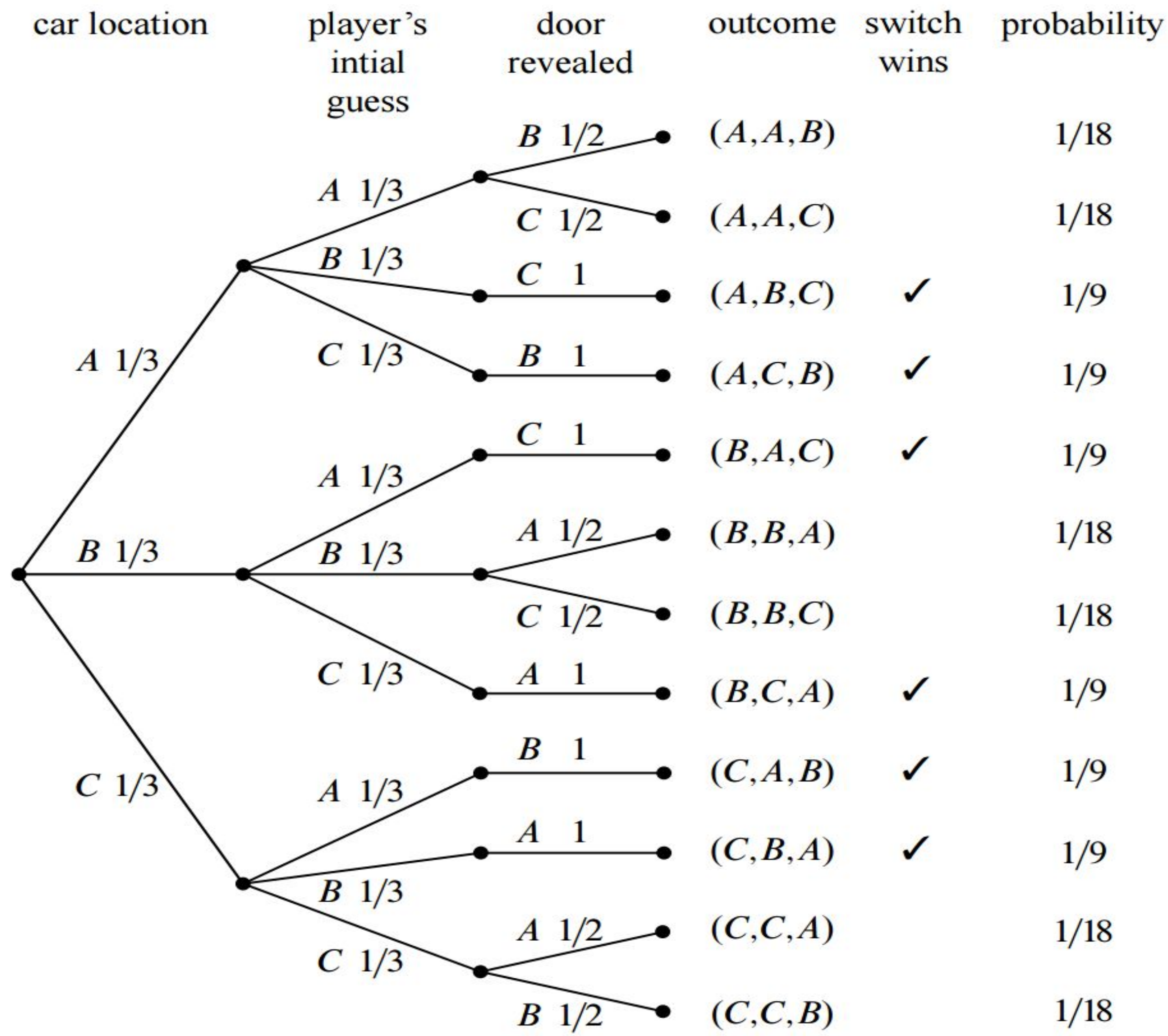
- **Multiplication Rule**

  - The **probability** that Events A and B both occur is equal to the **probability** that Event A occurs times the **probability** that Event B occurs, given that A has occurred.

You will learn more about this when I will teach you about conditional probabilities next week. For now, let's just use this rule!

Probability

❑ Outcome Probabilities



To understand, let's analyze the probability of the outcome (A, A, B).

Probability

❑ **Solving Problems Involving Probability---Cont.**

Step 4: Compute Event Probability

$$P(SwitchingWins) = P(A,B,C) + P(A,C,B) +$$

$$P(B,A,C) + P(B,C,A) + P\ (C,A,B) +$$

$$P(C,B,A)$$

$$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$$

$$= \frac{2}{3}$$

Probability

❑ Summary

To solve problems involving probability, that is, "**what is the probability that … ?**"

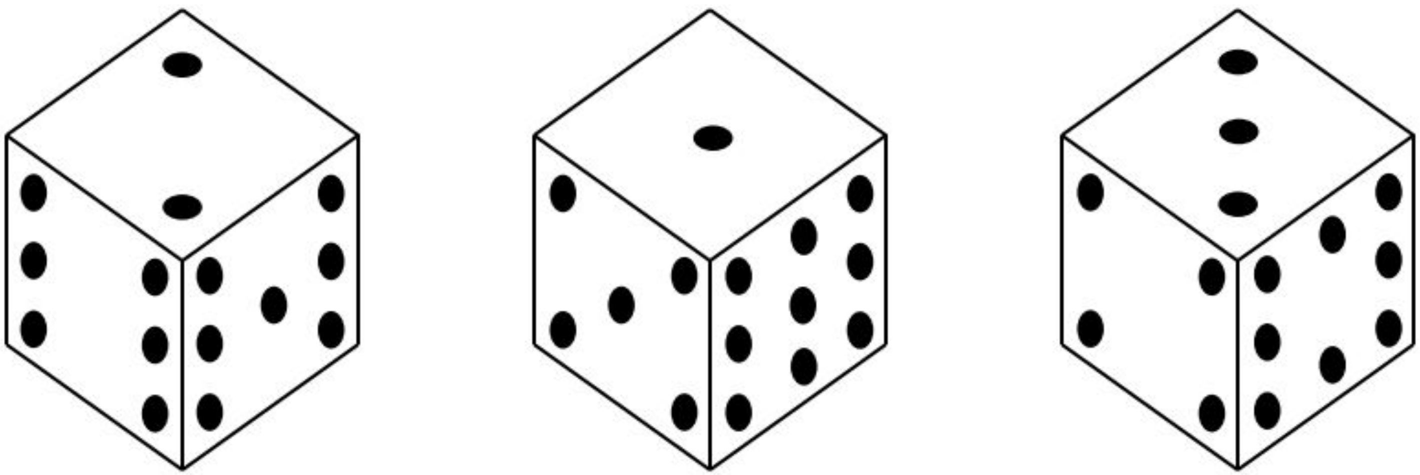
Perform the following **four steps**:

- ❖ Find the sample space
- ❖ Define event of interest
- ❖ Compute outcome probabilities
- ❖ Compute event probability

Probability

❑ Uniform Sample Space

Strange Dice

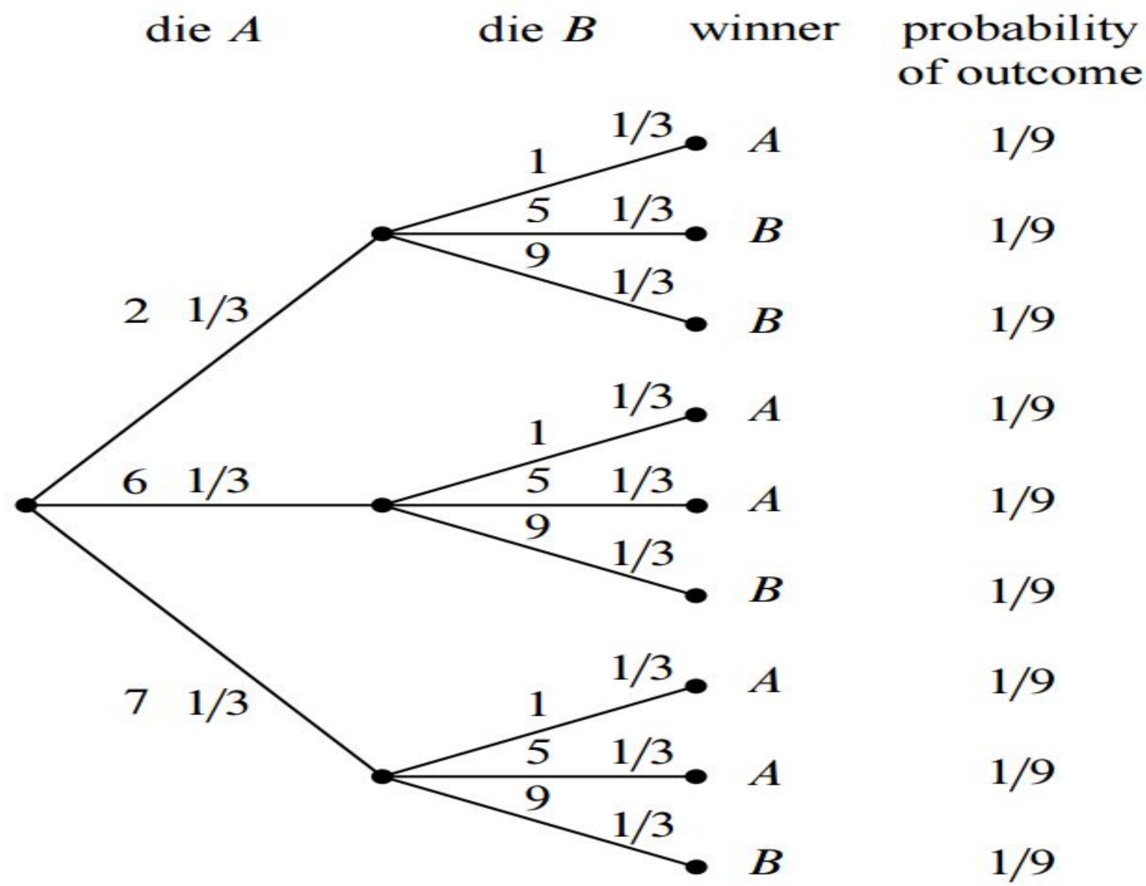


a *b* *c*

If we picked dices (a) and (b), rolled them once, what is the probability that (a) beats (b) (*has a higher value*)?

Probability

❑ Applying Four-Step Method



When the probability of every outcome is the same, we say such a sample space is uniform

Probability

❑ Applying Four-Step Method

- **Example--- Cont.**

So what is the **probability that (a) beats (b)?**

$$\boldsymbol{P(E) = \frac{|E|}{|S|}}$$

Which in this case $= \frac{5}{9}$

(a) Beats (b) more than half of the time.

Probability

❑ Applying Four-Step Method

- **Example--- Cont.**

What about the following:

- ❖ (a) vs. (c)
- ❖ (b) vs. (c)

Homework!

Set Theory and Probability

Sample Space S : A nonempty countable set.

An element $w \in S$ is called an outcome.

A subset of S is called an event to which a probability is assigned.

If you look closely, you will realize that to calculate this probability we first have to count the elements in these sets.

Probability

- ❑ **Counting**

Rules of counting the elements in a set

Probability

❑ The Addition Rule

- The basic rule underlying the calculation of the number of elements in a union or difference or intersection is the addition rule.
- This rule states that the number of elements in a union of mutually disjoint finite sets equals the sum of the number of elements in each of the component sets.

Theorem 9.3.1:

Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then

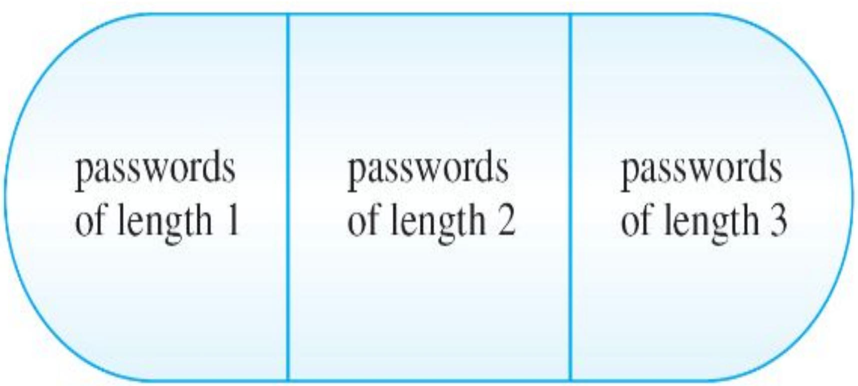
$$N(A) = N(A_1) + N(A_2) + \ldots + N(A_k)$$

Probability

❑ The Addition Rule---Cont.

Example: A computer access password consists of from one to three letters chosen from the 26 in the alphabet with repetitions allowed. How many different passwords are possible?

Solution: The set of all passwords can be partitioned into subsets consisting of those of length 1, those of length 2, and those of length 3 as shown in the figure below.



Probability

❑ The Addition Rule---Cont.

By the addition rule, the total number of passwords equals the number of passwords of length 1, plus the number of passwords of length 2, plus the number of passwords of length 3.

Now the,

Number of passwords of length 1= 26

Because there are 26 letters in the alphabet

Number of passwords of length 2 $=26^2$

Because forming such a word can be thought of as a two step process in which there are 26 ways to perform each step

Probability

❑ The Addition Rule---Cont.

Number of passwords of length 3 $=26^3$

Because forming such a word can be thought of as a three step process in which there are 26 ways to perform each step

Hence the total number of passwords$= 26^1+26^2+26^3=18{,}278$

Probability

❑ The Difference Rule

An important consequence of the addition rule is the fact that if the number of elements in a set *A* and the number in a subset *B* of *A* are both known, then the number of elements that are in *A* and not in *B* can be computed.

- **Theorem 9.3.2:** The Difference Rule:

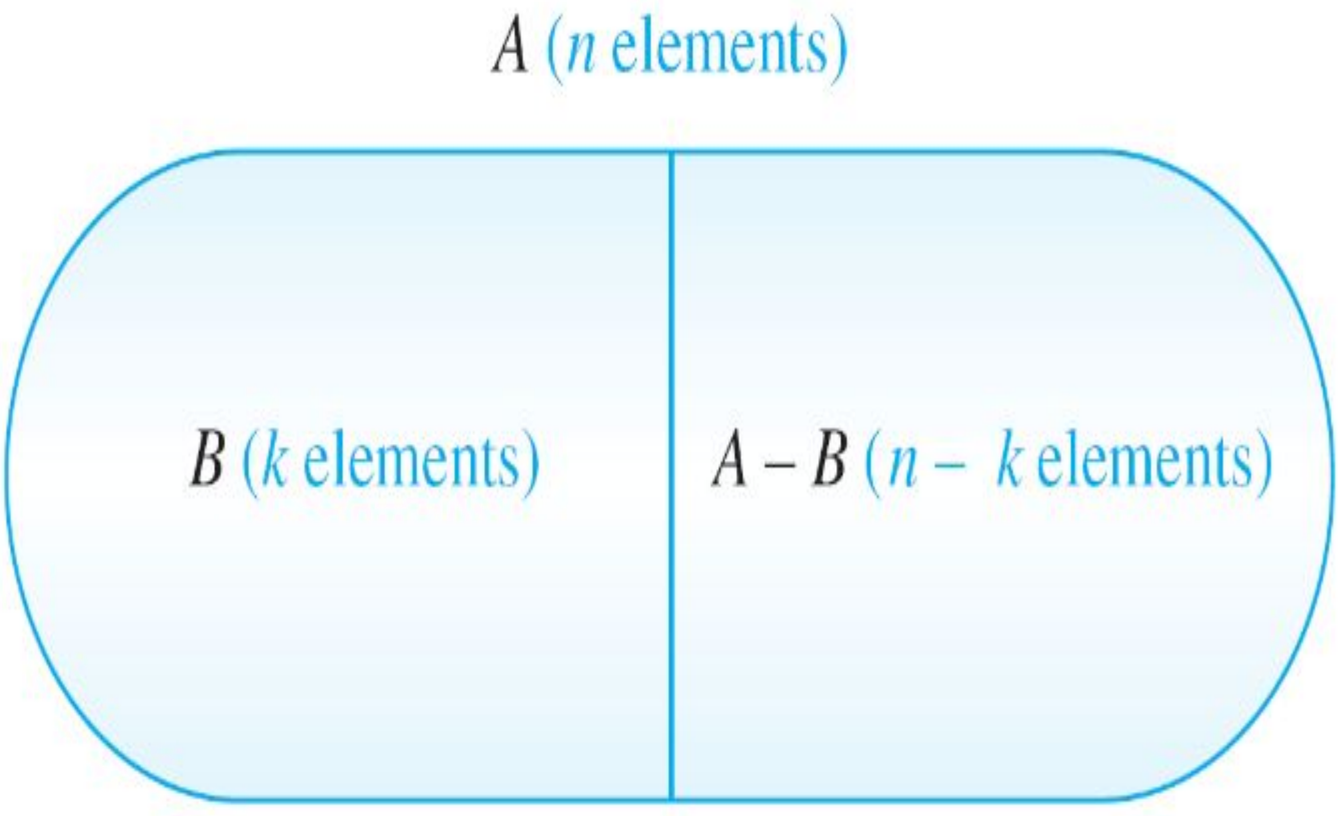
If A is finite set and B is a subset of A, then

$$N(A-B) = N(A) - N(B)$$

Probability

❑ The Difference Rule---Cont.

The difference rule is illustrated below.



Probability

❑ The Difference Rule---Cont.

- The difference rule holds for the following reason: If B is a subset of A, then the two sets B and $A - B$ have no elements in common and $B \cup (A - B) = A$. Hence, by the addition rule,

$$N(B) + N(A - B) = N(A).$$

Subtracting $N(B)$ from both sides gives the equation

$$N(A - B) = N(A) - N(B).$$

Probability

- **The Difference Rule---Cont.**
  - **Example:**

    A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition allowed.

    a. How many PINs contain repeated symbols?

    b. If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?

Probability

- **The Difference Rule---Cont.**

  a. How many PINs contain repeated symbols?

  Let's use the board to intuitively explain why the Difference Rule will work here!

Probability

- **The Difference Rule---Cont.**
  - **Example --- Cont.:**

    There are $36^4 = 1{,}679{,}616$ PINs when repetition is allowed,

    and there are $36 \bullet 35 \bullet 34 \bullet 33 = 1{,}413{,}720$ PINs when repetition is not allowed.

Probability

- **The Difference Rule---Cont.**
  - **Example --- Cont.:**

There are $36^4 = 1{,}679{,}616$ PINs when repetition is allowed,

and there are $36 \bullet 35 \bullet 34 \bullet 33 = 1{,}413{,}720$ PINs when repetition is not allowed.

Thus, by the difference rule, there are

$$1{,}679{,}616 - 1{,}413{,}720 = 265{,}896$$

PINs that contain at least one repeated symbol.

Probability

- **The Difference Rule---Cont.**

b. If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?

So, how would you figure this out?

Probability

- **The Difference Rule---Cont.**

  - **Example --- Cont.:**

There are 1,679,616 PINs in all, and by part (a) 265,896 of these contain at least one repeated symbol.

Thus, by the equally likely probability formula, the probability that a randomly chosen PIN contains a repeated

symbol is $\frac{\mathbf{265{,}896}}{\mathbf{1{,}679{,}616}} \cong \mathbf{0.158 = 15.8\%}$

Probability

- **The Difference Rule---Cont.**

  An alternative solution to Example 3(**b**) is based on the observation that if S is the set of all PINs and A is the set of all PINs with no repeated symbol, then $S - A$ is the set of all PINs with at least one repeated symbol.

Probability

❑ The Difference Rule---Cont.

An alternative solution to Example 3(**b**) is based on the observation that if S is the set of all PINs and A is the set of all PINs with no repeated symbol, then $S - A$ is the set of all PINs with at least one repeated symbol.

It follows that

$$P(S - A) = \frac{N(S-A)}{N(S)}$$

By definition of Probability in the equally likely case

Probability

❑ The Difference Rule---Cont.

An alternative solution to Example 3(**b**) is based on the observation that if S is the set of all PINs and A is the set of all PINs with no repeated symbol, then $S-A$ is the set of all PINs with at least one repeated symbol.

It follows that

$$P(S-A) = \frac{N(S-A)}{N(S)} \quad \boxed{\text{By definition of Probability in the equally likely case}}$$

$$= \frac{N(S)-N(A)}{N(S)} \quad \boxed{\text{By the difference rule}}$$

Probability

❑ The Difference Rule---Cont.

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It follows that

$$P(S-A) = \frac{N(S-A)}{N(S)} \quad \text{By definition of Probability in the equally likely case}$$

$$= \frac{N(S)-N(A)}{N(S)} \quad \text{By the difference rule}$$

$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)} \quad \text{By the law of fractions}$$

Probability

❑ The Difference Rule---Cont.

An alternative solution to Example 3(**b**) is based on the observation that if S is the set of all PINs and A is the set of all PINs with no repeated symbol, then $S - A$ is the set of all PINs with at least one repeated symbol.

It follows that

$$P(S-A) = \frac{N(S-A)}{N(S)}$$ *By definition of Probability in the equally likely case*

$$= \frac{N(S)-N(A)}{N(S)}$$ *By the difference rule*

$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)}$$ *By the law of fractions*

$$= 1 - P(A)$$ *By the definition of probability inthe equally likely case*

Probability

❑ The Difference Rule---Cont.

We know that the probability that a PIN chosen at random contains no repeated symbol is

$$P(A) = \frac{1{,}413{,}720}{1{,}679{,}616} \cong 0.8417$$

Probability

- **The Difference Rule---Cont.**

We know that the probability that a PIN chosen at random contains no repeated symbol is

$$P(A) = \frac{1,413,720}{1,679,616} \cong 0.8417$$

And hence

$$P(S - A) \cong 1 - 0.8417$$

$$\cong 0.158$$

$$= 15.8\%$$

Probability

- **The Difference Rule---Cont.**

This solution illustrates a more general property of probabilities: that the probability of the complement of an event is obtained by subtracting the probability of the event from the number 1.

Formula for the **Probability of the Complement** of an event!

If S is a finite sample space and A is an event in S, then

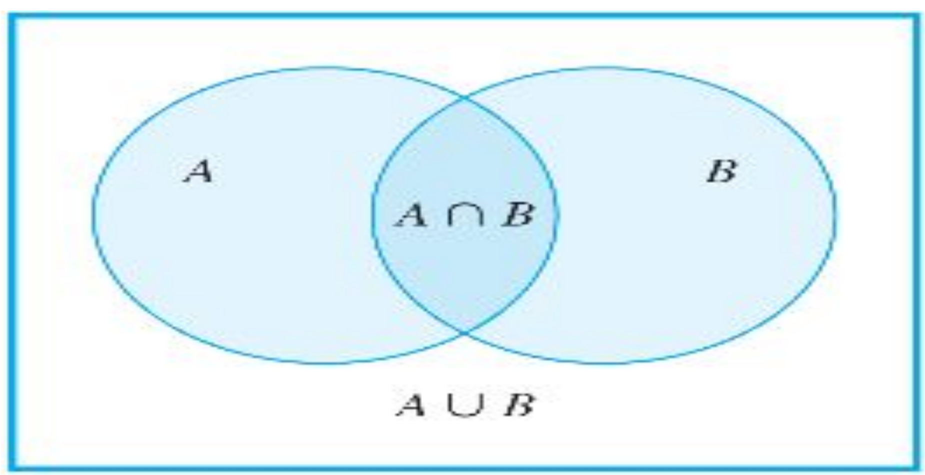
$$P(A^c) = 1 - P(A).$$

Probability

❑ The Inclusion/Exclusion Rule

The addition rule says how many elements are in a union of sets if the sets are mutually disjoint. Now consider the question of how to determine the number of elements in a union of sets when some of the sets overlap.

For simplicity, begin by looking at a union of two sets A and B, as shown below.



Probability

- **The Inclusion/Exclusion Rule--- Cont.**

To get an accurate count of the elements in $A \cup B$, it is necessary to subtract the number of elements that are in both A and B. Because these are the elements in $A \cap B$.

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

Probability

❑ Counting Rules in terms of Probabilities

If $\{E_0, E_1, \ldots.\}$ is collection of disjoint events, then

$$\mathrm{P}\left(\bigcup_{n \in N} E_n\right) = \sum_{n \in N} \mathrm{P}(E_n)$$

Probability

- **Counting Rules in terms of Probabilities---Cont.**

Complement Rule:

$$\mathrm{P}(\bar{A}) = 1 - \mathrm{P}(A)$$

Probability

❑ Counting Rules in terms of Probabilities---Cont.

$$P(\boldsymbol{B}-\boldsymbol{A})=\boldsymbol{P}(\boldsymbol{B})-P(\boldsymbol{A}\cap\boldsymbol{B})$$

Difference Rule

$$P(\boldsymbol{A}\cup\boldsymbol{B})=\boldsymbol{P}(\boldsymbol{A})+\boldsymbol{P}(\boldsymbol{B})-P(\boldsymbol{A}\cap\boldsymbol{B})$$

Inclusion – Exclusion

Further Counting

❑ Counting Subsets of a Set: Combinations:

Look at these examples:

- In how many ways, can I select 5 books from my collection of 100 to take on vacation?
- How many different ways 13-card Bridge hands can be dealt from a 52-card deck?
- In how many ways, can I select 5 toppings for my pizza if there are 14 available?

What is common in all these questions?

Further Counting

❑ Counting Subsets of a Set: Combinations:

Look at these examples:

- In how many ways, can I select 5 books from my collection of 100 to take on vacation?
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- In how many ways, can I select 5 toppings for my pizza if there are 14 available?

What is common in all these questions?

Each is trying to find "how many k-element subsets of an n-element set are there?"

❑ Counting Subsets of a Set: Combinations---Cont.

Why Count Subsets of Set?

- **Example:**

  Suppose we select 5 cards at random from a deck of 52 cards.

  What is the probability that we will end up having a full house?

  Doing this using the possibility tree will take some effort.

Counting Subsets of a Set: Combinations---Cont.

- How to calculate "n choose k"??
  - Permutations
  - Division rule

We will continue from here in the next lecture!