Ch8: Hypothesis Testing (2 Samples)

- 8.1 Testing the Difference Between Means
  (**Independent Samples, σ_1 and σ_2 Known**)
- 8.2 Testing the Difference Between Means
  (**Independent Samples, σ_1 and σ_2 Unknown**)
- 8.3 Testing the Difference Between Means
  (**Dependent Samples**)
- 8.4 Testing the Difference Between Proportions

8.1 Two Sample Hypothesis Test

- Compares two parameters from two populations.
- Two types of sampling methods:

Independent (unrelated) Samples

- Sample 1: Test scores for 35 statistics students.
- Sample 2: Test scores for 42 biology students who do not study statistics.

Dependent Samples (paired or matched samples)

Each member of one sample corresponds to a member of the other sample.

- Sample 1: Resting heart rates of 35 individuals before drinking coffee.
- Sample 2: Resting heart rates of the same individuals after drinking two cups of coffee.

Stating a Hypotheses in 2-Sample Hypothesis Test

Null hypothesis

- A statistical hypothesis H_0
- Statement of equality ($\leq$, $=$, or $\geq$).
- No difference between the parameters of two populations.

Alternative Hypothesis (H_a)
(Complementary to Null Hypothesis)

- A statement of inequality ($>$, $\neq$, or $<$).
- True when H_0 is false.

$$H_0: \mu_1 = \mu_2 \quad H_a: \mu_1 \neq \mu_2$$

OR

$$H_0: \mu_1 \leq \mu_2 \quad H_a: \mu_1 > \mu_2$$

OR

$$H_0: \mu_1 \geq \mu_2 \quad H_a: \mu_1 < \mu_2$$

Regardless of which hypotheses you use, you always assume there is no difference between the population means, or $\mu_1 = \mu_2$.

Two Sample *z*-Test for the Difference Between Means (μ_1 and μ_2.)

Three conditions are necessary

1. The samples must be <u>randomly selected</u>.
2. The samples must be <u>independent.</u>
3. <u>Each population must have a normal distribution with a known population standard deviation</u> OR each sample size must be at least 30.

Sampling distribution for $\bar{x}_1 - \bar{x}_2$ ***(difference of sample means) is a normal with:***

Mean: $\mu_{\bar{x}_1 - \bar{x}_2} = \mu_{\bar{x}_1} - \mu_{\bar{x}_2} = \mu_1 - \mu_2$

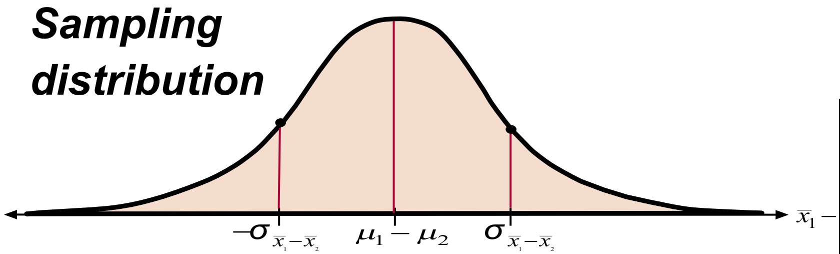
Standard error: $\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\sigma_{\bar{x}_1}^2 + \sigma_{\bar{x}_2}^2} = \sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}$

Test Statistic: $\bar{x}_1 - \bar{x}_2$

Standardized Test Statistic:

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}} \quad where \quad \sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Sampling distribution



- For large samples: use s_1 and s_2 in place of σ_1 and σ_2.
- For small samples: use a two-sample z-test if populations are normally distributed & pop. std deviations are known.

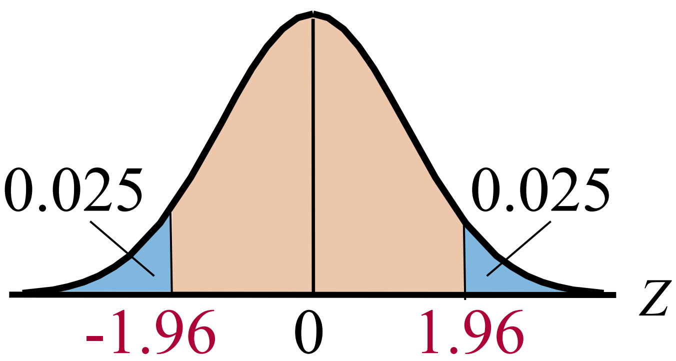
Using a Two-Sample z-Test for the Difference Between Means (Independent Samples ~~σ_1 and σ_2 known or n_1 and $n_2 \geq 30$~~)

In Words	In Symbols
1. State the claim mathematically. Identify the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Sketch the sampling distribution.	
4. Determine the critical value(s).	Use Table 4 in Appendix B.
5. Determine the rejection region(s).	
6. Find the standardized test statistic.	$z = \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}}$
7. Make a decision to reject or fail to reject the null hypothesis.	If z is in the rejection region, reject H_0.
8. Interpret the decision in the context of the original claim.	

Example1: Two-Sample *z*-Test for the Difference Between Means

A consumer education organization claims that there is a difference in the mean credit card debt of males and females in the United States. The results of a random survey of 200 individuals from each group are shown below. The two samples are independent. Do the results support the organization's claim? Use α = 0.05.

- H_0: $\mu_1 = \mu_2$
- H_a: $\mu_1 \neq \mu_2$
- α = .05
- n_1 = 200 , n_2 = 200
- **Rejection Region:**



Ti83/84
Stat-Tests
3:2-SampZTest

Females (1)	Males (2)
$\bar{x}_1 = \$2290$ $s_1 = \$750$ $n_1 = 200$	$\bar{x}_2 = \$2370$ $s_2 = \$800$ $n_2 = 200$

$$z = \frac{(2290 - 2370) - 0}{\sqrt{\frac{750^2}{200} + \frac{800^2}{200}}} = -1.03$$

Decision: Fail to reject H_0

At the 5% level of significance, there is not enough evidence to support the organization's claim that there is a difference in the mean credit card debt of males and females.

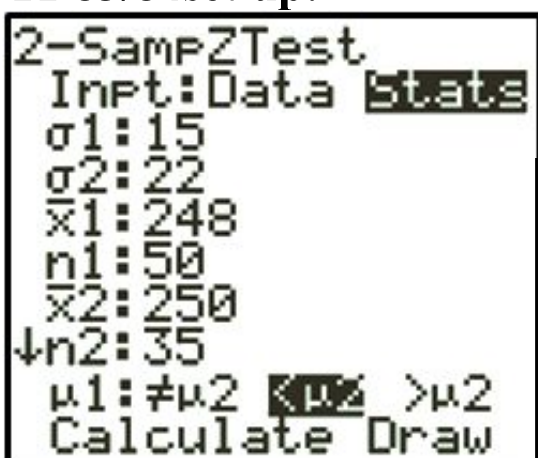
Example2: Using Technology to Perform a Two-Sample *z*-Test

The American Automobile Association claims that the average daily cost for meals and lodging for vacationing in Texas is less than the same average costs for vacationing in Virginia. The table shows the results of a random survey of vacationers in each state. The two samples are independent. At $\alpha = 0.01$, is there enough evidence to support the claim?

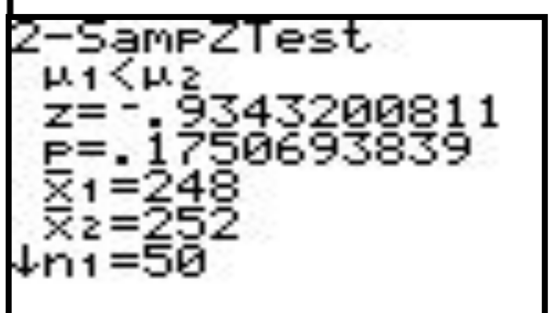
Texas (1)	Virginia (2)
$\bar{x}_1 = \$248$	$\bar{x}_2 = \$252$
$s_1 = \$15$	$s_2 = \$22$
$n_1 = 50$	$n_2 = 35$

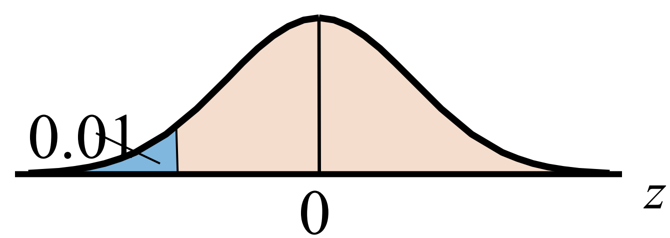
- H_0: $\mu_1 \geq \mu_2$
- H_a: $\mu_1 < \mu_2$

TI-83/84set up:



Calculate





Decision: Fail to reject H_0

At 1% level, Not enough evidence to support AAA's claim.

8.2 Two Sample *t*-Test for the Difference Between Means (σ_1 or σ_2 unknown)

- If (σ_1 or σ_2 is unknown and samples are taken from normally-distributed) **OR** If (σ_1 or σ_2 is unknown and both sample sizes are greater than or equal to 30) THEN a *t*-test may be used to test the difference between the population means μ_1 and μ_2.
- Three conditions are necessary to use a *t*-test for small independent samples.
  - The samples must be randomly selected.
  - The samples must be independent.
  - Each population must have a normal distribution.

Test Statistic:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}}$$

Two Sample *t*-Test for the Difference Between Means

The standard error and the degrees of freedom of the sampling distribution depend on whether the population variances and are equal.

- **Equal Variances**
- Information from the two samples is combined to calculate a **pooled estimate of the standard deviation** $\hat{\sigma}$.

$$\hat{\sigma} = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

  - The standard error for the sampling distribution of $\bar{x}_1 - \bar{x}_2$ is

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \hat{\sigma} \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

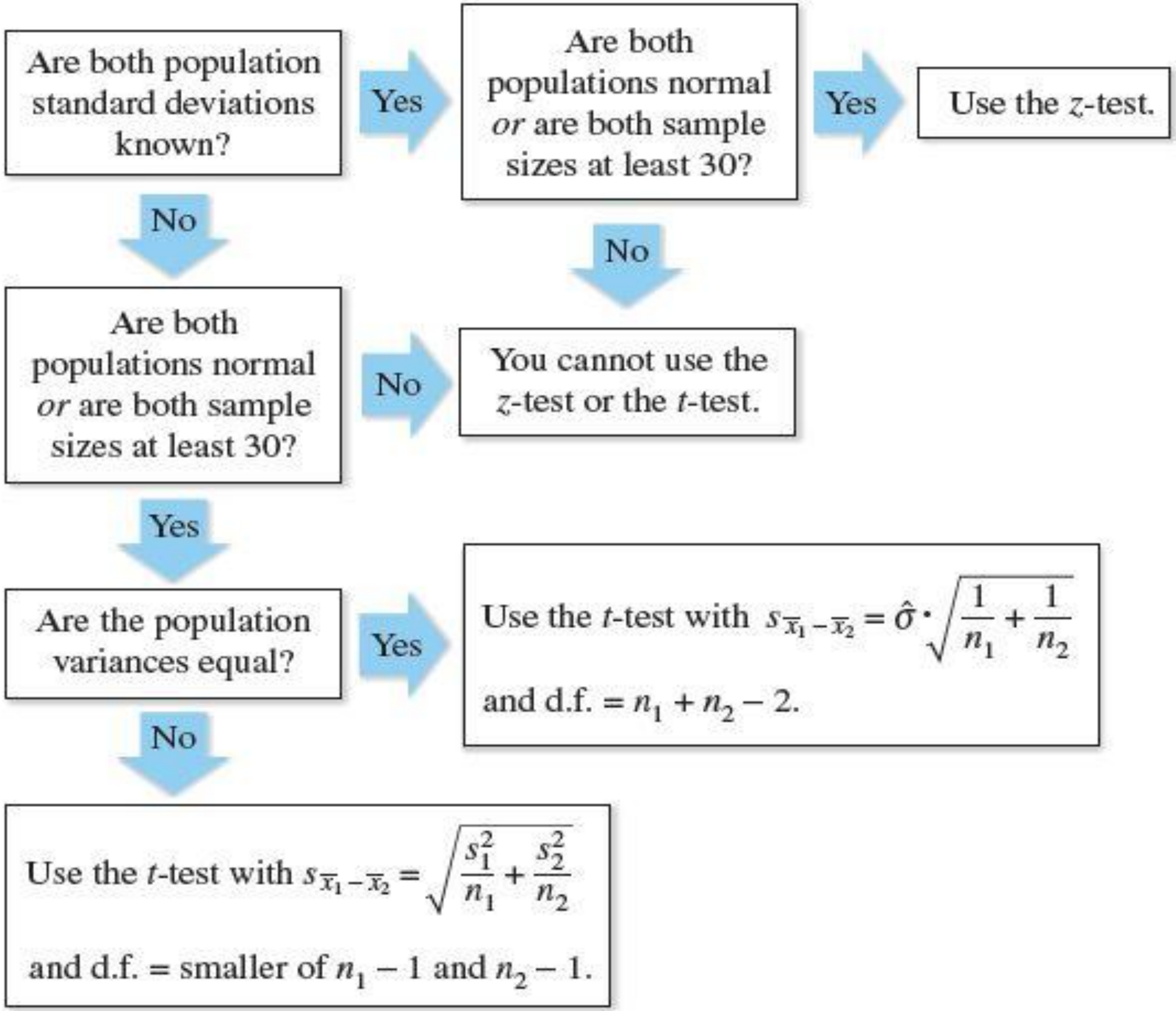
  - d.f.= $n_1 + n_2 - 2$

- **UnEqual Variances**
- **Standard Error is:**

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

d.f = smaller of $n_1 - 1$ or $n_2 - 1$

Normal or t-Distribution?



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Two-Sample *t*-Test for the Difference Between Means - Independent Samples (σ_1 or σ_2 unknown)

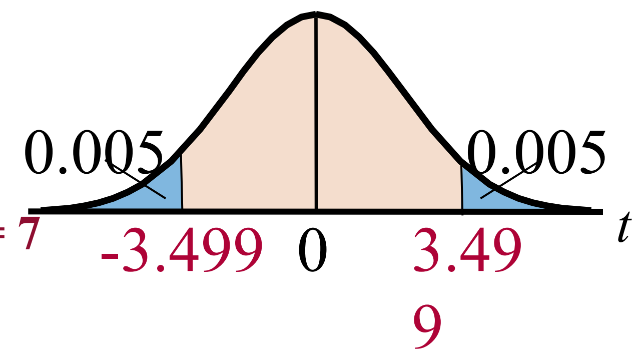
	In Words	In Symbols
1.	State the claim mathematically. Identify the null and alternative hypotheses.	State H_0 and H_a.
2.	Specify the level of significance.	Identify α.
3.	Identify the degrees of freedom and sketch the sampling distribution.	d.f. = $n_1 + n_2 - 2$ or d.f. = smaller of $n_1 - 1$ or $n_2 - 1$.
4.	Determine the critical value(s).	Use Table 5 in Appendix B.
5.	Determine the rejection region(s).	
6.	Find the standardized test statistic.	$t = \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}}$
7.	Make a decision to reject or fail to reject the null hypothesis and interpret the decision in the context of the original claim	If t is in the rejection region, reject H_0. Otherwise, fail to reject H_0.

Example: Two-Sample *t*-Test for the Difference Between Means

The braking distances of 8 Volkswagen GTIs and 10 Ford Focuses were tested when traveling at 60 miles per hour on dry pavement. The results are shown below. Can you conclude that there is a difference in the mean braking distances of the two types of cars? Use $\alpha = 0.01$. Assume the populations are normally distributed and the population variances are not equal. *(Consumer Reports)*

GTI (1)	Focus (2)
$\bar{x}_1 = 134$ ft	$\bar{x}_2 = 143$ ft
$s_1 = 6.9$ ft	$s_2 = 2.6$ ft
$n_1 = 8$	$n_2 = 10$

- H_0: $\mu_1 = \mu_2$
- H_a: $\mu_1 \neq \mu_2$
- $\alpha =$ 0.01
- d.f. = $8 - 1 = 7$



$$t = \frac{(134 - 143) - 0}{\sqrt{\frac{6.9^2}{8} + \frac{2.6^2}{10}}} = -3.496$$

- **Decision:** **Fail to Reject H_0**

At the 1% level of significance, there is not enough evidence to conclude that the mean braking distances of the cars are different.

Stat-Test
4: 2-SampTTest
Pooled: No

Example: Two-Sample *t*-Test for the Difference Between Means

A manufacturer claims that the calling range (in feet) of its 2.4-GHz cordless telephone is greater than that of its leading competitor. You perform a study using 14 randomly selected phones from the manufacturer and 16 randomly selected similar phones from its competitor. The results are shown below. At α = 0.05, can you support the manufacturer's claim? Assume the populations are normally distributed and the population variances are equal.

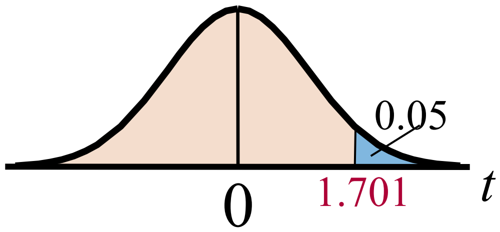
Manufacturer (1)	Competition (2)
$\bar{x}_1 = 1275$ ft $s_1 = 45$ ft $n_1 = 14$	$\bar{x}_2 = 1250$ ft $s_2 = 30$ ft $n_2 = 16$

H_0 = **$\mu_1 \le \mu_2$**

H_a = **$\mu_1 > \mu_2$** (Claim)

α = **.05**

d.f. = **14 + 16 – 2 = 28**



$$\sigma_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}} \cdot \sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

$$= \sqrt{\frac{(14-1)(45)^2+(16-1)(30)^2}{14+16-2}} \cdot \sqrt{\frac{1}{14}+\frac{1}{16}} \approx 13.8018$$

$$t = \frac{(\bar{x}_1-\bar{x}_2)-(\mu_1-\mu_2)}{\sigma_{\bar{x}_1-\bar{x}_2}} = \frac{(1275-1250)-0}{13.8018} \approx 1.811$$

Stat-Test
4: 2-SampTTest
Pooled: Yes

Decision: Reject H_0

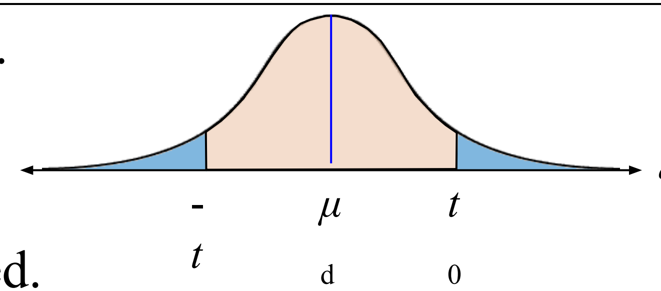
At 5% level of significance there is enough evidence to support the manufacturer's claim.

8.3 *t*-Test for the Difference Between Means (Paired Data/Dependent Samples)

- To perform a two-sample hypothesis test with dependent samples, the difference between each data pair is first found:
  - $d = x_1 - x_2$ Difference between entries for a data pair
- The test statistic is the mean $\bar{d}$ of these differences.
  - $\bar{d} = \frac{\sum d}{n}$ Mean of the differences between paired data entries in the dependent samples

Three conditions are required to conduct the test.

1. The samples must be randomly selected.
2. The samples must be dependent (paired).
3. Both populations must be normally distributed.



If these conditions are met, then the sampling distribution for is approximated by a *t*-distribution with $n - 1$ degrees of freedom, where n is the number of data pairs.

Symbols used for the *t*-Test for μ_d

Symbol	Description
n	The number of pairs of data **Degrees of Freedom (d.f.) = n - 1**
d	The difference between entries for a data pair, $d = x_1 - x_2$
μ_d	The hypothesized mean of the differences of paired data in the population
$\bar{d}$	Mean of the differences between paired data entries in dependent samples **(Test Statistic)** $\bar{d} = \frac{\sum d}{n}$
s_d	The standard deviation of the differences between the paired data entries in the dependent samples $s_d = \sqrt{\frac{\Sigma(d-\bar{d})^2}{n-1}} = \sqrt{\frac{\Sigma d^2 - \frac{(\Sigma d)^2}{n}}{n-1}}$

(Standardized Test Statistic) $t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$

t-Test for the Difference Between Means (Dependent Samples)

	In Words	*In Symbols*
1.	State the claim mathematically. Identify the null and alternative hypotheses.	State H_0 and H_a.
2.	Specify the level of significance.	Identify α.
3.	Identify the degrees of freedom and sketch the sampling distribution.	d.f. $= n - 1$
4.	Determine critical value(s) & rejection region	Use Table 5 in Appendix B If $n > 29$ use the last row (∞) .
5.	Calculate $\bar{d}$ and s_d. Use a table.	$\bar{d} = \frac{\sum d}{n}$ $s_d = \sqrt{\frac{\sum(d-\bar{d})^2}{n-1}} = \sqrt{\frac{\sum d^2 - \frac{(\sum d)^2}{n}}{n-1}}$
6.	Find the standardized test statistic.	$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$
7.	Make a decision to reject or fail to reject the null hypothesis and interpret the decision in the context of the original claim.	If t is in the rejection region, reject H_0. Otherwise, fail to reject H_0.

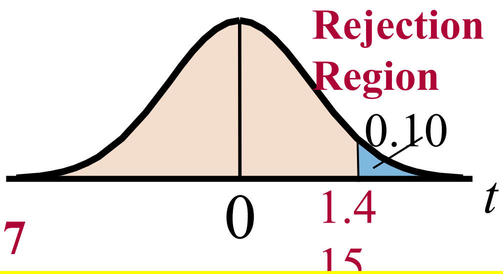
Example: *t*-Test for the Difference Between Means

A golf club manufacturer claims that golfers can lower their scores by using the manufacturer's newly designed golf clubs. Eight golfers are randomly selected, and each is asked to give his or her most recent score. After using the new clubs for one month, the golfers are again asked to give their most recent score. The scores for each golfer are shown in the table. Assuming the golf scores are normally distributed, is there enough evidence to support the manufacturer's claim at $\alpha = 0.10$?

Golfer	1	2	3	4	5	6	7	8
Score (old)	89	84	96	82	74	92	85	91
Score (new)	83	83	92	84	76	91	80	91

***d* = (old score) – (new score)**

- H_0: $\mu_d \le 0$
- H_a: $\mu_d > 0$
- α = 0.10
- d.f. = 8 – 1 = 7



$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}} \approx \frac{1.625 - 0}{3.0677/\sqrt{8}} \approx 1.498$$

Decision: Reject H_0

At the 10% level of significance, the results of this test indicate that after the golfers used the new clubs, their scores were significantly lower.

$\bar{d}$ & S_d calculations on next slide

Solution: Two-Sample *t*-Test for the Difference Between Means

d = (old score) – (new score)

Old	New	d	d^2
89	83	6	36
84	83	1	1
96	92	4	16
82	84	–2	4
74	76	–2	4
92	91	1	1
85	80	5	25
91	91	0	0
		$\Sigma = 13$	$\Sigma = 87$

$$\bar{d} = \frac{\sum d}{n} = \frac{13}{8} = 1.625$$

$$s_d = \sqrt{\frac{\Sigma d^2 - \frac{(\Sigma d)^2}{n}}{n-1}}$$

$$= \sqrt{\frac{87 - \frac{(13)^2}{8}}{8-1}}$$

$$\approx 3.0677$$

8.4 Two-Sample z-Test for Proportions

- Used to test the difference between two population proportions, p_1 and p_2.
- Three conditions are required to conduct the test.
  1. The samples must be randomly selected.
  2. The samples must be independent.
  3. The samples must be large enough to use a normal sampling distribution. That is,
     $n_1p_1 \geq 5,\ n_1q_1 \geq 5,\ n_2p_2 \geq 5,$ and $n_2q_2 \geq 5.$

- If these conditions are met, then the sampling distribution for $\hat{p}_1 - \hat{p}_2$ is normal
- **Mean:** $\mu_{\hat{p}_1-\hat{p}_2} = p_1 - p_2$ **(Test Statistic)**
- Find weighted estimate of p_1 and p_2 using

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2},\ \text{where } x_1 = n_1\hat{p}_1 \text{ and } x_2 = n_2\hat{p}_2$$

- Standard error: $\sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$

Standardized Test Statistic

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where $\bar{p} = \frac{x_1 + x_2}{n_1 + n_2}$ and $\bar{q} = 1 - \bar{p}$

Note: $n_1\bar{p},\ n_1\bar{q},\ n_2\bar{p},$ and $n_2\bar{q}$ must be at least 5.

Two-Sample *z*-Test for the Difference Between Proportions

In Words	*In Symbols*
1. State the claim. Identify the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Determine the critical value(s).	Use Table 4 in Appendix B.
4. Determine the rejection region(s).	
5. Find the weighted estimate of p_1 and p_2.	$\bar{p} = \dfrac{x_1 + x_2}{n_1 + n_2}$
6. Find the standardized test statistic.	$z = \dfrac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\bar{p}\bar{q}\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$
7. Make a decision to reject or fail to reject the null hypothesis and interpret decision in the context of the original claim.	If z is in the rejection region, reject H_0. Otherwise, fail to reject H_0.

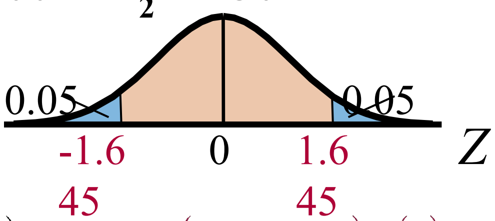
Example1: Two-Sample *z*-Test for the Difference Between Proportions

In a study of 200 randomly selected adult female**(1)** and 250 randomly selected adult male**(2)** Internet users, 30% of the females and 38% of the males said that they plan to shop online at least once during the next month. At α = 0.10 test the claim that there is a difference between the proportion of female and the proportion of male Internet users who plan to shop online.

$\alpha = .10$ $\quad n_1 = 200$ $\quad n_2 = 250$

$H_0: \mathbf{p_1 = p_2}$

$H_a: \mathbf{p_1 \neq p_2}$



$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\overline{p}\overline{q} \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \approx \frac{(0.30 - 0.38) - (0)}{\sqrt{(0.3444) \cdot (0.6556) \cdot \left(\frac{1}{200} + \frac{1}{250}\right)}}$$

$$\approx -1.77$$

$$x_1 = n_1 \hat{p}_1 = 60$$

$$x_2 = n_2 \hat{p}_2 = 95$$

$$\overline{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{60 + 95}{200 + 250} \approx 0.3444$$

$$\overline{q} = 1 - \overline{p} = 1 - 0.3444 = 0.6556$$

Note:

$n_1\overline{p} = 200(0.3444) \geq 5 \quad n_1\overline{q} = 200(0.6556) \geq 5$

$n_2\overline{p} = 250(0.3444) \geq 5 \quad n_2\overline{q} = 250(0.6556) \geq 5$

Decision: Reject H_o

At the 10% level of significance, there is enough evidence to conclude that there is a difference between the proportion of female and the proportion of male Internet users who plan to shop online.

Stat-Tests
5: 2-PropZTest

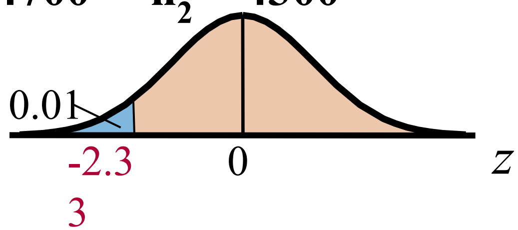
Example2: Two-Sample *z*-Test for the Difference Between Proportions

A medical research team conducted a study to test the effect of a cholesterol reducing medication(**1**). At the end of the study, the researchers found that of the 4700 randomly selected subjects who took the medication, 301 died of heart disease. Of the 4300 randomly selected subjects who took a placebo(**2**), 357 died of heart disease. At α = 0.01 can you conclude that the death rate due to heart disease is lower for those who took the medication than for those who took the placebo? *(New England Journal of Medicine)*

$\alpha = .01 \quad n_1 = 4700 \quad n_2 = 4300$

$H_0: p_1 \geq p_2$

$H_a: p_1 < p_2$



$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\bar{p}\bar{q}\cdot\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \approx \frac{(0.064 - 0.083) - (0)}{\sqrt{(0.0731)\cdot(0.9269)\cdot\left(\frac{1}{4700} + \frac{1}{4300}\right)}}$$

≈ -3.46

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{301}{4700} = 0.064$$

$$\hat{p}_2 = \frac{x_2}{n_2} = \frac{357}{4300} = 0.083$$

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{301 + 357}{4700 + 4300} \approx 0.0731$$

$$\bar{q} = 1 - \bar{p} = 1 - 0.0731 = 0.9269$$

Note:

$n_1\bar{p} = 4700(0.0731) \geq 5 \quad n_1\bar{q} = 4700(0.9269) \geq 5$

$n_2\bar{p} = 4300(0.0731) \geq 5 \quad n_2\bar{q} = 4300(0.9269) \geq 5$

Decision: Reject H_0

At the 1% level of significance, there is enough evidence to conclude that the death rate due to heart disease is lower for those who took the medication than for those who took the placebo.

Stat-Tests
5: 2-PropZTest