Tukey's 1-Degree of Freedom for Non-Additivity

Yields for 8 Business Indices Over 18 Years

K.V. Smith(1969). "Stock Price and Economic Indexes for Generating Efficient Portfolios," *The Journal of Business*, Vol. 42, #3, pp. 326-336

Experimental Setting

- 2-Way ANOVA with one measurement per combination of levels of factors A and B (N=a(b)(1))
- Additive Model: $E(Y_{ij}) = \mu + \alpha_i + \beta_j$
- Interaction Model (1 df): $E(Y_{ij}) = \mu + \alpha_i + \beta_j + \alpha\beta_{ij}$
  - Where $\alpha\beta_{ij} = \eta\alpha_i\beta_j/\mu = D\alpha_i\beta_j$
  - Procedure Involves Estimating D and testing whether the parameter equals 0

Data – 18 Years for 8 Business Indices

Year\Index	DJIA	POOR	NYSE	GNP	CPI	FRB	BWEEK	MONEY	Average
1965	1.103	1.099	1.095	1.086	1.017	1.083	1.093	1.048	1.078
1964	1.145	1.131	1.143	1.066	1.013	1.064	1.073	1.043	1.085
1963	1.169	1.201	1.180	1.050	1.012	1.051	1.060	1.038	1.095
1962	0.890	0.872	0.880	1.072	1.012	1.078	1.018	1.013	0.979
1961	1.207	1.231	1.240	1.032	1.011	1.099	1.154	1.031	1.126
1960	0.896	0.953	0.976	1.041	1.016	1.029	0.917	0.924	0.969
1959	1.184	1.094	1.097	1.086	1.008	1.127	1.073	1.006	1.084
1958	1.425	1.376	1.366	1.044	1.028	0.930	1.099	1.038	1.163
1957	0.833	0.856	0.866	1.056	1.035	1.008	0.906	0.993	0.944
1956	1.000	1.034	1.026	1.055	1.015	1.034	1.011	1.012	1.023
1955	1.231	1.301	1.222	1.095	0.997	1.126	1.125	1.020	1.140
1954	1.393	1.497	1.426	0.994	1.004	0.940	1.074	1.029	1.170
1953	0.965	0.925	0.938	1.053	1.008	1.083	0.952	1.010	0.992
1952	1.074	1.109	1.065	1.055	1.022	1.037	1.141	1.038	1.068
1951	1.174	1.178	1.132	1.156	1.080	1.085	1.012	1.055	1.109
1950	1.179	1.247	1.211	1.102	1.009	1.157	1.213	1.057	1.147
1949	1.114	1.091	1.102	0.996	0.990	0.945	0.995	0.994	1.028
1948	0.972	0.996	0.972	1.110	1.077	1.041	1.019	0.985	1.022
Average	1.109	1.122	1.108	1.064	1.020	1.051	1.052	1.019	1.068

Factor A: Year (a=18)   Factor B: Index (b=8)

Algorithm

- **Fit the additive Model and estimate μ, α_i and β_j**

$$\hat{\mu} = \overline{Y}_{\bullet\bullet} \qquad \hat{\alpha}_i = \overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet} \qquad \hat{\beta}_j = \overline{Y}_{\bullet j} - \overline{Y}_{\bullet\bullet}$$

- **Fit the interaction model with $\alpha\beta_{ij} = D\alpha_i\beta_j$**

$$Y_{ij} = \mu_{\bullet\bullet} + \alpha_i + \beta_j + D\alpha_i\beta_j + \varepsilon_{ij}$$

$$Y_{ij} = \hat{\mu}_{\bullet\bullet} + \hat{\alpha}_i + \hat{\beta}_j + D\hat{\alpha}_i\hat{\beta}_j + e_{ij}$$

- **Use Least Squares to estimate D**
- **Obtain Sum of Squares for Interaction and Remainder**
- **Conduct 1-degree of freedom F-test of H_0: D=0**

OLS Estimation of D

$$Q=\sum_{i=1}^{a}\sum_{j=1}^{b}e_{ij}^{2}=\sum_{i=1}^{a}\sum_{j=1}^{b}\left(Y_{ij}-\hat{\mu}_{\bullet\bullet}-\hat{\alpha}_{i}-\hat{\beta}_{j}-D\hat{\alpha}_{i}\hat{\beta}_{j}\right)^{2}$$

$$\frac{\partial Q}{\partial D}=2\sum_{i=1}^{a}\sum_{j=1}^{b}\left(Y_{ij}-\hat{\mu}_{\bullet\bullet}-\hat{\alpha}_{i}-\hat{\beta}_{j}-D\hat{\alpha}_{i}\hat{\beta}_{j}\right)\left(-\hat{\alpha}_{i}\hat{\beta}_{j}\right)$$

$$=-2\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\left(Y_{ij}\hat{\alpha}_{i}\hat{\beta}_{j}\right)-\hat{\mu}_{\bullet\bullet}\sum_{i=1}^{a}\hat{\alpha}_{i}\sum_{j=1}^{b}\hat{\beta}_{j}-\sum_{i=1}^{a}\hat{\alpha}_{i}^{2}\sum_{j=1}^{b}\hat{\beta}_{j}-\sum_{i=1}^{a}\hat{\alpha}_{i}\sum_{j=1}^{b}\hat{\beta}_{j}^{2}-D\sum_{i=1}^{a}\hat{\alpha}_{i}^{2}\sum_{j=1}^{b}\hat{\beta}_{j}^{2}\right]$$

$$=-2\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\left(Y_{ij}\hat{\alpha}_{i}\hat{\beta}_{j}\right)-0-0-0-D\sum_{i=1}^{a}\hat{\alpha}_{i}^{2}\sum_{j=1}^{b}\hat{\beta}_{j}^{2}\right]$$

Setting $\frac{\partial Q}{\partial D}=0 \quad \Rightarrow 0=\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\left(Y_{ij}\hat{\alpha}_{i}\hat{\beta}_{j}\right)-\hat{D}\sum_{i=1}^{a}\hat{\alpha}_{i}^{2}\sum_{j=1}^{b}\hat{\beta}_{j}^{2}\right] \quad \Rightarrow \quad \hat{D}=\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\left(Y_{ij}\hat{\alpha}_{i}\hat{\beta}_{j}\right)}{\sum_{i=1}^{a}\hat{\alpha}_{i}^{2}\sum_{j=1}^{b}\hat{\beta}_{j}^{2}}$

$$\Rightarrow \quad \hat{D}=\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\left(Y_{ij}\left(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet}\right)\left(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet}\right)\right)}{\sum_{i=1}^{a}\left(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet}\right)^{2}\sum_{j=1}^{b}\left(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet}\right)^{2}}$$

Sum of Squares for Interaction ($\alpha\beta_{ij} = D\alpha_i\beta_j$)

$$SSAB^* = \sum_{i=1}^{a}\sum_{j=1}^{b}\hat{\alpha\beta}_{ij}^{2} = \sum_{i=1}^{a}\sum_{j=1}^{b}\hat{D}^{2}\hat{\alpha}_i^{2}\hat{\beta}_j^{2}$$

$$= \left[\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\left(Y_{ij}\left(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet}\right)\left(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet}\right)\right)}{\sum_{i=1}^{a}\left(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet}\right)^2\sum_{j=1}^{b}\left(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet}\right)^2}\right]^2 \sum_{i=1}^{a}\left(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet}\right)^2\sum_{j=1}^{b}\left(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet}\right)^2$$

$$= \frac{\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\left(Y_{ij}\left(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet}\right)\left(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet}\right)\right)\right]^2}{\sum_{i=1}^{a}\left(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet}\right)^2\sum_{j=1}^{b}\left(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet}\right)^2}$$

$$\text{"Remainder" SS}: SSREM^* = SSTO - SSA - SSB - SSAB^*$$

$$\text{Under } H_0: D = 0 \text{ (No interactio ns exist of the form } D\alpha_i\beta_j\text{)}$$

$$SSAB^* \sim \chi_1^2 \qquad SSREM^* \sim \chi^2_{(a-1)(b-1)-1} \qquad SSAB^* \perp SSREM^*$$

$$\Rightarrow F^* = \frac{\left(SSAB^*/1\right)}{\left(SSREM^*/[(a-1)(b-1)-1]\right)} \sim F_{1,(a-1)(b-1)-1}$$

mu
1.0679

Business Index Example

$$\sum_{i=1}^{18}\left(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet}\right)^2 = 0.081676$$

$$\sum_{j=1}^{8}\left(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet}\right)^2 = 0.011447$$

$$\sum_{i=1}^{18}\sum_{j=1}^{8} Y_{ij}\left(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet}\right)\left(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet}\right) = 0.022112$$

$$\hat{D} = \frac{0.022112}{(0.081676)(0.011447)} = 23.65$$

i	alpha_i
1	0.0101
2	0.0169
3	0.0273
4	-0.0885
5	0.0578
6	-0.0989
7	0.0165
8	0.0954
9	-0.1237
10	-0.0445
11	0.0718
12	0.1018
13	-0.0761
14	-0.0002
15	0.0411
16	0.0790
17	-0.0395
18	-0.0464

j	beta_j
1	0.0407
2	0.0539
3	0.0398
4	-0.0040
5	-0.0482
6	-0.0169
7	-0.0159
8	-0.0493

Y →

1.103	1.099	1.095	1.086	1.017	1.083	1.093	1.048
1.145	1.131	1.143	1.066	1.013	1.064	1.073	1.043
1.169	1.201	1.180	1.050	1.012	1.051	1.060	1.038
0.890	0.872	0.880	1.072	1.012	1.078	1.018	1.013
1.207	1.231	1.240	1.032	1.011	1.099	1.154	1.031
0.896	0.953	0.976	1.041	1.016	1.029	0.917	0.924
1.184	1.094	1.097	1.086	1.008	1.127	1.073	1.006
1.425	1.376	1.366	1.044	1.028	0.930	1.099	1.038
0.833	0.856	0.866	1.056	1.035	1.008	0.906	0.993
1.000	1.034	1.026	1.055	1.015	1.034	1.011	1.012
1.231	1.301	1.222	1.095	0.997	1.126	1.125	1.020
1.393	1.497	1.426	0.994	1.004	0.940	1.074	1.029
0.965	0.925	0.938	1.053	1.008	1.083	0.952	1.010
1.074	1.109	1.065	1.055	1.022	1.037	1.141	1.038
1.174	1.178	1.132	1.156	1.080	1.085	1.012	1.055
1.179	1.247	1.211	1.102	1.009	1.157	1.213	1.057
1.114	1.091	1.102	0.996	0.990	0.945	0.995	0.994
0.972	0.996	0.972	1.110	1.077	1.041	1.019	0.985

Sums of Squares & F-Test for Interaction

$$SSTO = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(Y_{ij} - \overline{Y}_{\bullet\bullet}\right)^2 = 1.78991$$

$$SSA = b\sum_{i=1}^{a}\left(\overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet}\right)^2 = 0.653406$$

$$SSB = a\sum_{j=1}^{b}\left(\overline{Y}_{\bullet j} - \overline{Y}_{\bullet\bullet}\right)^2 = 0.206039$$

$$SSAB^* = \frac{\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\left(Y_{ij}\left(\overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet}\right)\left(\overline{Y}_{\bullet j} - \overline{Y}_{\bullet\bullet}\right)\right)\right]^2}{\sum_{i=1}^{a}\left(\overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet}\right)^2\sum_{j=1}^{b}\left(\overline{Y}_{\bullet j} - \overline{Y}_{\bullet\bullet}\right)^2} = \frac{(0.022112)^2}{(0.081676)(0.011447)} = 0.522991$$

$$SSREM^* = SSTO - SSA - SSB - SSAB^* = 0.407475$$

Test Statistic for Testing $H_0 : D = 0$:

$$F^* = \frac{(SSAB^*/1)}{(SSREM^*/((18-1)(8-1))-1)} = \frac{SSAB^*}{(SSREM^*/118)} = \frac{0.522991}{(0.407475/118)} = 151.45$$

$F(.05,1,118) = 3.92 \quad P(F \geq F^*) = .0000$

Plot of Residuals From additive Model vs (a_i)(b_j)