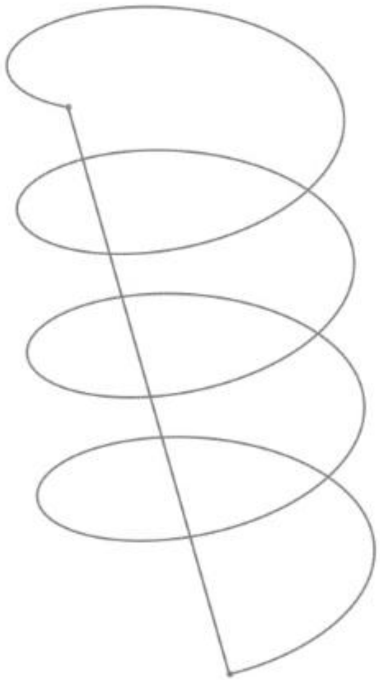
Week 7: Geometric Modeling - Parametric Representation of Synthetic Curves

Spring 2018, AUA

Zeid, I., *Mastering CAD/CAM, Chapter 6*

Planar vs. Space



Analytic (known form) vs. Synthetic (free form)

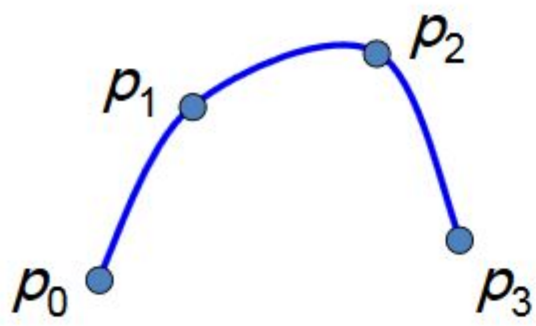
We can create simplistic objects such as the forklift given below by using known equations.

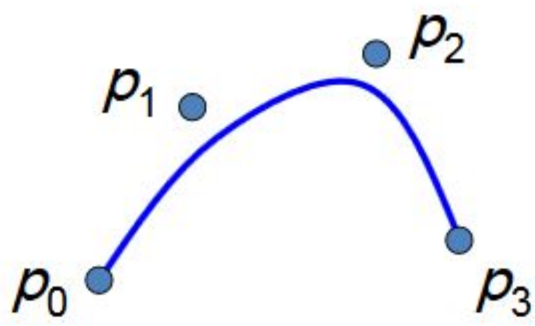
Creating these curves by using known analytic curve equations is not reasonable all the time. Sometimes – impossible.

Interpolation vs. Approximation

The curve passing through given data (control) points - interpolation curve.

The curve not necessarily passing but controlled by data points - approximation curve

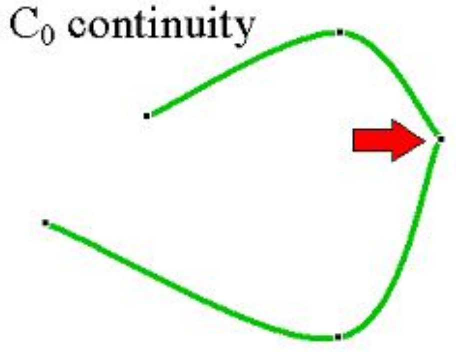


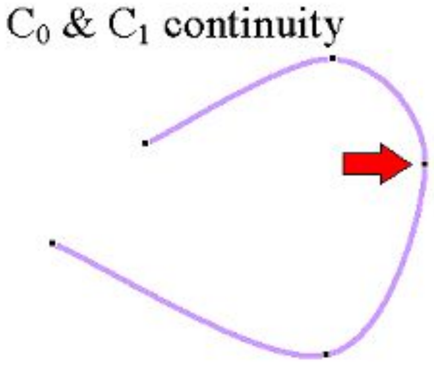


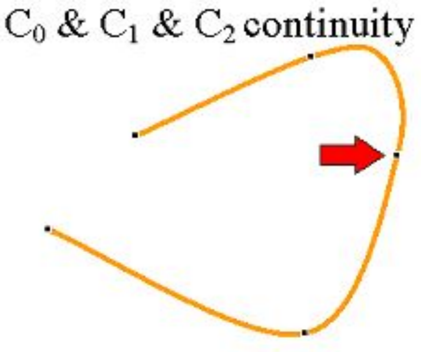
Continuity

The smoothness of the connection of two curves or surfaces at the connection points or edges.

- C^0: simple connection of two curves
- C^1: the geometric slopes at the joint must be same
- C^2: curvature continuity that not only the gradients but also the center of curvature is the same







Cubic Curves

Parametric equation of a cubic spline segment:

$$\mathbf{P}(u) = \sum_{i=0}^{3} \mathbf{a}_i \cdot u^i,$$ where 0□u □1

In an expanded vector form:

$$\mathbf{P}(u) = \mathbf{a}_0 + \mathbf{a}_1 \cdot u + \mathbf{a}_2 \cdot u^2 + \mathbf{a}_3 \cdot u^3$$

The tangent vector:

$$\mathbf{P}^{'}(u) = \sum_{i=0}^{3} \mathbf{a}_i \cdot i \cdot u^{i-1}$$

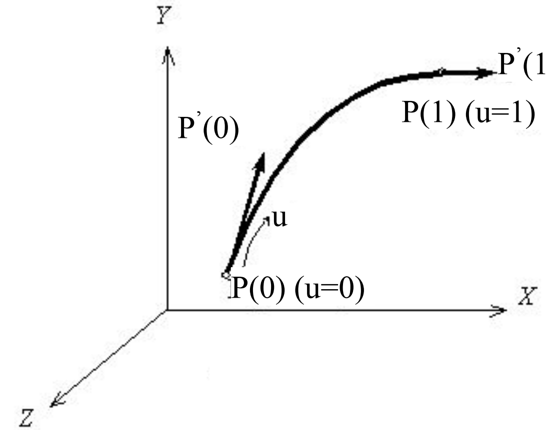
In an expanded vector form:

$$\mathbf{P}'(u) = \mathbf{a}_1 + 2 \cdot \mathbf{a}_2 \cdot u + 3 \cdot \mathbf{a}_3 \cdot u^2$$

Hermite Cubic Splines

Charles Hermite
(1822 - 1901)

Hermite form of a general cubic spline is defined by positions and tangent vectors at two data points.



Applying the boundary conditions at u = 0 and u = 1 and performing the necessary substitutions,

$$\mathbf{P}(u) = \mathbf{P}(0)\cdot(2\cdot u^3 - 3\cdot u^2 + 1) + \mathbf{P}(1)\cdot(-2\cdot u^3 + 3\cdot u^2) + \mathbf{P}'(0)\cdot(u^3 - 2\cdot u^2 + u) + \mathbf{P}'(1)\cdot(u^3 - u^2)$$

$$\mathbf{P}'(u) = \mathbf{P}(0)\cdot(6\cdot u^2 - 6\cdot u) + \mathbf{P}(1)\cdot(-6\cdot u^2 + 6\cdot u) + \mathbf{P}'(0)\cdot(3\cdot u^2 - 4\cdot u + 1) + \mathbf{P}'(1)\cdot(3\cdot u^2 - 2\cdot u)$$

Hermite Cubic Splines

$$\mathbf{P}(u) = \mathbf{P}(0)\cdot(2\cdot u^3 - 3\cdot u^2 + 1) + \mathbf{P}(1)\cdot(-2\cdot u^3 + 3\cdot u^2) + \mathbf{P}'(0)\cdot(u^3 - 2\cdot u^2 + u) + \mathbf{P}'(1)\cdot(u^3 - u^2)$$

→ Vector form

$$X(u) = X(0)\cdot(2\cdot u^3 - 3\cdot u^2 + 1) + X(1)\cdot(-2\cdot u^3 + 3\cdot u^2) + X'(0)\cdot(u^3 - 2\cdot u^2 + u) + X'(1)\cdot(u^3 - u^2)$$

$$Y(u) = Y(0)\cdot(2\cdot u^3 - 3\cdot u^2 + 1) + Y(1)\cdot(-2\cdot u^3 + 3\cdot u^2) + Y'(0)\cdot(u^3 - 2\cdot u^2 + u) + Y'(1)\cdot(u^3 - u^2)$$

$$Z(u) = Z(0)\cdot(2\cdot u^3 - 3\cdot u^2 + 1) + Z(1)\cdot(-2\cdot u^3 + 3\cdot u^2) + Z'(0)\cdot(u^3 - 2\cdot u^2 + u) + Z'(1)\cdot(u^3 - u^2)$$

Scalar form

$$\boldsymbol{P}(u) = \left|\boldsymbol{P}(0) \quad \boldsymbol{P}(1) \quad \boldsymbol{P}'(0) \quad \boldsymbol{P}'(1)\right| \cdot \begin{vmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} u^3 \\ u^2 \\ u \\ 1 \end{vmatrix}$$

→ Vector form

$$\boldsymbol{P}(u) = \begin{vmatrix} X(0) & X(1) & X'(0) & X'(1) \\ Y(0) & Y(1) & Y'(0) & Y'(1) \\ Z(0) & Z(1) & Z'(0) & Z'(1) \end{vmatrix} \cdot \begin{vmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} u^3 \\ u^2 \\ u \\ 1 \end{vmatrix}$$

Scalar form

Hermite Cubic Spline – Tangent Vector

$$\mathbf{P}'(u) = \mathbf{P}(0)\cdot(6\cdot u^2 - 6\cdot u) + \mathbf{P}(1)\cdot(-6\cdot u^2 + 6\cdot u) + \mathbf{P}'(0)\cdot(3\cdot u^2 - 4\cdot u + 1) + \mathbf{P}'(1)\cdot(3\cdot u^2 - 2\cdot u)$$

Vector form

$$X'(u) = X(0)\cdot(6\cdot u^2 - 6\cdot u) + X(1)\cdot(-6\cdot u^2 + 6\cdot u) + X'(0)\cdot(3\cdot u^2 - 4\cdot u + 1) + X'(1)\cdot(3\cdot u^2 - 2\cdot u)$$

$$Y'(u) = Y(0)\cdot(6\cdot u^2 - 6\cdot u) + Y(1)\cdot(-6\cdot u^2 + 6\cdot u) + Y'(0)\cdot(3\cdot u^2 - 4\cdot u + 1) + Y'(1)\cdot(3\cdot u^2 - 2\cdot u)$$

$$Z'(u) = Z(0)\cdot(6\cdot u^2 - 6\cdot u) + Z(1)\cdot(-6\cdot u^2 + 6\cdot u) + Z'(0)\cdot(3\cdot u^2 - 4\cdot u + 1) + Z'(1)\cdot(3\cdot u^2 - 2\cdot u)$$

Scalar form

$$\boldsymbol{P}(u) = |\boldsymbol{P}(0) \quad \boldsymbol{P}(1) \quad \boldsymbol{P}'(0) \quad \boldsymbol{P}'(1)| \cdot \begin{vmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} u^3 \\ u^2 \\ u \\ 1 \end{vmatrix}$$

Vector form

$$\boldsymbol{P}(u) = \begin{vmatrix} X(0) & X(1) & X'(0) & X'(1) \\ Y(0) & Y(1) & Y'(0) & Y'(1) \\ Z(0) & Z(1) & Z'(0) & Z'(1) \end{vmatrix} \cdot \begin{vmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} u^3 \\ u^2 \\ u \\ 1 \end{vmatrix}$$

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$$\boldsymbol{P}(u) = \begin{vmatrix} X(0) & X(1) & X'(0) & X'(1) \\ Y(0) & Y(1) & Y'(0) & Y'(1) \\ Z(0) & Z(1) & Z'(0) & Z'(1) \end{vmatrix} \cdot \begin{vmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} u^3 \\ u^2 \\ u \\ 1 \end{vmatrix}$$

Scalar form

Hermite Cubic Splines - example

The Hermite curve fits the points:

$\mathbf{P}_0 = [1,1]^T$,

$\mathbf{P}_1 = [3,5]^T$

and the tangent vectors:

$\mathbf{P}_0' = [0,4]^T$,

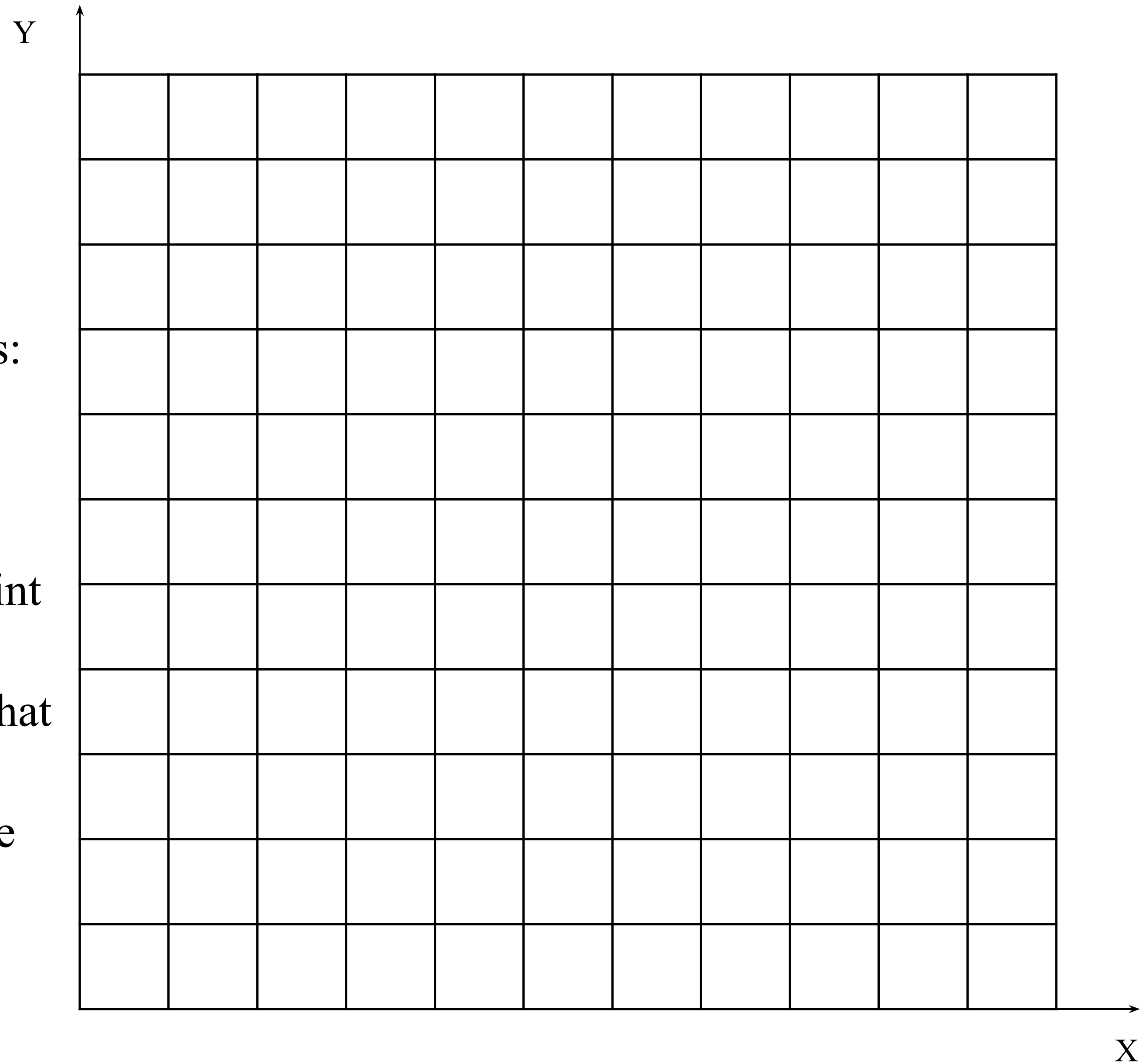
$\mathbf{P}_1' = [4,0]^T$.

Calculate

a) the parametric mid-point of the curve,

b) the tangent vector on that point.

c) Sketch the curve on the grid



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Bezier Curves - sl. 1

Pierre Bezier
(1910-1999)
Renault

Paul de Casteljau
(1930)
Citroën

Parametric equation of Bezier curve

$$\mathbf{P}(u) = \sum_{i=0}^{n} \mathbf{P}_i \cdot B_{i,n}(u), \qquad 0 \le u \le 1$$

where **P**(u) is the position vector of a point on the curve, $\mathbf{P}_i$ are control points, and $B_{i,n}$ are the Bernstein polynomials (blending functions for the curve).

$$B_{i,n}(u) = C(n,i) \cdot u^i \cdot (1-u)^{n-i}$$

and C(n,i) are the binomial coefficients:

$$C(n,i) = \frac{n!}{i! \cdot (n-i)!}$$

In an expanded form:

$$\mathbf{P}(u) = \mathbf{P}_0 \cdot (1-u)^n + \mathbf{P}_1 \cdot C(n,1) \cdot u \cdot (1-u)^{n-1} + \mathbf{P}_2 \cdot C(n,2) \cdot u^2 \cdot (1-u)^{n-2} + \cdots$$
$$+ \mathbf{P}_{n-1} \cdot C(n,n-1) \cdot u^{n-1} \cdot (1-u) + \mathbf{P}_n \cdot u^n$$

$$\mathbf{P}(u) = \mathbf{P}_0 \cdot (1-u)^3 + \mathbf{P}_1 \cdot 3 \cdot u \cdot (1-u)^2 + \mathbf{P}_2 \cdot 3 \cdot u^2 \cdot (1-u) + \mathbf{P}_3 \cdot u^{n3}$$

Bezier Curves - sl. 3

For n = 3:

$$\mathbf{P}(u)=\mathbf{P}_0\cdot(1-3\cdot u+3\cdot u^2-u^3)+3\cdot\mathbf{P}_1\cdot(0+u-2\cdot u^2+u^3)+3\cdot\mathbf{P}_2\cdot(0+0+u^2-u^3)+\mathbf{P}_3\cdot(0+0+0+u^3)$$

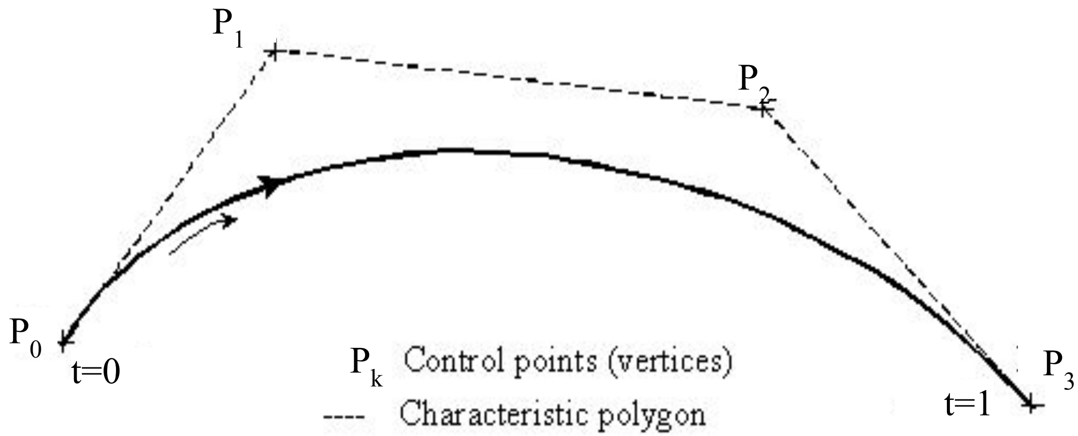
Or, in matrix form:

$$\mathbf{P}(u)=\begin{bmatrix}\mathbf{P}_0 & \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}\cdot\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\cdot\begin{bmatrix}u^3\\ u^2\\ u^1\\ 1\end{bmatrix}$$

Bezier geometric matrix
$\mathbf{G_B}$

Bezier basis matrix
$\mathbf{M_B}$

Bezier Curves - sl. 2



General Characteristics

- The Bezier curve is defined by n+1 points
- Only P_0 and P_{n+1} lie on the curve
- The curve is tangent to the first and last polygon segments
- The curve shape tends to follow the polygon shape.
- Convex hull property.
- The sum of $B_{i,n}$ functions is always equal to unity.

Bezier vs. Hermite Cubic Spline

- The Bezier curve is controlled by data points. No derivatives
- The order is variable: n+1 points define n^{th} order curve . -> higher order continuity

Bezier Curves - sl. 5
Practice

- The coordinates of 4 control points are given:
  $\mathbf{P}_0 = [2,2]^T$, $\mathbf{P}_1 = [2,3]^T$, $\mathbf{P}_3 = [3,3]^T$, $\mathbf{P}_4 = [3,2]^T$
  - *Find the equation of the resulting Bezier curve,*
  - *Find the points on the curve for $u = 0, ¼, ½, ¾, 1$,*
  - *Sketch the curve.*

Bezier Curves - sl. 4

- More complicated shapes require higher order Bezier curves. This is not good, because higher order curves take longer to evaluate and suffer from oscillations.

- Bezier curves also can not be modified locally. Moving any one control point will affect the whole curve.

B-spline Curves - sl. 1

See: http://www.ibiblio.org/e-notes/Splines/Basis.htm

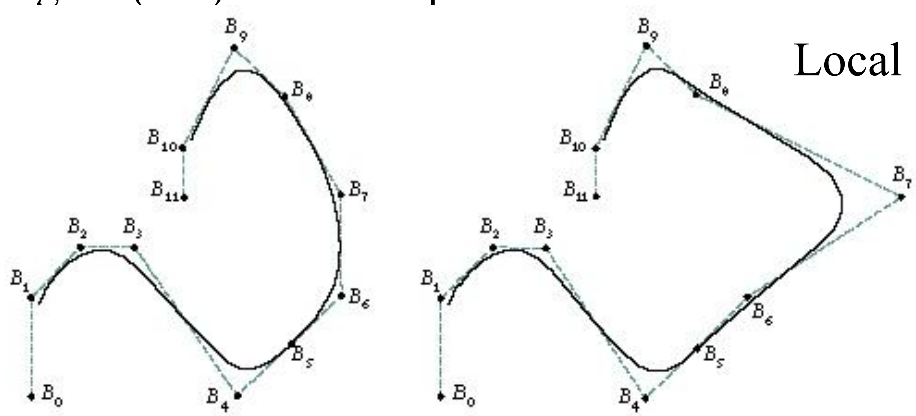
Powerful generalization of Bezier curves

- local control
- opportunity to add control points without increasing the degree of the curve
- ability to interpolate or approximate data points

The B-spline curve defined by n+1 control points P_i consists of n – 2 curve segments and is given by:

$$\mathbf{P}(u) = \sum_{i=0}^{n} \mathbf{P}_i \cdot N_{i,k}(u), \qquad 0 \le u \le u_{max}$$

where $N_{i,k}(u)$ are the B-spline (blending or basis) functions. The parameter k controls the degree (k-1) of the B-spline curve.



Local control

B-spline Curves - sl. 2

See: http://www.ibiblio.org/e-notes/Splines/Basis.htm

The B-spline curve defined by n+1 control points P_i consists of n – 2 curve segments and is given by:

$$\mathbf{P}(u) = \sum_{i=0}^{n} \mathbf{P}_i \cdot N_{i,k}(u), \qquad 0 \le u \le u_{max}$$

where $N_{i,k}(u)$ are the B-spline (blending or basis) functions. The parameter k controls the degree (k-1) of the B-spline curve.

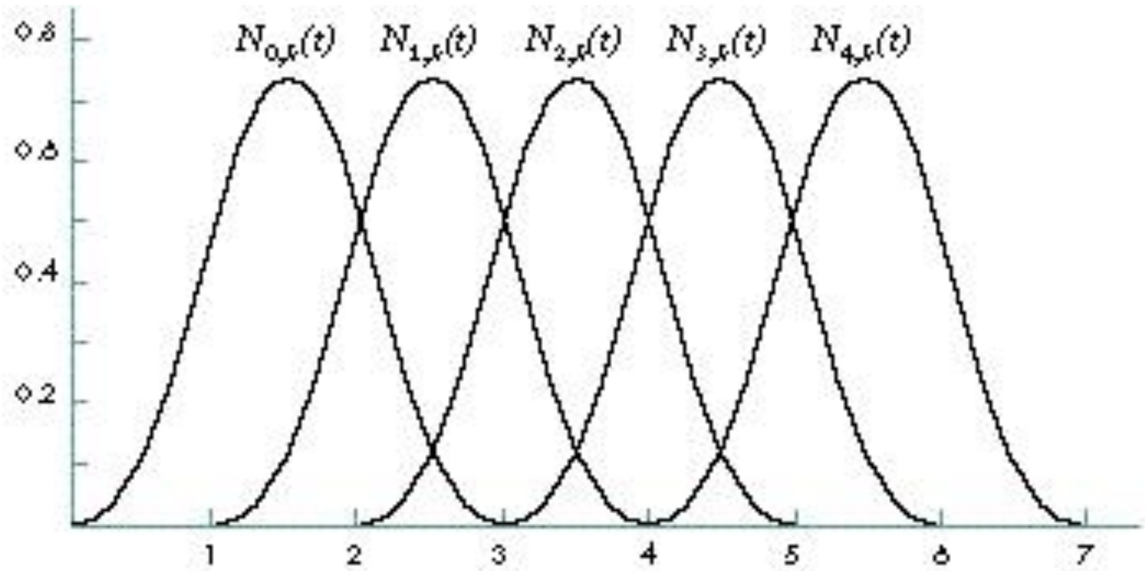
$$N_{i,k}(u) = (u - u_i)\frac{N_{i,k-1}(u)}{u_{i+k-1} - u_i} + (u_{i+k} - u)\frac{N_{i+1,k-1}(u)}{u_{i+k} - u_{i+1}}$$

where

$$N_{i,1} = \begin{cases} 1, & u_i \le u \le u_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

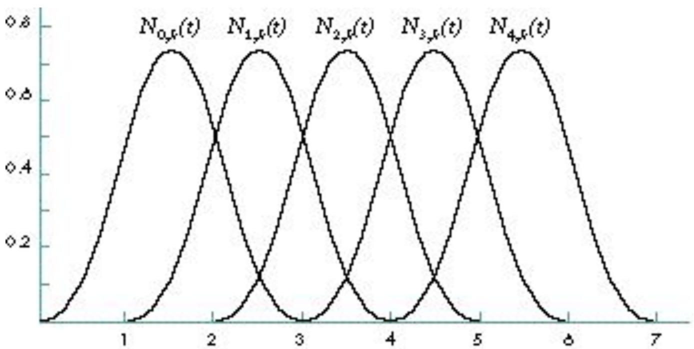
B-spline Curves - sl. 3
Basis Functions

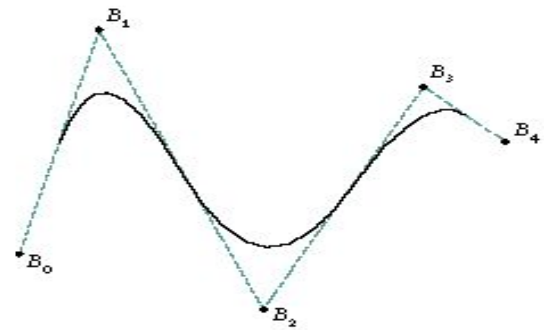
- The function $N_{i,k}$ determines how strongly **control point** P_i influences the **curve** at t. Its value is a real number – 0.25, 0.5…

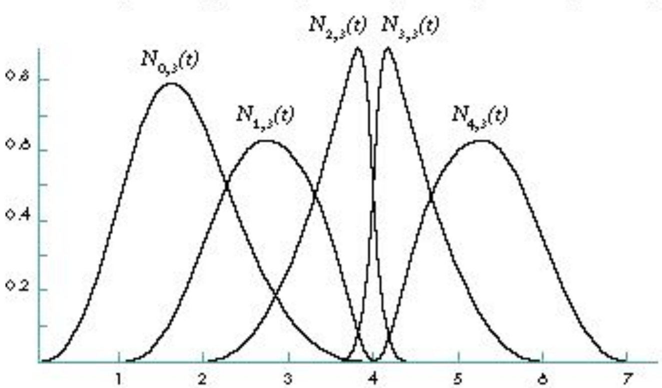


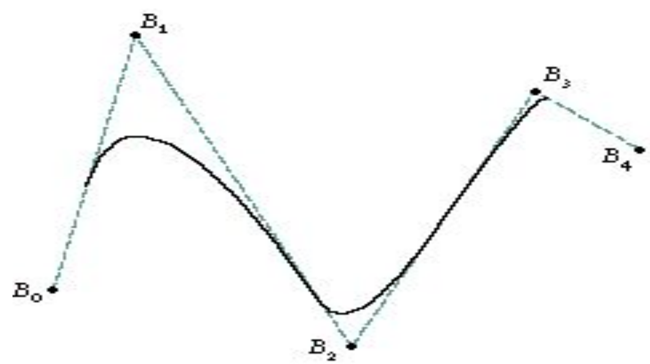
NURBS Curves - sl. 1

NURBS (Non-uniform Rational B-spline) curves are the generalization of uniform B-spline curves.









NURBS Curves - sl. 2

- NURBS curves are useful because they allow exact representation of conic curves.
- To create a circular arc (less than 180º) using a NURBS curve:
  - use $k = 3$ (degree = 2)
  - arrange control points in triangle with two equal angles as shown
  - use weightings: $h_0 = h_2 = 1$ and $h_1 = \cos\theta$