Solving linear recurrence relations

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- Generating functions
- Using generating functions to solve recurrence relations

Generating functions

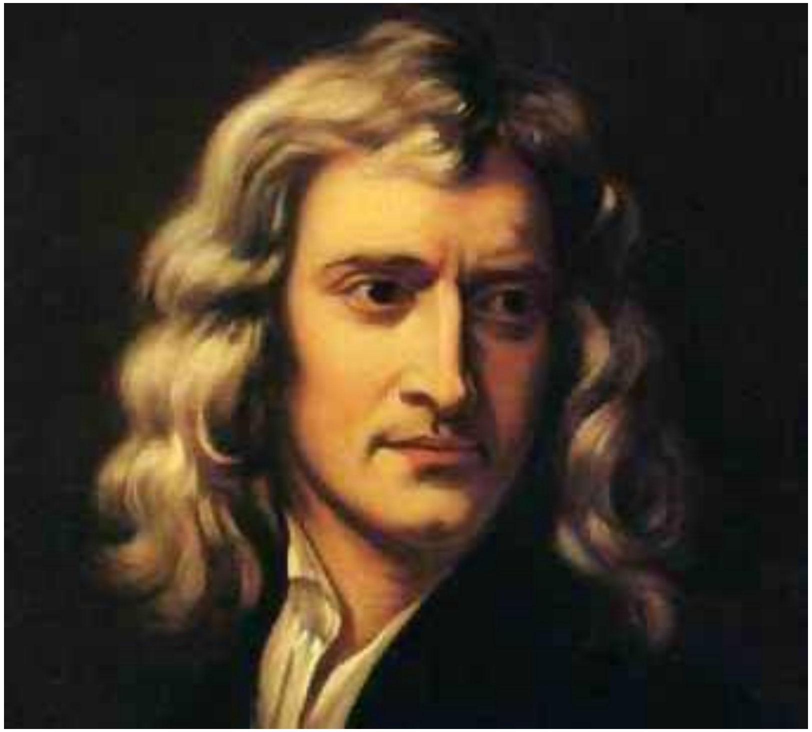
Generating functions provide a powerful tool for solving LHRRWCCs, as will be seen shortly.

They were invented in 1718 by the French mathematician Abraham De Moivre, when he used them to solve the Fibonacci recurrence relation. Generating functions can also solve combinatorial problems.

Generating functions

Abraham De Moivre (1667-1754), son of a surgeon, was born in Vitry-le-Francois, France.

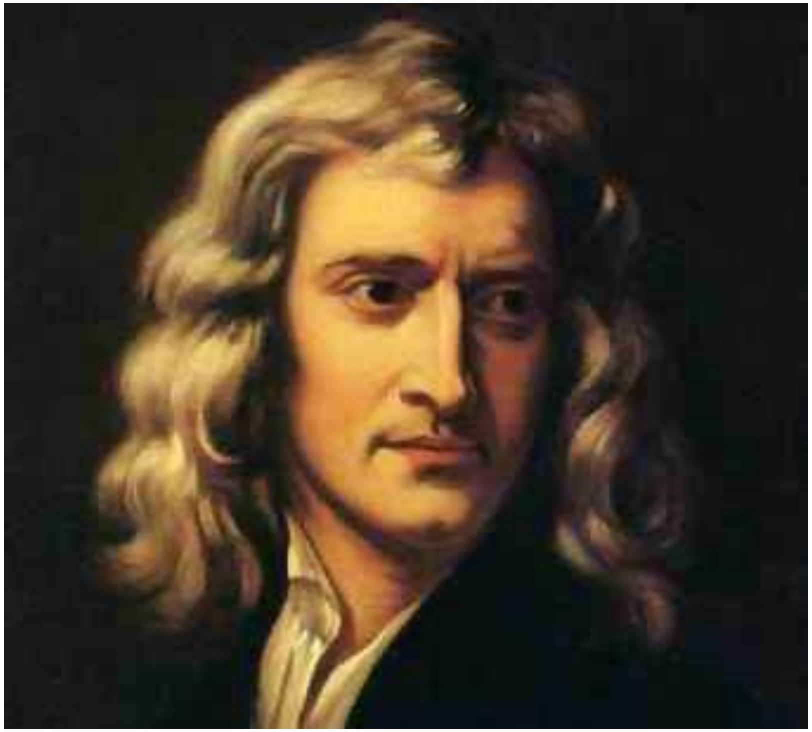
His formal education began at the Catholic village school, and then continued at the Protestant Academy at Sedan and later at Saumur.



Abraham De Moivre

Generating functions

He did not receive good training in mathematics until he moved to Paris in 1684, where he studied Euclid's later books and other texts.

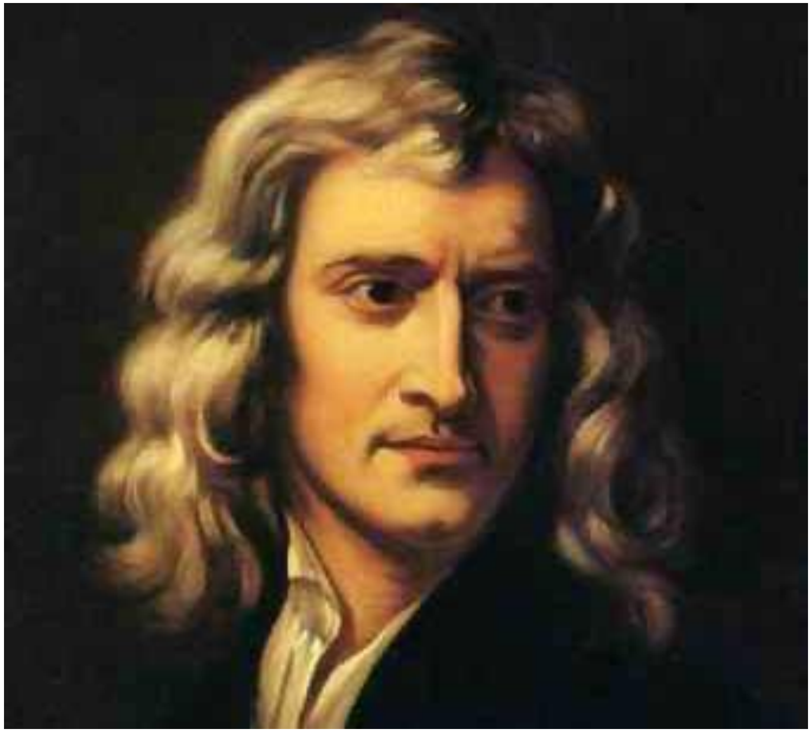


Abraham De Moivre

Generating functions

Around 1686, De Moivre emigrated to England, where he began his lifelong profession, tutoring in mathematics, and mastered Newton's Principia Mathematica.

In 1695 he presented a paper, his first, on Newton's theory of fluxions to the Royal Society of London and 2 years later he was elected a member of the Society.

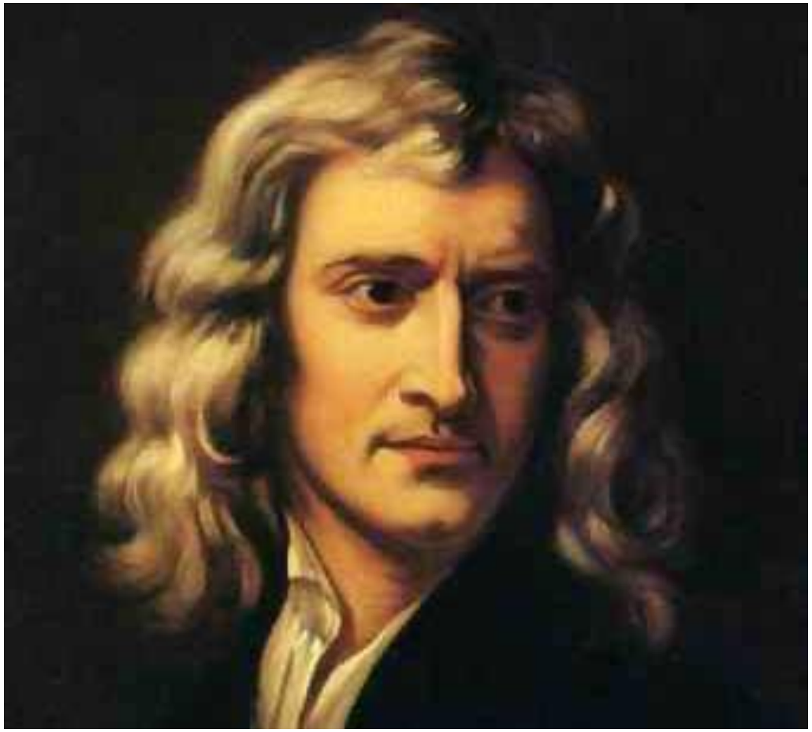


Abraham De Moivre

Generating functions

Unfortunately, despite his influential friends, he could not find an academic position.

He had to earn a living as a tutor, author, and expert on applications of probability to gambling and annuities.



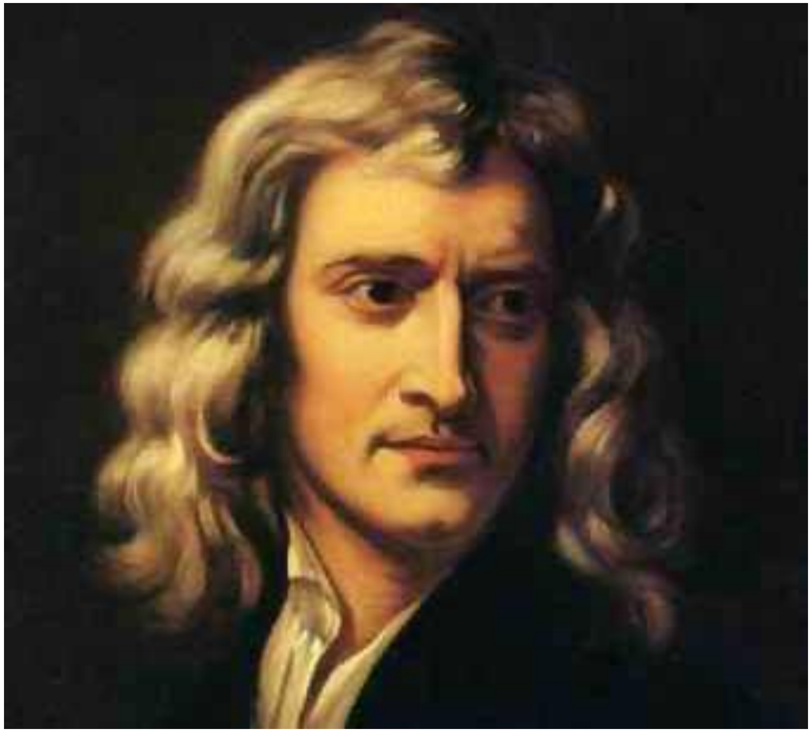
Abraham De Moivre

Generating functions

He dedicated his first book, a masterpiece, The Doctrine of Chances, to Newton.

His most notable discovery concerns probability theory: The binomial probability distribution can be approximated by the normal distribution.

De Moivre died in London.



Abraham De Moivre

Generating functions

To begin with, notice that the polynomial

$$1 + x + x^2 + x^3 + x^4 + x^5$$

can be written as

$$\frac{x^6 - 1}{x - 1}$$

You may verify this by the familiar long division method.

Accordingly,

$$f(x) = \frac{x^6 - 1}{x - 1}$$

is called the generating function of the sequence of coefficients $1, 1, 1, 1, 1, 1$ in the polynomial.

Generating functions

- More generally, we make the following definition.

Definition 1

The **generating function** for the sequence

$$a_0, a_1, \dots, a_n, \dots$$

of real numbers is the infinite series

$$g(x) = a_0 + a_1 x + \dots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

Generating functions

- We can define generating functions for finite sequences of real numbers by extending a finite sequence

$$a_0, a_1, \dots, a_n,$$

into an infinite sequence by setting

$$a_{n+1} = 0, a_{n+2} = 0, \text{ and so on.}$$

The generating function of this infinite sequence is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$g(x) = a_0 + a_1 x + \dots + a_n x^n.$$

Generating functions

For example,

$$1 + 2x + \ \ldots + (n+1)x^n + \cdots = \sum_{n=0}^{\infty}(n+1)x^n$$

is the generating function for the sequence of positive integers.

Generating functions

Since

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}$$

then

$$g(x) = \frac{x^n - 1}{x - 1}$$

is the generating function for the sequence of n ones.

Generating functions

A word of caution: The RHS of Equation

$$g(x) = a_0 + a_1 x + \ ... + a_n x^n + \cdots$$

is a formal power series in x.

The letter x does not represent anything.

The various powers x^n of x are simply used to keep track of the corresponding terms a_n of the sequence.

In other words, think of the powers x^n as placeholders.

Consequently, unlike in calculus, the convergence of the series is of no interest to us.

Generating functions

Definition 2

Two generating functions

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and

$$g(x) = \sum_{n=0}^{\infty} b_n x^n$$

are equal if

$a_n = b_n$ for every $n \geq 0$.

Generating functions

For example, let $f(x) = 1 + 3x + 6x^2 + 10x^3 + \cdots$ and $g(x) = 1 + \frac{2 \cdot 3}{2} x + \frac{3 \cdot 4}{2} x^2 + \frac{4 \cdot 5}{2} x^3 + \cdots$. Then $f(x) = g(x)$.

Generating functions

A generating function we will use frequently is

$$\frac{1}{1-ax} = 1 + ax + a^2x^2 + \cdots + a^nx^n + \cdots$$

Then
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

Generating functions

Can we add and multiply generating functions?

Yes!

Such operations are performed exactly the same way as polynomials are combined.

Generating functions

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be two generating functions.

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n \text{ and } f(x)g(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) x^n$$

Generating functions

For example,

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{1}{1-x}\cdot\frac{1}{1-x} \\
&= \left(\sum_{i=0}^{\infty} x^i\right)\left(\sum_{i=0}^{\infty} x^i\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} 1\cdot 1\right)x^n \\
&= \sum_{n=0}^{\infty}(n+1)x^n \\
&= 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{(1-x)^3} &= \frac{1}{1-x} \cdot \frac{1}{(1-x)^2} \\
&= \left(\sum_{n=0}^{\infty} x^n \right) \left[\sum_{n=0}^{\infty} (n+1) x^n \right] \\
&= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} 1 \cdot (n+1-i) \right] x^n \\
&= \sum_{n=0}^{\infty} [(n+1) + n + \cdots + 1] \, x^n \\
&= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n \\
&= 1 + 3x + 6x^2 + 10x^3 + \cdots
\end{aligned}
$$

Generating functions

- Before exploring how valuable generating functions are in solving LHRRWCCs, we illustrate how the technique of partial fraction decomposition, used in integral calculus, enables us to express the quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$ as a sum of proper fractions, where $\deg(p(x)) < \deg(q(x))$.

Generating functions

For example,

$$\frac{6x+1}{(2x-1)(2x+3)} = \frac{1}{2x-1} + \frac{2}{2x+3}$$

Generating functions

Fraction Decomposition Rule for $\frac{p(x)}{q(x)}$, where $\deg p(x) < \deg q(x)$

If $q(x)$ has a factor of the form $(ax+b)^m$, then the decomposition contains a sum of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

where A_i is a rational number.

Generating functions

- Examples $1 - 3$ illustrate the partial fraction decomposition technique.

We use their results to solve the recurrence relations in Examples $4 - 6$.

Example 1

Express $\frac{x}{(1-x)(1-2x)}$ as a sum of partial fractions.

SOLUTION:

Since the denominator contains two linear factors, we let

$$\frac{x}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$$

To find the constants A and B, multiply both sides by $(1-x)(1-2x)$:

$$x = A(1-2x) + B(1-x)$$

Example 1

Express $\dfrac{x}{(1-x)(1-2x)}$ as a sum of partial fractions.

$$x = A(1-2x) + B(1-x)$$

Now give convenient values to x. Setting $x = 1$ yields $A = -1$ and setting $x = 1/2$ yields $B = 1$. (The values of A and B can also be found by equating coefficients of like terms from either side of the equation and solving the resulting linear system.)

$$\frac{x}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{1}{1-2x} \quad \blacksquare$$

Example 2

Express $\dfrac{x}{1-x-x^2}$ as a sum of partial fractions.

SOLUTION:
First, factor $1 - x - x^2$:

$$1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$. (Notice that $\alpha + \beta = 1, \alpha\beta = -1$, $\alpha - \beta = \sqrt{5}$.)

Example 2

Express $\dfrac{x}{1-x-x^2}$ as a sum of partial fractions.

Let

$$\frac{x}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

Then

$$x = A(1-\beta x) + B(1-\alpha x)$$

Example 2

Express $\dfrac{x}{1 - x - x^2}$ as a sum of partial fractions.

$$x = A(1 - \beta x) + B(1 - \alpha x)$$

Equating coefficients of like terms, we get:

$$A + B = 0$$
$$-\beta A - \alpha B = 1$$

Example 2

Express $\dfrac{x}{1 - x - x^2}$ as a sum of partial fractions.

$$A + B = 0$$

$$-\beta A - \alpha B = 1$$

Solving this linear system yields $A = \dfrac{1}{\sqrt{5}} = -B$ (Verify this.).

Thus

$$\frac{x}{(1 - x - x^2)} = \frac{1}{\sqrt{5}}\left[\frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x}\right] \quad \blacksquare$$

Example 3

Express $\frac{2 - 9x}{1 - 6x + 9x^2}$ as a sum of partial fractions.

SOLUTION:
Again, factor the denominator:

$$1 - 6x + 9x^2 = (1 - 3x)^2$$

Example 3

Express $\dfrac{2-9x}{1-6x+9x^2}$ as a sum of partial fractions.

By the decomposition rule, let

$$\frac{2-9x}{1-6x+9x^2} = \frac{A}{1-3x} + \frac{B}{(1-3x)^2}$$

Then

$$2-9x = A(1-3x) + B$$

Example 3

Express $\frac{2-9x}{1-6x+9x^2}$ as a sum of partial fractions.

$$2 - 9x = A(1 - 3x) + B$$

This yields $A = 3$ and $B = -1$ (Verify this.).

Thus

$$\frac{2-9x}{1-6x+9x^2} = \frac{3}{1-3x} - \frac{1}{(1-3x)^2} \quad \blacksquare$$

Generating functions

Now we are ready to use partial fraction decompositions and generating functions to solve recurrence relations in the next three examples.

Example 4

Use generating functions to solve the recurrence relation $b_n = 2b_{n-1} + 1$, where $b_1 = 1$.

SOLUTION:

First, notice that the condition $b_1 = 1$ yields $b_0 = 0$. To find the sequence $\{b_n\}$ that satisfies the recurrence relation, consider the corresponding generating function

$$g(x) = b_0 + b_1 x + \quad b_2 x^2 + \quad b_3 x^3 + \cdots + \quad b_n x^n + \cdots$$

Example 4

Use generating functions to solve the recurrence relation $b_n = 2b_{n-1} + 1$, where $b_1 = 1$.

$$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n + \cdots$$

Then

$$2xg(x) = 2b_1 x^2 + 2b_2 x^3 + \cdots + 2b_{n-1} x^n + \cdots$$

Also,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

$$g(x) = b_0 + b_1 x + \quad b_2 x^2 + \quad b_3 x^3 + \cdots + \quad b_n x^n + \cdots$$

$$2xg(x) = \quad 2b_1 x^2 + 2b_2 x^3 + \cdots + 2b_{n-1} x^n + \cdots$$

$$\frac{1}{1-x} = 1 + x + \quad x^2 + \quad x^3 + \cdots + \quad x^n + \cdots$$

Then

$$\begin{aligned} g(x) - 2xg(x) - \frac{1}{1-x} &= -1 + (b_1 - 1)x + (b_2 - 2b_1 - 1)x^2 + \cdots \\ &\quad + (b_n - 2b_{n-1} - 1)x^n + \cdots \\ &= -1 \end{aligned}$$

since $b_1 = 1$ and $b_n = 2b_{n-1} + 1$ for $n \geq 2$. That is,

$$(1-2x)g(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

$$(1-2x)g(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

Then

$$g(x) = \frac{x}{(1-x)(1-2x)}$$

$$= -\frac{1}{1-x} + \frac{1}{1-2x}, \text{ by Example 1}$$

$$= -\left(\sum_{n=0}^{\infty} x^n\right) + \left(\sum_{n=0}^{\infty} 2^n x^n\right), \text{ by slide 11}$$

$$= \sum_{n=0}^{\infty} (2^n - 1)x^n$$

But $g(x) = \sum_{n=0}^{\infty} b_n x^n$, so $b_n = 2^n - 1$, $n \geq 1$. ■

Example 5

Using generating functions, solve the Fibonacci recurrence relation $F_n = F_{n-1} + F_{n-2}$, where $F_1 = 1 = F_2$.

SOLUTION:
Notice that the two initial conditions yield $F_0 = 0$. Let

$$g(x) = F_0 + F_1 x + F_2 x^2 + \cdots + F_n x^n + \cdots$$

be the generating function of the Fibonacci sequence. Since the orders of F_{n-1} and F_{n-2} are 1 and 2 less than the order of F_n, respectively, we find $xg(x)$ and $x^2 g(x)$:

$$xg(x) = F_1 x^2 + F_2 x^3 + F_3 x^4 + \cdots + F_{n-1} x^n + \cdots$$

$$x^2 g(x) = F_1 x^3 + F_2 x^4 + F_3 x^5 + \cdots + F_{n-2} x^n + \cdots$$

Example 5

$$g(x) - xg(x) - x^2 g(x) = F_1 x + (F_2 - F_1)x^2 + (F_3 - F_2 - F_1)x^3 + \cdots$$
$$+ (F_n - F_{n-1} - F_{n-2})x^n + \cdots$$
$$= x$$

That is,

$$(1 - x - x^2)g(x) = x$$
$$g(x) = \frac{x}{1 - x - x^2}$$
$$= \frac{1}{\sqrt{5}}\left[\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right], \text{ by Example 2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$

$$g(x) = \frac{x}{1 - x - x^2}$$

$$= \frac{1}{\sqrt{5}} \left[\frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x} \right]$$

Then

$$\sqrt{5} g(x) = \frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x}$$

$$= \sum_{n=0}^{\infty} \alpha^n x^n - \sum_{n=0}^{\infty} \beta^n x^n = \sum_{n=0}^{\infty} (\alpha^n - \beta^n) x^n$$

So

$$g(x) = \sum_{n=0}^{\infty} \frac{(\alpha^n - \beta^n)}{\sqrt{5}} x^n$$

Example 5

$$g(x) = \sum_{n=0}^{\infty} \frac{(\alpha^n - \beta^n)}{\sqrt{5}} x^n$$

Therefore, by the equality of generating functions,

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

(Recall that this is the **Binet form** of F_n.) ■

Example 6

Using generating functions, solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, where $a_0 = 2$ and $a_1 = 3$.

SOLUTION:
Let

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

Then

$$6xg(x) = 6a_0 x + 6a_1 x^2 + 6a_2 x^3 + \cdots + 6a_{n-1} x^n + \cdots$$

$$9x^2 g(x) = 9a_0 x^2 + 9a_1 x^3 + 9a_2 x^4 + \cdots + 9a_{n-2} x^n + \cdots$$

Example 6

$$6xg(x) = 6a_0x + 6a_1x^2 + 6a_2x^3 + \cdots + 6a_{n-1}x^n + \cdots$$
$$9x^2g(x) = 9a_0x^2 + 9a_1x^3 + 9a_2x^4 + \cdots + 9a_{n-2}x^n + \cdots$$

Then

$$\begin{aligned} g(x) - 6xg(x) + 9x^2g(x) &= a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 + 9a_0)x^2 + \cdots \\ &\quad + (a_n - 6a_{n-1} + 9a_{n-2})x^n + \cdots \\ &= 2 - 9x \end{aligned}$$

using the given conditions.

Example 6

$$g(x) - 6xg(x) + 9x^2g(x) = a_0 + (a_1 - 6a_0)x + (a_2 - 6a_1 + 9a_0)x^2 + \cdots$$
$$+ (a_n - 6a_{n-1} + 9a_{n-2})x^n + \cdots$$
$$= 2 - 9x$$

Thus

$$(1 - 6x + 9x^2)g(x) = 2 - 9x$$

Therefore,

$$g(x) = \frac{2 - 9x}{1 - 6x + 9x^2}$$

$$
\begin{aligned}
g(x) &= \frac{2 - 9x}{1 - 6x + 9x^2} \\
&= \frac{3}{1 - 3x} - \frac{1}{(1 - 3x)^2}, \text{ by Example 3} \\
&= 3\left(\sum_{n=0}^{\infty} 3^n x^n\right) - \sum_{n=0}^{\infty} (n + 1)3^n x^n \\
&= \sum_{n=0}^{\infty} [3^{n+1} - (n + 1)3^n]x^n \\
&= \sum_{n=0}^{\infty} 3^n (2 - n)x^n
\end{aligned}
$$

Thus

$$a_n = (2 - n)3^n, \quad n \geq 0 \qquad \blacksquare$$