Discrete Mathematics

Lecture 9

Recurrences

Let $(x_0, x_1, x_2, ...)$ be an infinite sequence of numbers in that

- several initial elements $x_0, x_1, ..., x_k$ are given;
- each next element is defined by previous elements according to some rule.

This rule is named a *recurrence* (*recurrence equation* or *recurrence relation*).

We shall consider the *linear recurrences with constant coefficients*, i.e. equations in which the rule is described by a linear expression.

Examples

1) Initial element: $x_0 = 1$,
recurrence: $x_n = x_{n-1} + 1$ for $n > 0$.

We find:

$$x_1 = x_0 + 1 = 2,$$
$$x_2 = x_1 + 1 = 3,$$
$$x_3 = x_2 + 1 = 4,$$
$$\ldots$$

Obviously, that is the sequence of all natural numbers.

2) Initial element: $x_0 = 1$,
recurrence: $x_n = 2x_{n-1}$ for $n > 0$.

We find: $x_1 = 2x_0 = 2$,
$x_2 = 2x_1 = 4$,
$x_3 = 2x_2 = 8$,
...

Obviously, that is the sequence of degrees of 2:

$$x_n = 2^n.$$

First order linear recurrences

The general linear recurrence of the first order has the form

$$x_n = ax_{n-1} + b,$$

where a and b are given constants, $n > 0$.

If an initial element x_0 is also given then we can compute sequentially the other elements:

$$x_1 = ax_0 + b,$$

$$x_2 = ax_1 + b = a(ax_0 + b) + b = a^2 x_0 + ab + b,$$

...

Any element x_n, $n > 0$, is uniquely defined by a, b, and x_0.

Can we write a general formula for it?

First we shall consider the following two special cases:

1. $a = 1$.

2. $b = 0$.

1. $a = 1$.

The equation has the form

$$x_n = x_{n-1} + b.$$

We can find

$$x_1 = x_0 + b,$$
$$x_2 = x_1 + b = x_0 + 2b,$$
$$x_3 = x_2 + b = x_0 + 3b,$$
$$\dots$$

Obviously, $x_n = x_0 + nb$ for any n.

(It can be easy proved by induction on n).

This sequence is an *arithmetic progression*.

2. $b = 0$.

The equation has the form

$$x_n = a x_{n-1}.$$

We can find

$$x_1 = a x_0,$$
$$x_2 = a x_1 = a^2 x_0,$$
$$x_3 = a x_2 = a^3 x_0,$$
$$\ldots$$

Obviously, $x_n = a^n x_0$ for any n.

This sequence is a *geometric progression*.

Now we consider the general case

$$x_n = ax_{n-1} + b. \qquad (1)$$

First we shall reduce the equation to a simplified form with a help of the change of unknown

$$x_n = y_n + s, \qquad (2)$$

where y_n is a new unknown, s is a constant which value we shall determine later.

Substituting of (2) in (1) we get

$$y_n + s = a(y_{n-1} + s) + b,$$

or

$$y_n = ay_{n-1} + as + b - s.$$

Let us select such s that

$$as + b - s = 0,$$

i.e.

$$s = \frac{b}{1-a}.$$

Note that for $a = 1$ this expression is not defined.
But the case $a = 1$ had been considered previously.

Further we suppose that $a \neq 1.$

Then we obtain the equation

$$y_n = ay_{n-1}.$$

This equation has a simplest form (case 2), its solution is

$$y_n = a^n y_0.$$

Because of $x_n = y_n + s$ we obtain

$$x_n - s = a^n (x_0 - s)$$

and it remains to substitute the expression for s:

$$x_n = a^n \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}.$$

We don't recommend you to remember this formula. It is rather the solution method. It includes the following three stages.

1. Reduction of equation to the simplest form by the change of unknown $x_n = y_n + s$ and choice of suitable value for s.

2. Solving of the obtained simplest equation.

3. Return to the former unknown x_n.

Note that the solution has the form

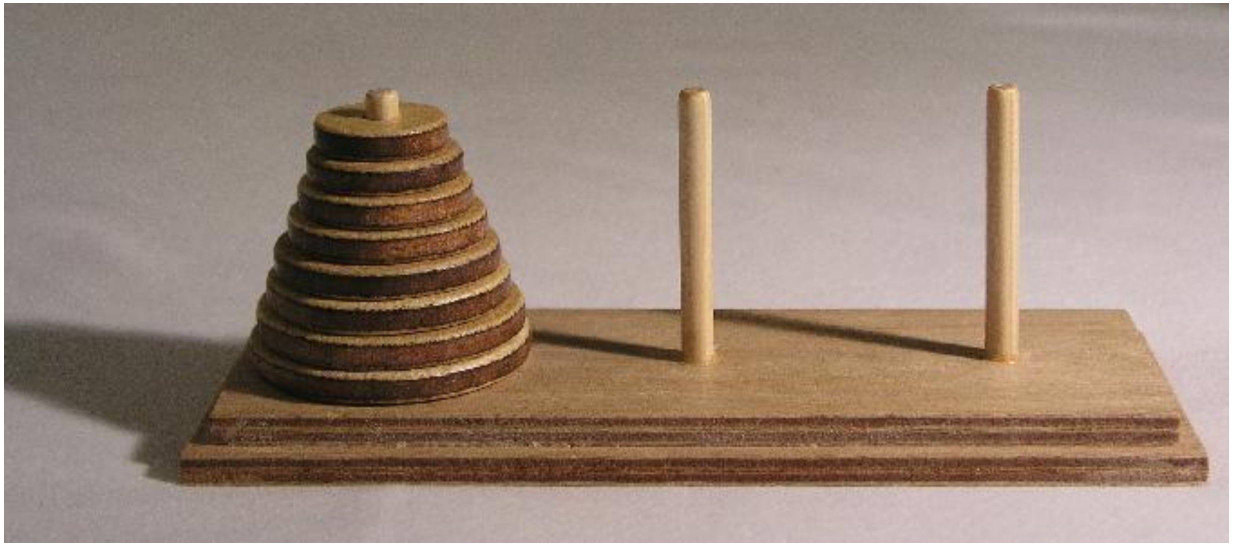
$$x_n = c_1 a^n + c_2$$

where c_1 and c_2 are some constants.

We see that the dependence of x_n on n is expressed as an exponential function.

Example: Towers of Hanoi

The French mathematician Edouard Lucas invents in 1883 the following problem. Eight disks of different sizes are threaded on one of three pegs in order of size decreasing. Goal is to transpose the disks in the same order onto another peg. It is allowed to move only one disk at a time and it is forbidden to put a larger disk on the smaller one. How many steps are needed for this? (A step is a movement of a disk)

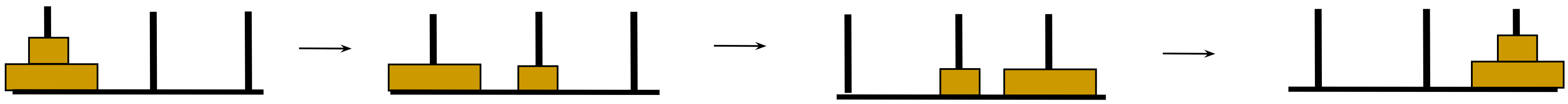


Let us consider this problem in a general form when there are n disks.

Let T_n be the minimum number of steps needed to move n disks from one peg to another.

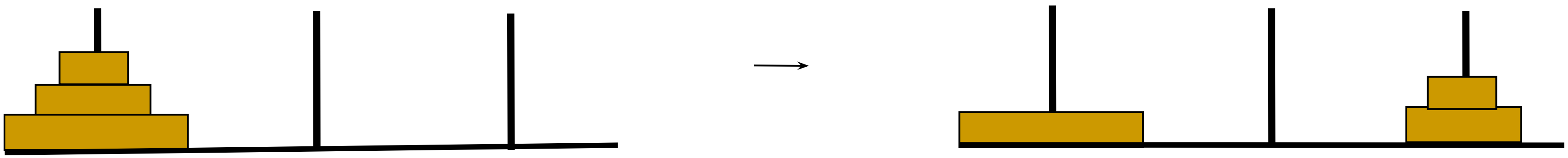
Obviously, $T_1 = 1$.

It is easy to see that $T_2 = 3$:

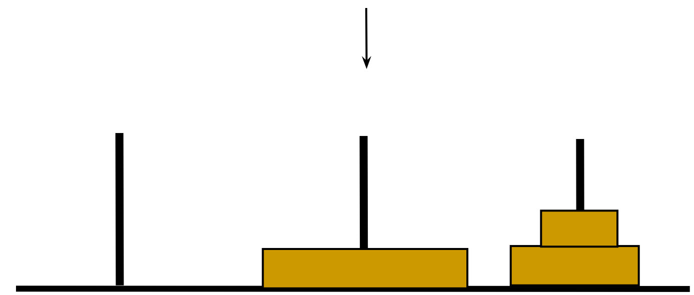


We can transpose three disks in the following manner:

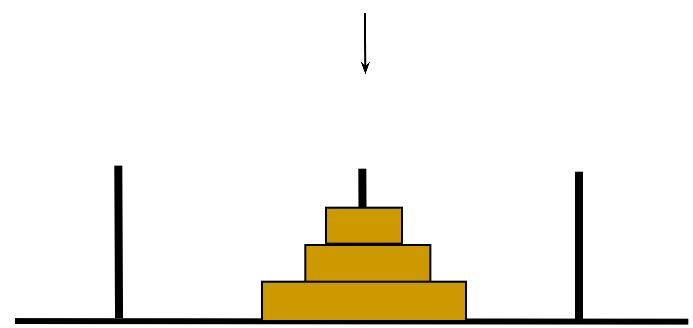
1) move two disks as above (3 steps)



2) move the largest disk (1 step)



3) move two disks again (3 steps)



Thus $T_3 = 3 + 1 + 3 = 7.$

In general case we must transpose $n - 1$ smaller disks before we move the largest disk.
It requires T_{n-1} steps.

After moving the largest disk it is necessary again to transpose $n - 1$ smaller disks.
It requires also T_{n-1} steps.

Hence we have the recurrence

$$T_n = 2T_{n-1} + 1, \quad n > 1.$$

If we let $T_0 = 0$ then the equality is also valid for $n = 1$.

We solve this equation in three described stage.

1. Reduction. Let $T_n = Y_n + s$

then $Y_n + s = 2Y_{n-1} + 2s + 1.$

Take $s = -1$ and get the equation $Y_n = 2Y_{n-1}.$

2. Solving of simplified equation. $Y_n = 2^n Y_0.$

3. Return to the former unknown. $Y_n = T_n + 1,$

$Y_0 = T_0 + 1 = 1.$

Answer: $T_n = 2^n - 1.$

Second order linear recurrences

General linear recurrence equation of second order has the form

$$x_n = ax_{n-1} + bx_{n-2} + c.$$

where $a, b,$ and c are given constants, $n > 1$.

We shall consider first the *homogeneous* equation $(c = 0)$:

$$x_n = ax_{n-1} + bx_{n-2}. \qquad (3)$$

If we know the two initial elements x_0 and x_1 then we are able to compute sequentially the other elements:

$$x_2 = ax_1 + bx_0,$$

$$x_3 = ax_2 + bx_1 = a(ax_1 + bx_0) + bx_1 = (a^2 + b)x_1 + abx_0$$

and so on.

Every element x_n, $n > 1$, is uniquely defined by a, b, x_0, x_1.

We also can obtain a general formula for x_n in this case .

We shall look for a solution in the exponential form:

$$x_n = \alpha^n$$

where α is unknown constant (the idea goes from the solutions form of the first order recurrence).

Substitution of this expression in the equation (3) gives

$$\alpha^n = a\alpha^{n-1} + b\alpha^{n-2}.$$

It may be reduced by α^{n-2} and we get

$$\alpha^2 = a\alpha + b.$$

Thus α must be a root of the equation

$$\alpha^2 - a\alpha - b = 0$$

called a *characteristic equation*.

There are the two possibilities.

A. The characteristic equation has two distinct roots α_1 and α_2.

B. There is one root $\alpha_1 = \alpha_2$.
(the discriminant is zero: $a^2 + 4b = 0$).

A. The characteristic equation has two distinct roots α_1 and α_2.

Then both sequences

$$x_n' = \alpha_1^n \qquad \text{and} \qquad x_n'' = \alpha_2^n$$

satisfy the equation (3).

But we need solution with given x_0, x_1.

Can we achieve it?

The homogeneous linear equation has two following important properties:

1) if a sequence

$$x_0, x_1, \ldots, x_n, \ldots$$

satisfies the equation (3) and a is some constant then the sequence

$$ax_0, ax_1, \ldots, ax_n, \ldots$$

also satisfies this equation

(to verify this statement it is sufficient to substitute ax_n in the equation instead of x_n);

2) if sequences

$$x'_0, x'_1, \dots, x'_n, \dots \qquad \text{and} \qquad x''_0, x''_1, \dots, x''_n, \dots$$

both satisfy the equation (3) then the sequence

$$x'_0 + x''_0,\ x'_1 + x''_1, \dots, x'_n + x''_n, \dots$$

also satisfies this equation.

(If we substitute $x'_n + x''_n$ in the equation instead of x_n

then we get $\quad x'_n + x''_n = ax'_{n-1} + ax''_{n-1} + bx'_{n-2} + bx''_{n-2}$

and it is fulfilled since $\quad x'_n = ax'_{n-1} + bx'_{n-2}$

and $\quad x''_n = ax''_{n-1} + bx''_{n-2}$)

Due to properties 1), 2) we may affirm that the sequence

$$c_1\alpha_1^n + c_2\alpha_2^n$$

satisfies the equation (3) for any constants c_1, c_2. This sequence is called a *general solution* of an equation (3).

Can we fit c_1, c_2 in such a way that two initial elements of the sequence would be x_0 and x_1? We shall try:

$$n = 0 \qquad x_0 = c_1\alpha_1^0 + c_2\alpha_2^0 = c_1 + c_2,$$

$$n = 1 \qquad x_1 = c_1\alpha_1^1 + c_2\alpha_2^1 = c_1\alpha_1 + c_2\alpha_2.$$

Thus we get a system of two linear equations with two unknowns c_1, c_2:

$$c_1 + c_2 = x_0,$$

$$c_1\alpha_1 + c_2\alpha_2 = x_1.$$

The determinant of this system is

$$\begin{vmatrix} 1 & 1 \\ \alpha_1 & \alpha_2 \end{vmatrix} = \alpha_2 - \alpha_1 \neq 0 \quad \text{as} \quad \alpha_1 \neq \alpha_2.$$

Hence there exists a unique solution c_1, c_2 and we obtain a solution of equation (3) with given start-up values.

B. The characteristic equation has a single root α.

In this case the general solution has the form

$$x_n = (c_1 + c_2 n)\alpha^n$$

(without a proof).

Constants c_1 and c_2 may be found using the initial values:

$$n = 0 \qquad x_0 = c_1,$$

$$n = 1 \qquad x_1 = (c_1 + c_2)\alpha.$$

So we have the following algorithm for solving of a linear homogeneous recurrence equation of the second order.

1. Write the characteristic equation and solve it.

2a. If the characteristic equation has two distinct roots α_1 and α_2 then write the general solution in the form

$$x_n = c_1\alpha_1^n + c_2\alpha_2^n.$$

2b. If the characteristic equation has a unique root α then write the general solution in the form

$$x_n = (c_1 + c_2 n)\alpha^n.$$

3. Write equation system for c_1, c_2 using given initial values x_0, x_1 and solve it.
4. Substitute the found values of constants in the general solution.

Examples

1) $x_n = 2x_{n-1} + 3x_{n-2}$, $\quad x_0 = 1, \quad x_1 = 2.$

The characteristic equation: $\alpha^2 - 2\alpha - 3 = 0.$

Roots: $\alpha_1 = 3, \quad \alpha_2 = -1.$

The general solution: $c_1 3^n + c_2 (-1)^n.$

Equations for c_1, c_2:

$$\begin{cases} c_1 + c_2 = 1, \\ 3c_1 - c_2 = 2. \end{cases} \quad \Bigg| \quad c_1 = \frac{3}{4}, \quad c_2 = \frac{1}{4}$$

Solution of the equation:

$$x_n = \frac{3}{4} 3^n + \frac{1}{4}(-1)^n = \frac{3^{n+1} + (-1)^n}{4}.$$

Examples

2) $x_n = 4x_{n-1} - 4x_{n-2}$, $\quad x_0 = 1, \quad x_1 = 4.$

The characteristic equation: $\alpha^2 - 4\alpha + 4 = 0.$

Unique root: $\alpha = 2.$

The general solution: $(c_1 + c_2 n)2^n.$

Equations for c_1, c_2:

$$\begin{cases} c_1 = 1, \\ 2c_1 + 2c_2 = 4. \end{cases} \quad \Bigg| \quad c_1 = 1, \quad c_2 = 1$$

Solution of the equation :

$$x_n = (1+n)\,2^n.$$

Inhomogeneous equation

An inhomogeneous linear recurrence of the second order

$$x_n = ax_{n-1} + bx_{n-2} + c$$

may be reduced to homogeneous equation in the same way as in the case of recurrences of the first order. We introduce a new unknown y_n

$$x_n = y_n + s$$

and select such s that the constant term vanishes:

$$s = \frac{c}{1 - a - b}.$$

If $a + b \neq 1$ then s exist and we obtain a homogeneous equation, solve it, and then return to former unknown.

If $a + b = 1$ then s does not exist. In this case one has to make the other change of unknown:

$$x_n = y_n + sn$$

and again select such s that equation becomes homogeneous.

Fibonacci numbers

Leonardo Fibonacci (1170 – 1250) also known as **Leonardo Pisano**, was an Italian mathematician.

He is the best known due to the discovery of the Fibonacci numbers and because of his role in the introduction of the modern Arabic decimal system for writing numbers in Europe.

The *Fibonacci numbers* are elements of the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots .$$

In this sequence each element beginning with the third is equal to the sum of two preceding elements:

$$\begin{aligned}
1 &= 1 + 0,\\
2 &= 1 + 1,\\
3 &= 2 + 1,\\
5 &= 3 + 2,\\
&\ldots
\end{aligned}$$

If we denote the n-th element of the sequence by F_n $(n = 0, 1, 2, ...)$ then the rule may be written as

$$F_n = F_{n-1} + F_{n-2}.$$

It is a linear recurrence equation of the second order. There are also initial values

$$F_0 = 0, \quad F_1 = 1$$

and we can find a general formula for the Fibonacci number.

The characteristic equation for this recurrence is

$$\alpha^2 - \alpha - 1 = 0.$$

It has two roots

$$\alpha_1 = \frac{1+\sqrt{5}}{2}, \qquad \alpha_2 = \frac{1-\sqrt{5}}{2}.$$

The general solution of the recurrence is

$$c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Now we use the initial values for writing the equations for the constants c_1 and c_2:

$$\begin{cases} c_1 + c_2 = 0, \\ c_1\alpha_1 + c_2\alpha_2 = 1. \end{cases}$$

Solving this system we find

$$c_1 = \frac{1}{\alpha_1 - \alpha_2} = \frac{1}{\sqrt{5}}, \qquad c_2 = -\frac{1}{\sqrt{5}},$$

and obtain the formula for the Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Example: Sparse words

A binary word containing no two consecutive 1's will be called a *sparse* word.

For example, the word 0100101000101 is sparse

while words 0101100011 and 0011110 are not sparse.

Denote the number of all sparse words of length n by U_n.

For small n we have:

n	0	1	2	3	4
Sparse words	λ	0	00	000	0000
		1	01	001	0001
			10	010	0010
				100	0100
				101	0101
					1000
					1001
					1010
U_n	1	2	3	5	8

What is U_5?

If α is a sparse word of length 5 and the first letter of α is 0 then

$$\alpha = 0\beta$$

where β is a sparse word of length 4.
There are 8 of such words:

00000
00001
00010
00100
00101
01000
01001
01010

If the first letter in α is 1 then the second letter must be 0 and

$$\alpha = 10\gamma$$

where γ is any sparse word of length 3.
There are 5 of such words:

10000
10001
10010
10100
10101

At all we have $U_5 = U_4 + U_3 = 8 + 5 = 13.$

In general case let α be a sparse word of the length n.

- If the first letter of α is 0 then $\alpha = 0\beta$ where β is any sparse word of length $n - 1$.

  There are U_{n-1} such words.

- If the first letter in α is 1 then the second letter must be 0 and $\alpha = 10\gamma$ where γ is any sparse word of length $n - 2$.

  There are U_{n-2} such words.

Thus we obtain the recurrence for these numbers:

$$U_n = U_{n-1} + U_{n-2}$$

for $n \geq 2$.
It is the same recurrence as for Fibonacci numbers.

However the start-up values are other: $U_0 = 1,\ U_1 = 2.$

But we see that U_0 and U_1 coincide with two successive Fibonacci numbers:

$$U_0 = F_2, \quad U_1 = F_3.$$

Consequently,

$$U_n = F_{n+2} \qquad \text{for } n = 0, 1, 2, \ldots .$$