Common Probability Distributions

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DISCRETE RANDOM VARIABLES

- **A discrete random variable** can take on at most a countable number of possible values. For example, a discrete random variable X can take on a limited number of outcomes x1, x2, …, xn (n possible outcomes), or a discrete random variable Y can take on an unlimited number of outcomes y1, y2, … (without end).1 Because we can count all the possible outcomes of X and Y (even if we go on forever in the case of Y), both X and Y satisfy the definition of a discrete random variable
- We can view a probability distribution in two ways:

1. The **probability function** specifies the probability that the random variable will take on a specific value. The probability function is denoted p(x) for a discrete random variable and f(x) for a continuous random variable. For any probability function p(x), $0 \leq p(x) \leq 1$, and the sum of p(x) over all values of X equals 1.
2. The **cumulative distribution function**, denoted F(x) for both continuous and discrete random variables, gives the probability that the random variable is less than or equal to x.

The Discrete Uniform Distribution

- The discrete uniform and the continuous uniform distributions are the distributions of equally likely outcomes.

The Binomial Distribution

- The binomial random variable is defined as the number of successes in *n* **Bernoulli trials**, where the probability of success, *p, is constant for all trials and* the trials are independent. A Bernoulli trial is an experiment with two outcomes, which can represent success or failure, an up move or a down move, or another binary (two-fold) outcome.

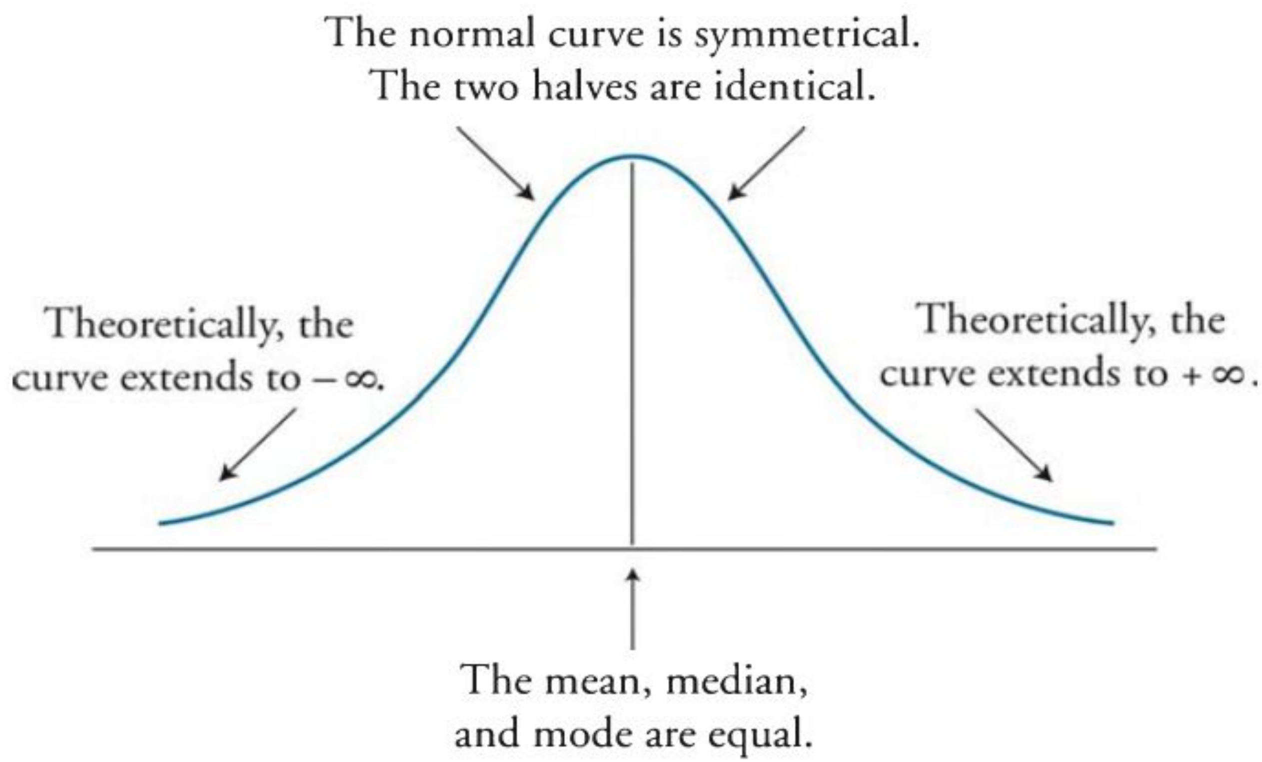
$$p(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \qquad \textbf{(1)}$$

- A binomial random variable has an expected value or **mean** equal to np and **variance** equal to np(1 − p).
- A binomial tree is the graphical representation of a model of asset price dynamics in which, at each period, the asset moves up with probability p or down with probability (1 − p). **The binomial tree** is a flexible method for modelling asset price movement and is widely used in pricing options.

CONTINUOUS RANDOM VARIABLES

- The Normal Distribution

The normal distribution is a continuous symmetric probability distribution that is completely described by two parameters: its mean, μ, and its variance, σ^2.



Having established that the normal distribution is the appropriate model for a variable of interest, we can use it to make the following probability statements:

- Approximately 50 percent of all observations fall in the interval $\mu \pm (2/3)\sigma$.
- Approximately 68 percent of all observations fall in the interval $\mu \pm \sigma$.
- Approximately 95 percent of all observations fall in the interval $\mu \pm 2\sigma$.
- Approximately 99 percent of all observations fall in the interval $\mu \pm 3\sigma$.

Calculating Probabilities Using the Standard Normal Distribution

The *z-value* "standardizes" an observation from a normal distribution and represents the number of standard deviations a given observation is from the population mean.

$$z = \frac{\text{observation} - \text{population mean}}{\text{standard deviation}} = \frac{x - \mu}{\sigma}$$

Shortfall risk and Safety-first Ratio

Shortfall risk. The probability that a portfolio's return or value will be below a specified (target) return or value over a specified period.

Roy's safety-first criterion states that the optimal portfolio minimizes the probability that the return of the portfolio falls below some minimum acceptable "threshold" level.

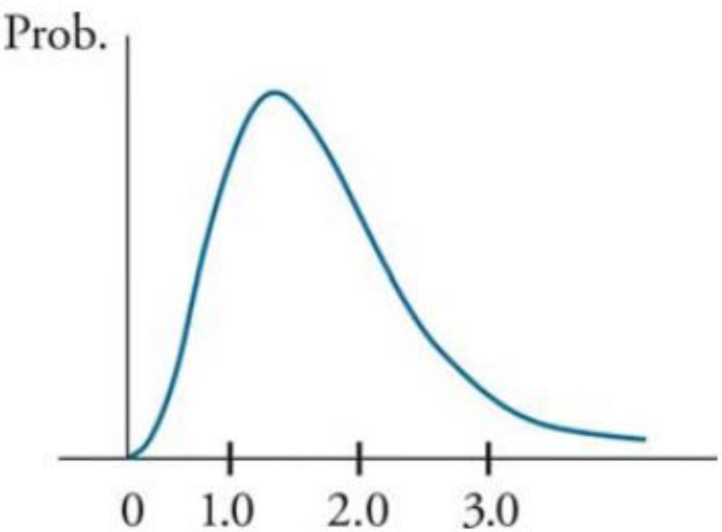
Roy's safety-first ratio (SFRatio) is similar to the Sharpe ratio. In fact, the Sharpe ratio is a special case of Roy's ratio where the "threshold" level is the risk-free rate of return.

$$\text{SFRatio} = \frac{E(R_p) - R_L}{\sigma_p}$$

Lognormal distribution

If x is normally distributed, $Y = e^x$ is lognormally distributed. Values of a lognormal distribution are always positive so it is used to model asset prices (rather than rates of return, which can be negative). The lognormal distribution is positively skewed as shown in the following figure.

Figure 6: Lognormal Distribution



- Mean (μ_L) of a lognormal random variable = $\exp(\mu + 0.50\sigma^2)$
- Variance (σ_L^2) of a lognormal random variable = $\exp(2\mu + \sigma^2) \times [\exp(\sigma^2) - 1]$

Continuously compounded return

$$\text{CCR} = \ln(1+\text{HPR}) = \ln\left(\frac{\text{ending value}}{\text{beginning value}}\right)$$

$$\text{HPR} = \frac{\text{ending value}}{\text{beginning value}} - 1 = e^{\text{CCR}} - 1$$

When the holding period is one year, so that HPR is also the effective annual return, CCR is the annual continuously compounded rate of return.

One property of continuously compounded rates is that they are additive over multiple periods. If the continuously compounded rate of return is 8%, the holding period return over a 2-year horizon is $e^{2(0.08)} - 1$, and \$1,000 will grow to 1,000 $e^{2.5(0.08)}$ over two and one-half years.