SYSTEM OF LINEAR EQUATIONS

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OVERVIEW

- Rank of a matrix
- Systems of linear equations
- Matrix representation of SLEs and solution.
- Elementary row and column operations

RANK OF A MATRIX

- A matrix of *r* rows and *c* columns is said to be of ***order*** *r* by *c*. If it is a square matrix, *r* by *r*, then the matrix is of ***order*** *r.*
- The ***rank*** of a matrix equals the order of highest-order nonsingular submatrix.

- Nonsingular matrices have non-zero determinants
- Singular matrices have zero determinants

RANK OF 2×2 MATRIX

Let us see how to compute 2×2 matrix:

EXAMPLE:

The rank of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

- $rk(A) = 2$ *if* $det(A) = \text{ad} - \text{bc} \neq 0$, since both column vectors are independent in this case.
- $\text{rk}(A) = 1$ if $\det(A) = 0$ but $A \neq 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, since both column vectors are not linearly independent, but there is a single column vector that is linearly independent (i.e. non-zero).

- $\text{rk}(A) = 0$ if $A = 0$

COMPUTING RANK BY VARIOUS METHODS

- By Gauss elimination
- By determinants
- By minors

1. USING GAUSS ELIMINATION

GAUSS ELIMINATION:

Use elementary row operations to reduce A to echelon form. The rank of A is the number of pivots or leading coefficients in the echelon form. In fact, the pivot columns (i.e. the columns with pivots in them) are linearly independent.

Note that it is not necessary to and the reduced echelon form –any echelon form will do since only the pivots matter.

POSSIBLE RANKS:

Counting possible number of pivots, we see that $rk(A) \leq m$ and $rk(A) \leq n$ for any m n matrix A.

ELEMENTARY ROW AND COLUMN OPERATIONS

Key Idea 1

Elementary Row Operations

1. Add a scalar multiple of one row to another row, and replace the latter row with that sum
2. Multiply one row by a nonzero scalar
3. Swap the position of two rows

ELEMENTARY ROW AND COLUMN OPERATIONS

Definition 3

Reduced Row Echelon Form

A matrix is in *reduced row echelon form* if its entries satisfy the following conditions.

1. The first nonzero entry in each row is a 1 (called a *leading 1*).
2. Each leading 1 comes in a column to the right of the leading 1s in rows above it.
3. All rows of all 0s come at the bottom of the matrix.
4. If a column contains a leading 1, then all other entries in that column are 0.

A matrix that satisfies the first three conditions is said to be in *row echelon form*.

EXAMPLE

Gauss elimination:

* Find the rank of a matrix

$$A=\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

2. USING DETERMINANTS

Definition:

Let A be an m n matrix. A minor of A of order k is a determinant of a k X k sub-matrix of A.

We obtain the minors of order k from A by first deleting m rows and n -k columns, and then computing the determinant. There are usually many minors of A of a give order.

Example:

Find the minors of order 3 of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

3. USING MINORS

Proposition:

Let A be an m X n matrix. The rank of A is the maximal order of a non-zero minor of A.

Idea of proof:

If a minor of order k is non-zero, then the corresponding columns of A are linearly independent.

Computing the rank:

Start with the minors of maximal order k. If there is one that is non-zero, then $rk(A) = k$. If all maximal minors are zero, then $rk(A) < k$, and we continue with the minors of order k-1 and so on, until we and a minor that is non-zero. If all minors of order 1 (i.e. all entries in A) are zero, then $rk(A) = 0$.

EXAMPLE 1: RANK OF MATRIX

2×3 order matrix, $\mathbf{R} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

3 square submatrices:

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}, \quad \mathbf{R}_3 = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

Each of these has a determinant of 0, so the rank is less than 2. Thus the rank of **R** is 1.

EXAMPLE 2: RANK OF MATRIX

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 9 & 10 & 11 \end{bmatrix}$$

Since $|\mathbf{A}|=0$, the rank is not 3. The following submatrix has a nonzero determinant:

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2(3) - 4(1) = 2$$

Thus, the rank of **A** is 2.

SYSTEMS OF LINEAR EQUATIONS

Definition 1

Linear Equation

A *linear equation* is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

where the x_i are variables (the unknowns), the a_i are coefficients, and c is a constant.

A *system of linear equations* is a set of linear equations that involve the same variables.

A *solution* to a system of linear equations is a set of values for the variables x_i such that each equation in the system is satisfied.

MATRIX REPRESENTATION OF SLES

- Any SLEs can be formulated in the matrix form:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots$$
$$a_{n1}x_1 + a_{n1}x_2 + \ldots + a_{nn}x_n = b_n$$

a_{ij} -elements of the coefficient matrix A, b - load vector

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots\ldots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b \end{pmatrix} \qquad A \cdot x = b$$

METHODS OF SOLVING SLE

SLE in a compact matrix form: $A \cdot x = b$

Inverse matrix A^{-1} : $A \cdot A^{-1} = I = A^{-1} \cdot A$

$$A^{-1} \cdot A \cdot x = A^{-1} \cdot b \quad \rightarrow \quad x = A^{-1} \cdot b$$

Thus, to solve SLE we need to invert the matrix.

METHODS OF SOLVING SLE

CRAMER'S RULE (2 X 2)

- Used to solve linear systems.
- Linear System: $\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$
- Coefficient Matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$
- If det A $\neq$ 0,

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\det A} \quad and \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det A}$$

- **Replace coefficients** for the variable you are finding **with constants**.

GAUSS ELIMINATION

- Two steps
- 1. Forward Elimination
- 2. Back Substitution

FORWARD ELIMINATION

A set of *n* equations and *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

(n-1) steps of forward elimination

FORWARD ELIMINATION

Step 1

For Equation 2, divide Equation 1 by a_{11} and multiply by a_{21}.

$$\left[\frac{a_{21}}{a_{11}}\right](a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

FORWARD ELIMINATION

Subtract the result from Equation 2.

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$-\quad a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or $\quad a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2$

FORWARD ELIMINATION

Repeat this procedure for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n = b'_3$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n = b'_n$$

End of Step 1

FORWARD ELIMINATION

Step 2

Repeat the same procedure for the 3rd term of Equation 3.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
\vdots \qquad\qquad & \quad \vdots \\
a''_{n3}x_3 + \ldots + a''_{nn}x_n &= b''_n
\end{aligned}
$$

End of Step 2

FORWARD ELIMINATION

At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

End of Step (n-1)

MATRIX FORM AT END OF FORWARD ELIMINATION

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

BACK SUBSTITUTION STARTING EQNS

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

BACK SUBSTITUTION

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

BACK SUBSTITUTION

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)} x_{i+1} - a_{i,i+2}^{(i-1)} x_{i+2} - \ldots - a_{i,n}^{(i-1)} x_n}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

Example 4 Put the augmented matrix of the following system of linear equations into reduced row echelon form.

$$\begin{array}{rcrcrcr} -3x_1 & - & 3x_2 & + & 9x_3 & = & 12 \\ 2x_1 & + & 2x_2 & - & 4x_3 & = & -2 \\ & & -2x_2 & - & 4x_3 & = & -8 \end{array}$$

Solution We start by converting the linear system into an augmented matrix.

$$\begin{bmatrix} \boxed{-3} & -3 & 9 & 12 \\ 2 & 2 & -4 & -2 \\ 0 & -2 & -4 & -8 \end{bmatrix}$$

Our next step is to change the entry in the box to a 1. To do this, let's multiply row 1 by $-\frac{1}{3}$.

$$-\frac{1}{3}R_1 \to R_1 \qquad \begin{bmatrix} 1 & 1 & -3 & -4 \\ 2 & 2 & -4 & -2 \\ 0 & -2 & -4 & -8 \end{bmatrix}$$

$$\tfrac{1}{2}R_3 \to R_3 \qquad \begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This ends what we will refer to as the *forward steps*. Our next task is to use the elementary row operations and go back and put zeros above our leading 1s. This is referred to as the *backward steps*. These steps are given below.

$$\begin{array}{c} 3R_3 + R_1 \to R_1 \\ -2R_3 + R_2 \to R_2 \end{array} \qquad \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$-R_2 + R_1 \to R_1 \qquad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

It is now easy to read off the solution as $x_1 = 7, x_2 = -2$ and $x_3 = 3$.

Definition 4

Gaussian Elimination

Gaussian elimination is the technique for finding the reduced row echelon form of a matrix using the above procedure. It can be abbreviated to:

1. Create a leading 1.
2. Use this leading 1 to put zeros underneath it.
3. Repeat the above steps until all possible rows have leading 1s.
4. Put zeros above these leading 1s.

EXISTENCE AND UNIQUENESS OF SOLUTIONS

Theorem 1

Solution Forms of Linear Systems

Every linear system of equations has exactly one solution, infinite solutions, or no solution.

This leads us to a definition. Here we don't differentiate between having one solution and infinite solutions, but rather just whether or not a solution exists.

Definition 5

Consistent and Inconsistent Linear Systems

A system of linear equations is *consistent* if it has a solution (perhaps more than one). A linear system is *inconsistent* if it does not have a solution.

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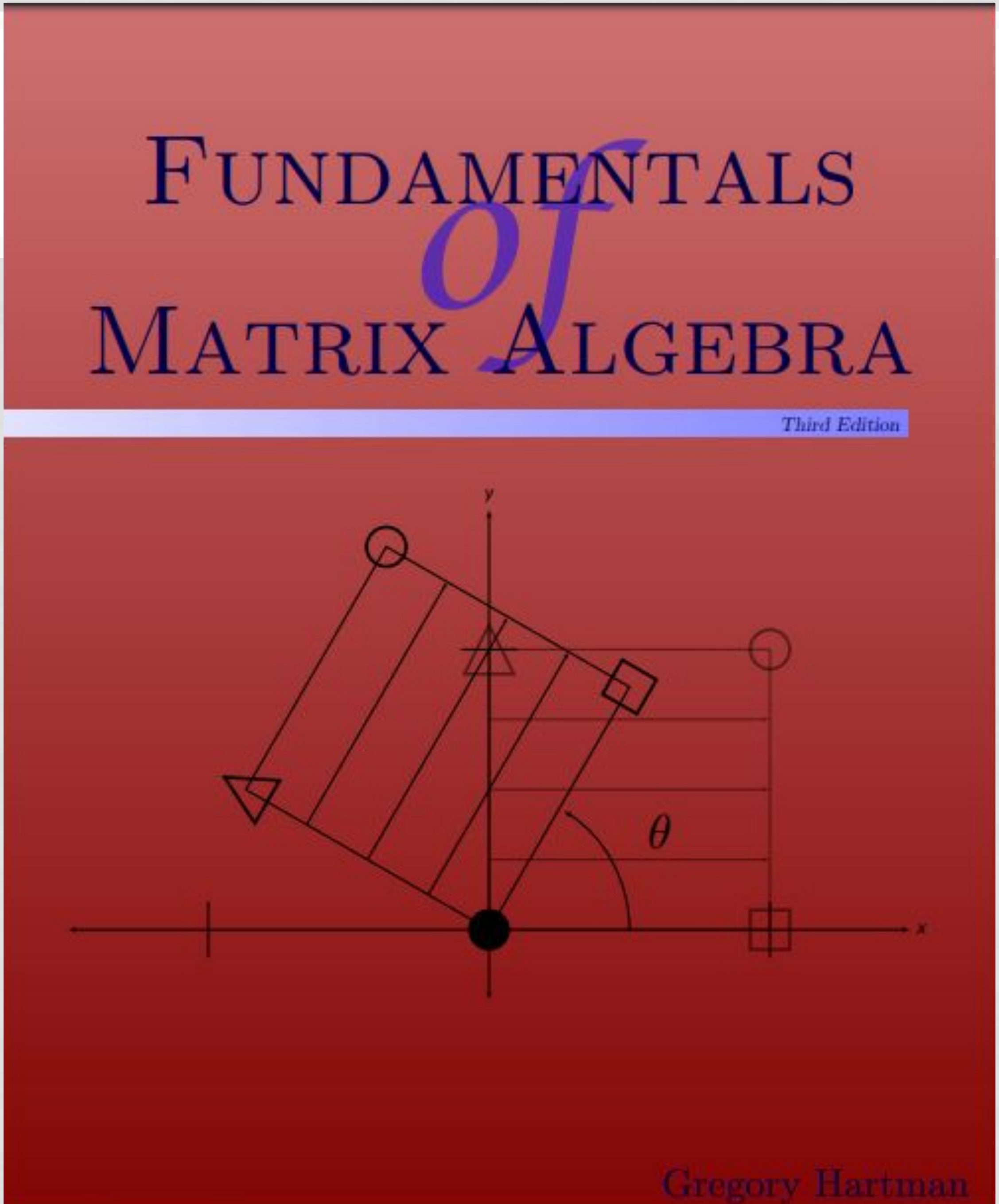
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