Graph theory

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- Labeled graphs
- Operations on graphs
- Intersection graphs
- Metrical characteristics of graphs
- König's theorem

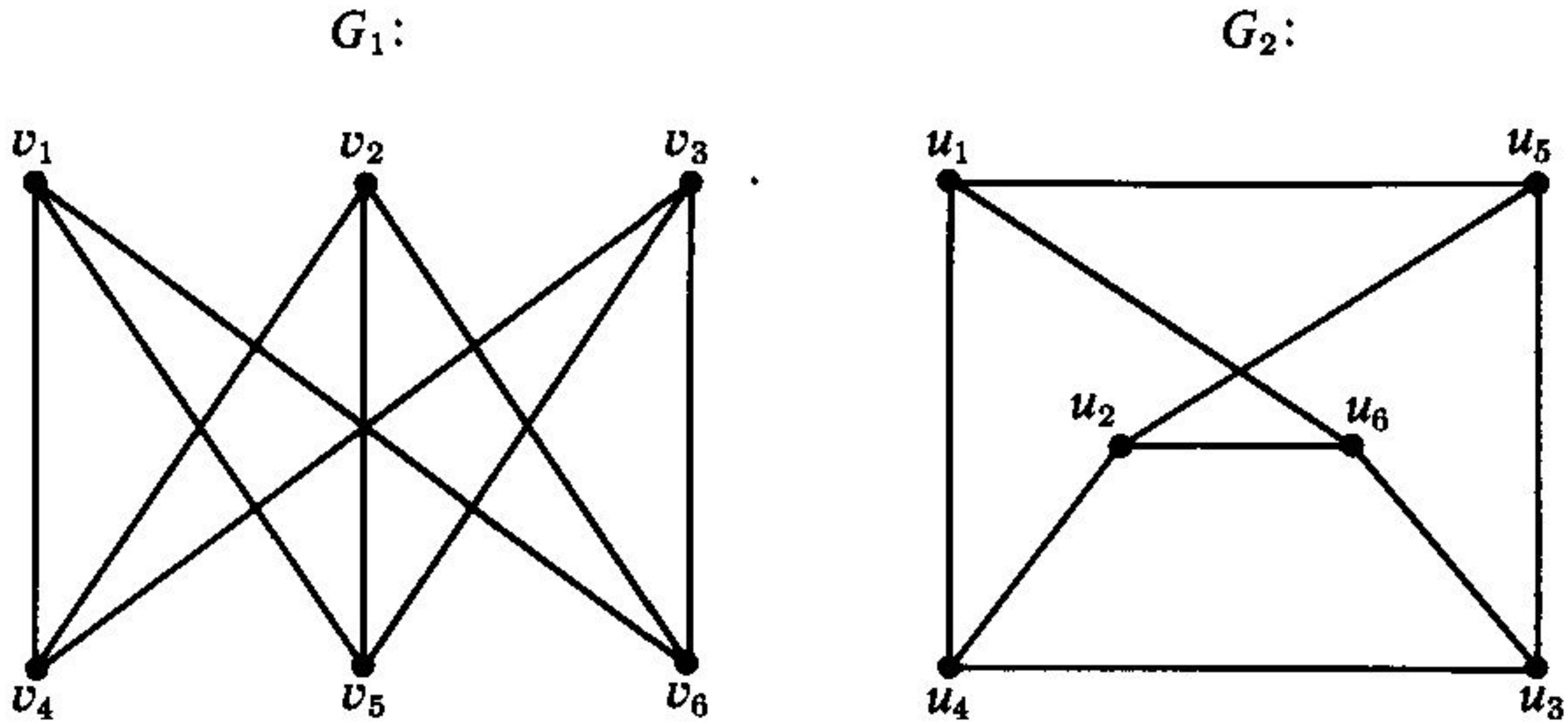
Labeled graphs

Definition 1

A graph G is **labeled** when the p points are distinguished from one another by names such as v_1, $v_2, \dots, v_p$.

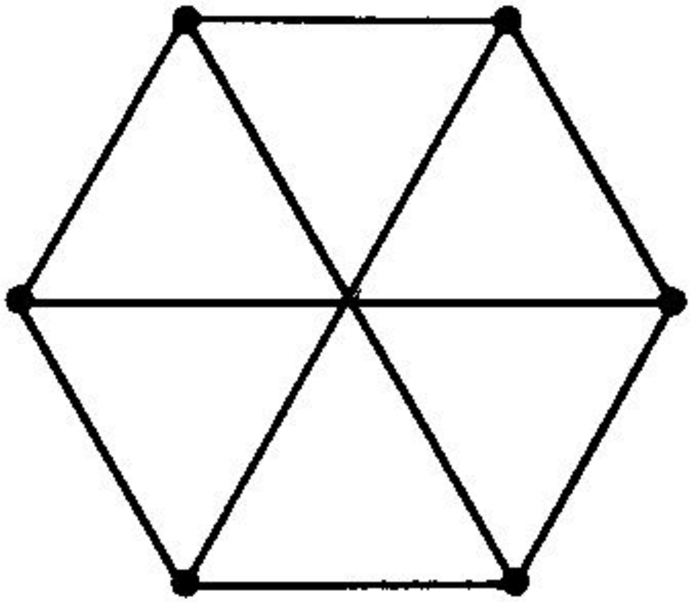
For example, the two graphs G_1 and G_2 of the following figures are labeled but G_3 is not.

Labeled graphs



Unlabeled graph

G_3:



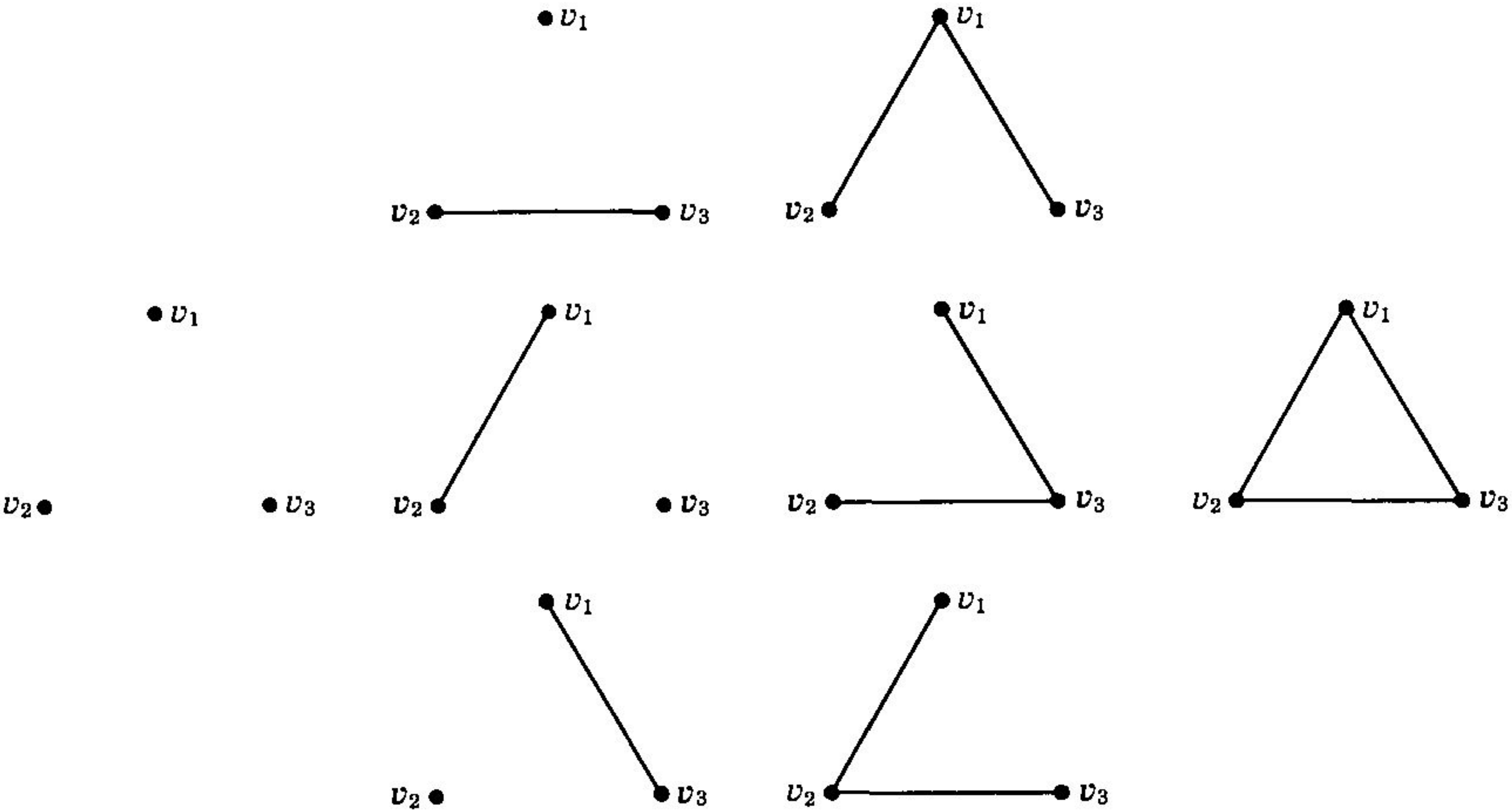
Labeled graphs

Theorem 1 The number of labeled graphs with p points is $2^{\binom{p}{2}}$.

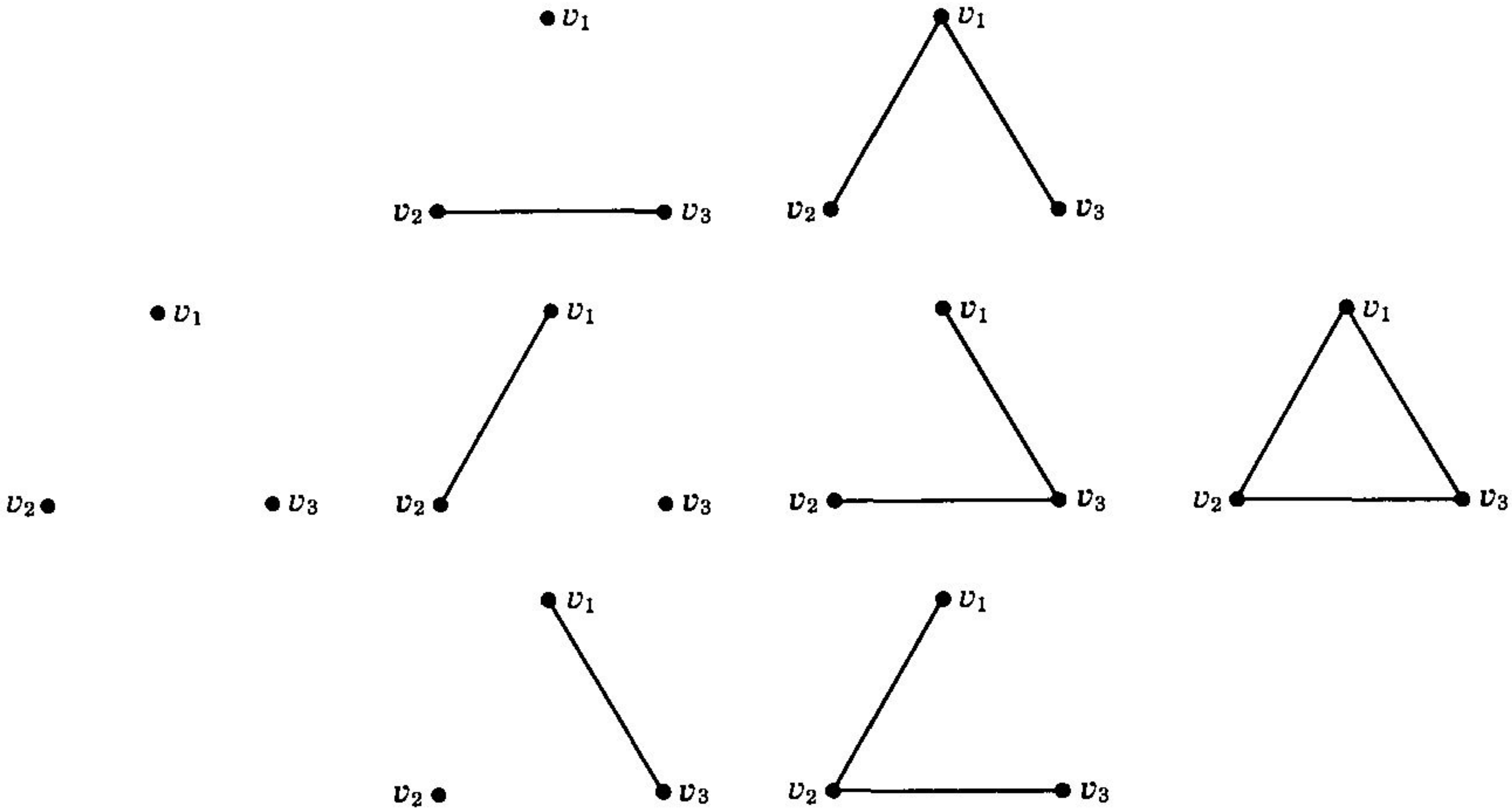
Proof

All of the labeled graphs with three points are shown in the following figure.

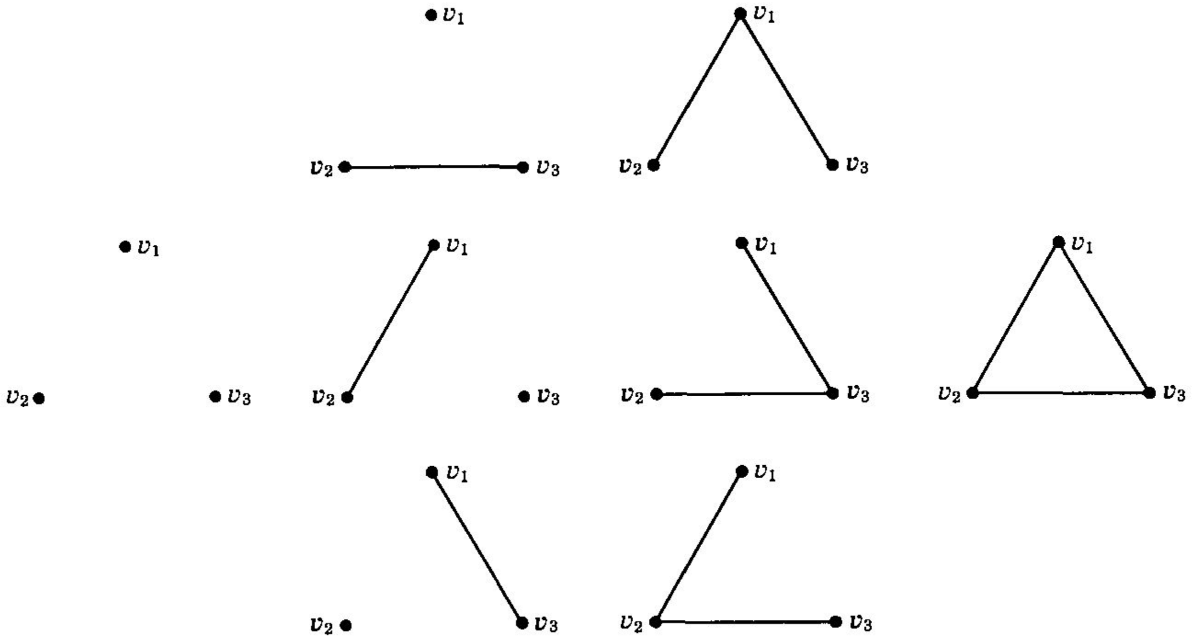
The labeled graphs with three points



We see that the 4 different graphs with 3 points become 8 different labeled graphs.



We see that the 4 different graphs with 3 points become 8 different labeled graphs.

- 

To obtain the number of labeled graphs with 3 points, we need only observe that each of the $\binom{3}{2}$ possible lines is either present or absent.

Labeled graphs

- To obtain the number of labeled graphs with p points, we need only observe that each of the $\binom{p}{2}$ possible lines is either present or absent.

Operations on graphs

A **subgraph** of G is a graph having all of its points and lines in G.

If G_1 is a subgraph of G, then G is a **supergraph** of G_1.

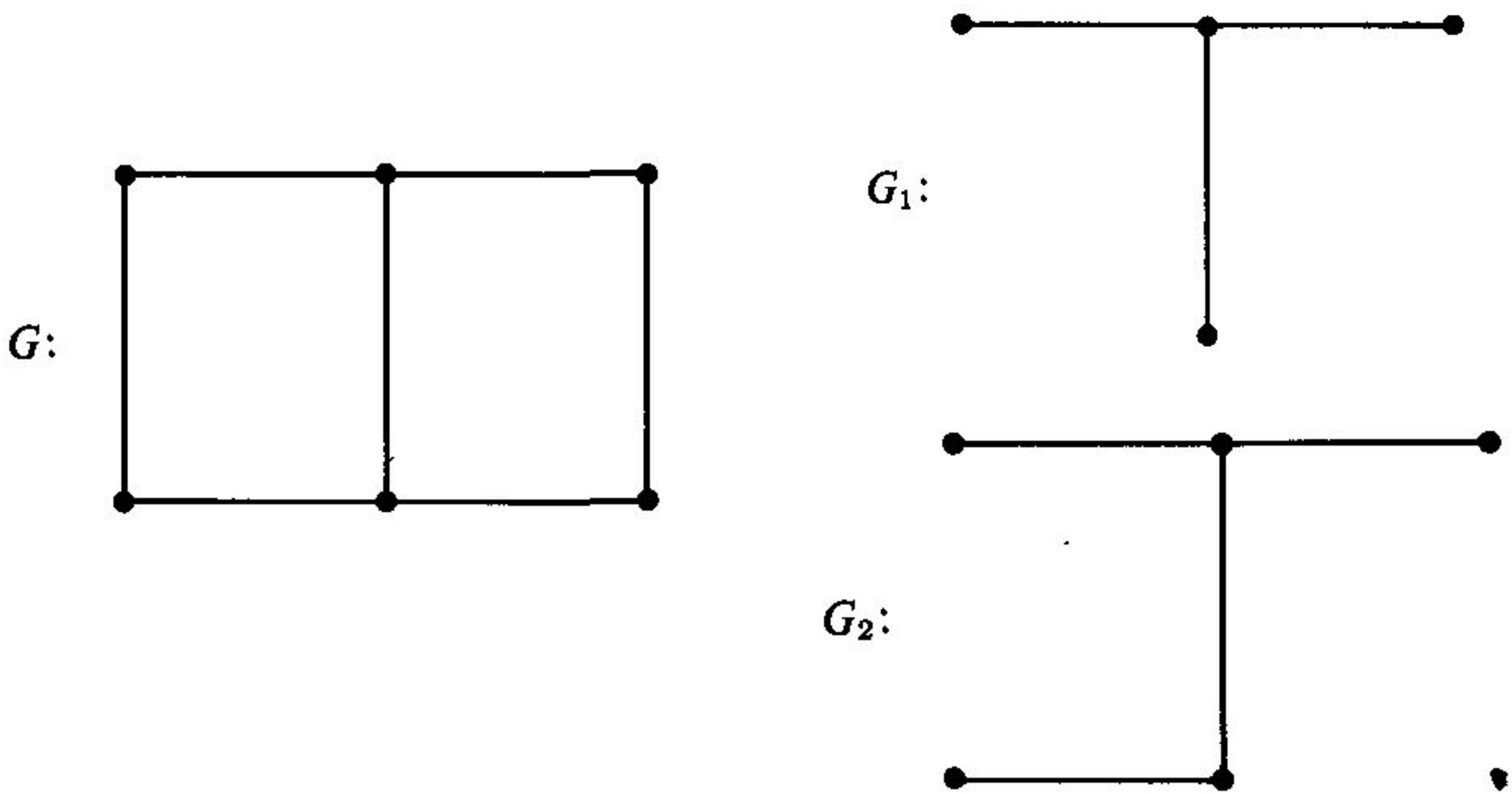
A **spanning subgraph** is a subgraph containing all the points of G.

For any set S of points of G, the **induced** subgraph $< S >$ is the maximal subgraph of G with point set S.

Thus two points of S are adjacent in $< S >$ if and only if they are adjacent in G.

Operations on graphs

G_2 is a spanning subgraph of G but G_1 is not; G_1 is an induced subgraph but G_2 is not.



Operations on graphs

- The **removal of a point** v_i from a graph G results in that subgraph $G - v_i$ of G consisting of all points of G except v_i and all lines not incident with v_i.

  Thus $G - v_i$ is the maximal subgraph of G not containing v_i.

Operations on graphs

- On the other hand, the **removal of a line** x_j from G yields the spanning subgraph $G - x_j$ containing all lines of G except x_j.

Thus $G - x_j$ is the maximal subgraph of G not containing x_j.

Operations on graphs

- The removal of a set of points or lines from G is defined by the removal of single elements in succession.

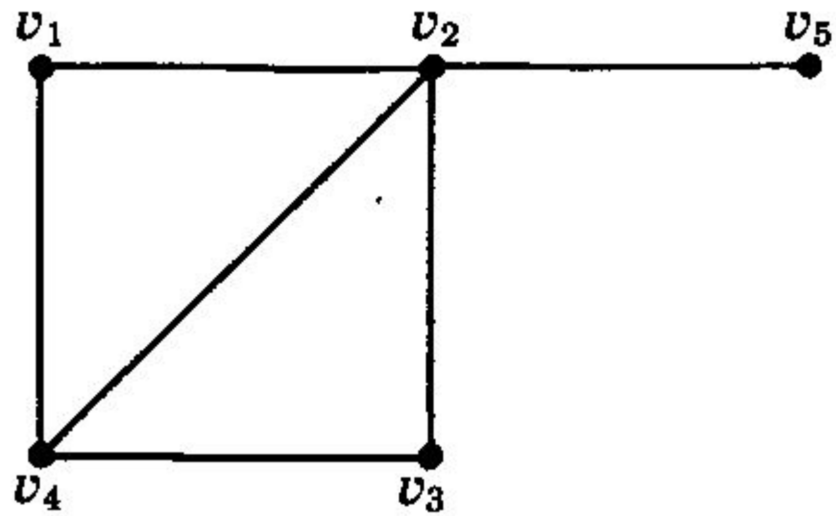
Operations on graphs

- On the other hand, if v_i and v_j are not adjacent in G, the addition of line $v_i v_j$ results in the smallest supergraph of G containing the line $v_i v_j$.

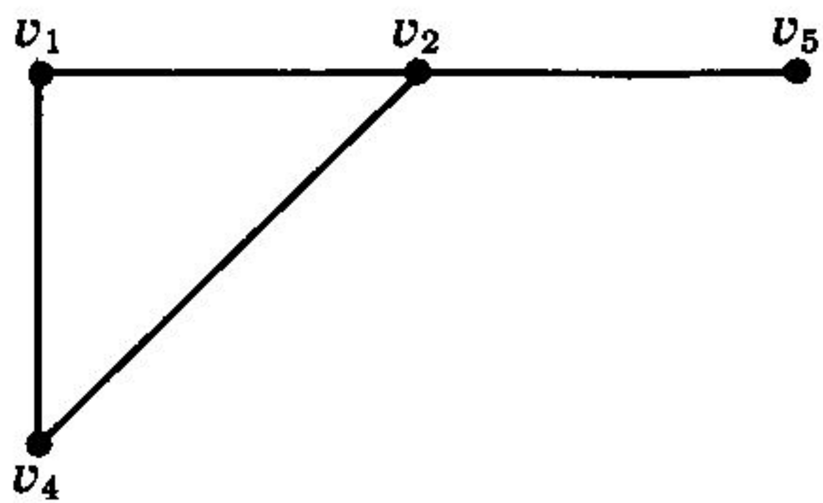
These concepts are illustrated in the following figures.

A graph plus or minus a specific point or line

G:

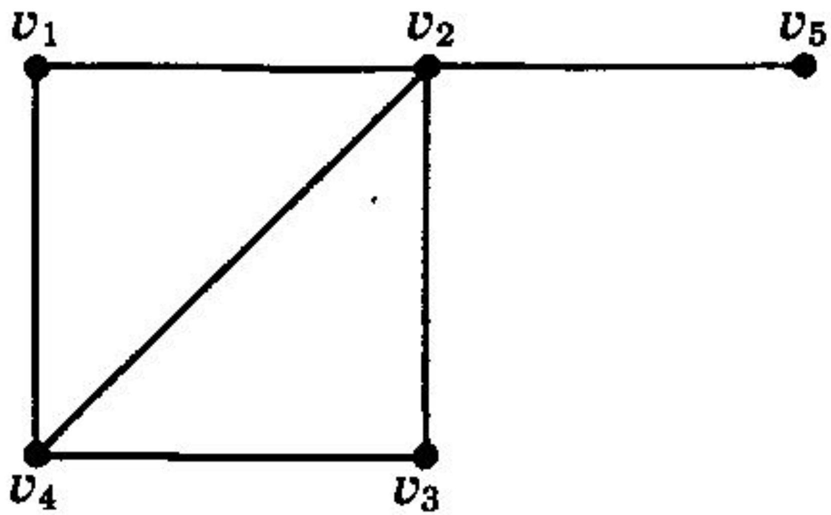


$G - v_3$:

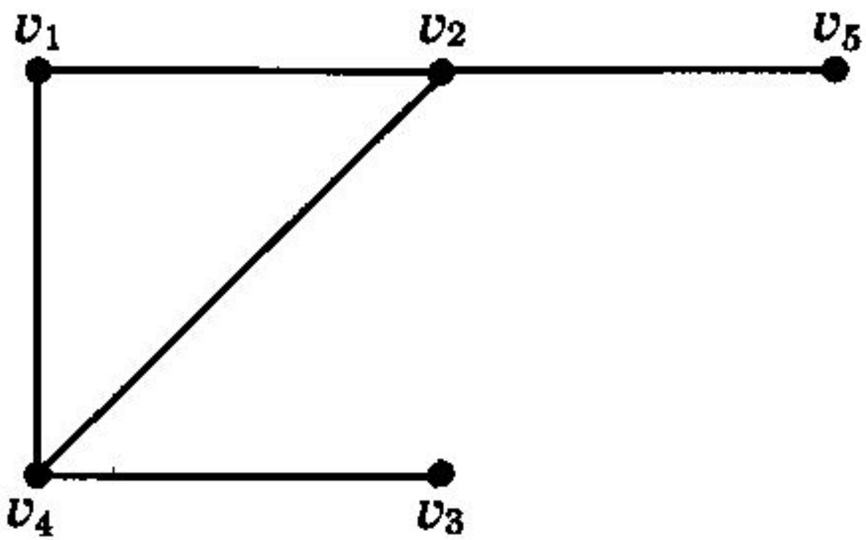


A graph plus or minus a specific point or line

G:

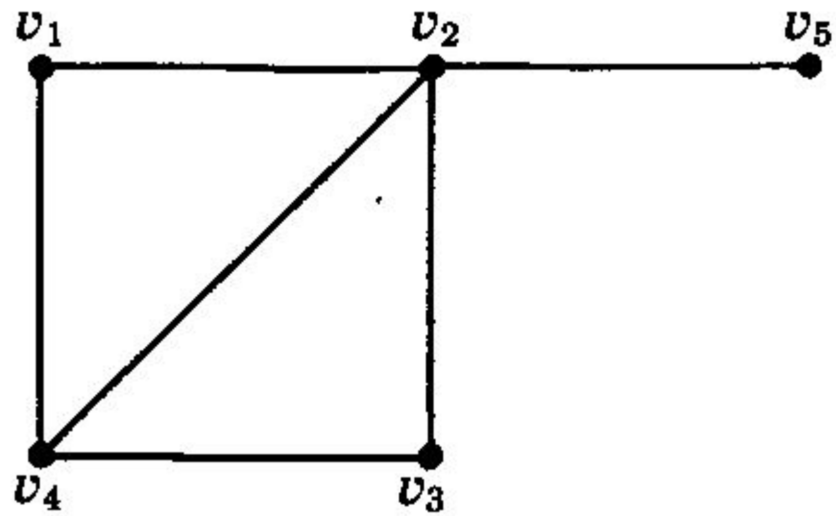


$G - v_2v_3$:

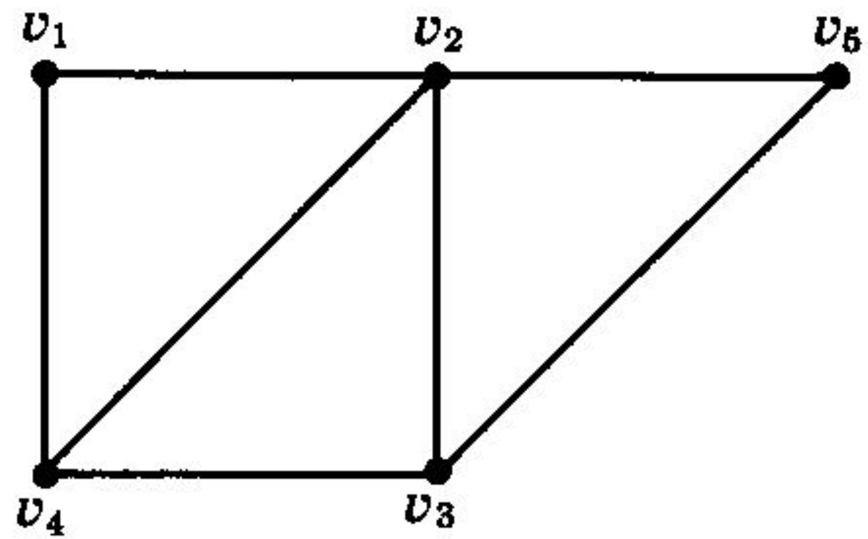


A graph plus or minus a specific point or line

G:

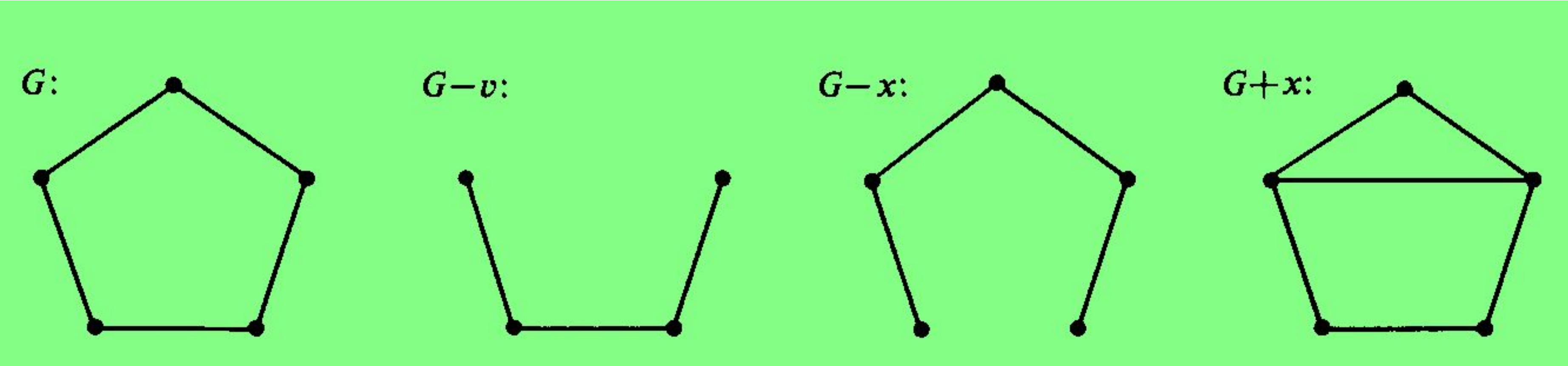


$G+v_3v_5$:



Operations on graphs

There are certain graphs for which the result of deleting a point or line, or adding a line, is independent of the particular point or line selected.



A graph plus or minus a point or line.

Operations on graphs

It was suggested by Ulam in the following conjecture that the collection of subgraphs $G - v_i$ of G gives quite a bit of information about G itself.

Ulam's Conjecture Let G have p points v_i and H have p points u_i, with $p \geq 3$. If for each i, the subgraphs $G_i = G - v_i$ and $H_i = H - u_i$ are isomorphic, then the graphs G and H are isomorphic.

Operations on graphs

It is rather convenient to be able to express the structure of a given graph in terms of smaller and simpler graphs.

Operations on graphs

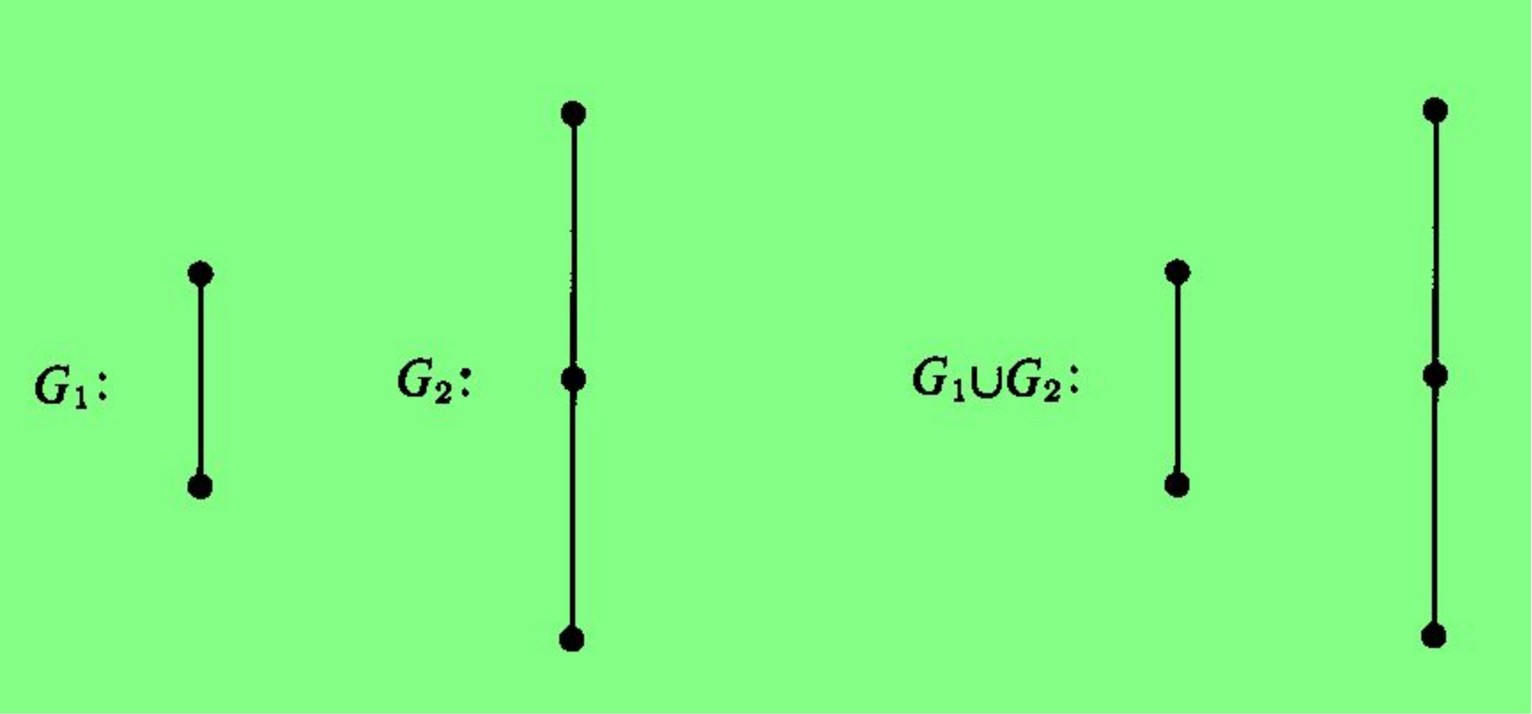
- Let graphs G_1 and G_2 have disjoint point sets V_1 and V_2 and line sets X_1 and X_2 respectively.

Their **union** $G = G_1 \cup G_2$ has, as expected, $V = V_1 \cup V_2$ and $X = X_1 \cup X_2$.

Their **join** is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all lines joining V_1 with V_2.

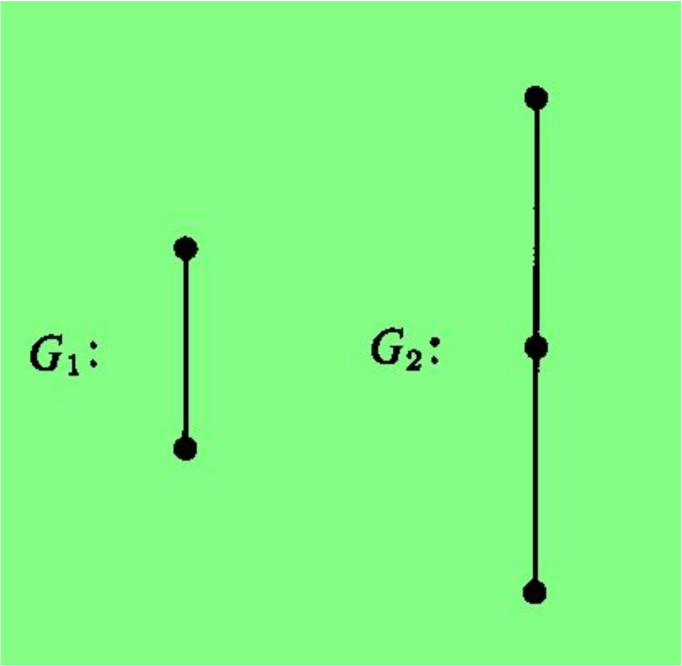
These operations are illustrated in the following figure.

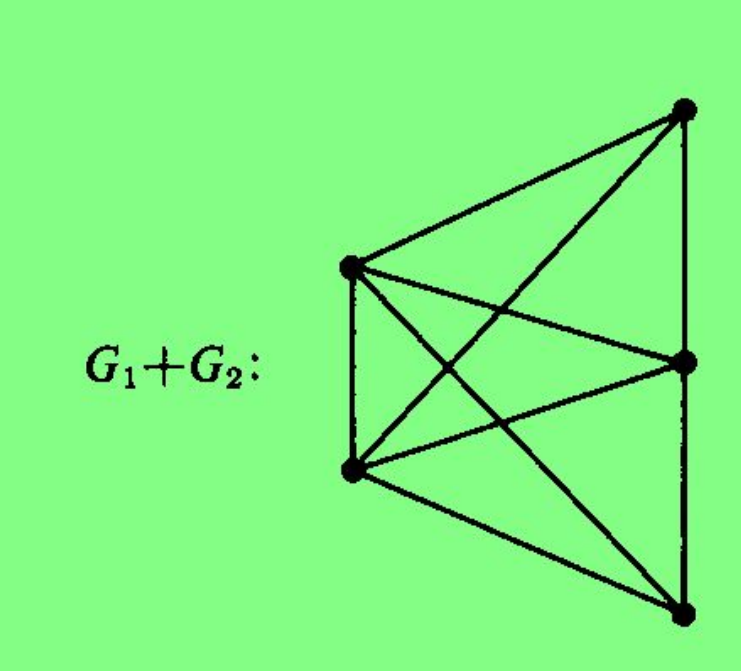
Operations on graphs



The union of two graphs.

Operations on graphs





The join of two graphs.

Operations on graphs

- There are several operations on G_1 and G_2 which result in a graph G whose set of points is the cartesian product $V_1 \times V_2$.

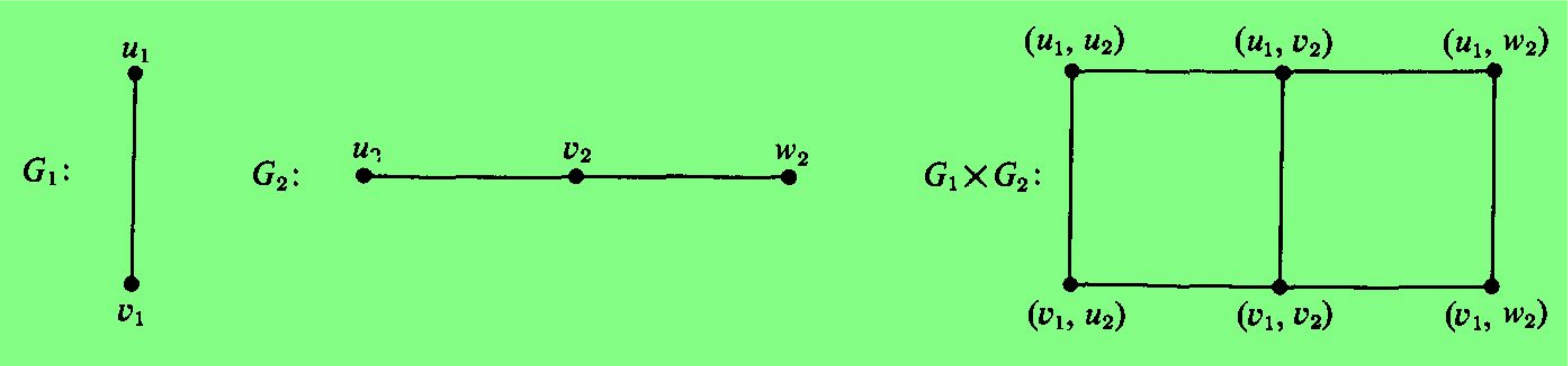
These include the **product** (or **cartesian product**), and the **composition** (or **lexicographic product**).

Operations on graphs

- To define the **product** $G_1 \times G_2$ consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V_1 \times V_2$.

Then u and v are adjacent in $G_1 \times G_2$ whenever ($u_1 = v_1$ and u_2 and v_2 are adjacent in G_2) or ($u_2 = v_2$ and u_1 and v_1 are adjacent in G_1).

Operations on graphs

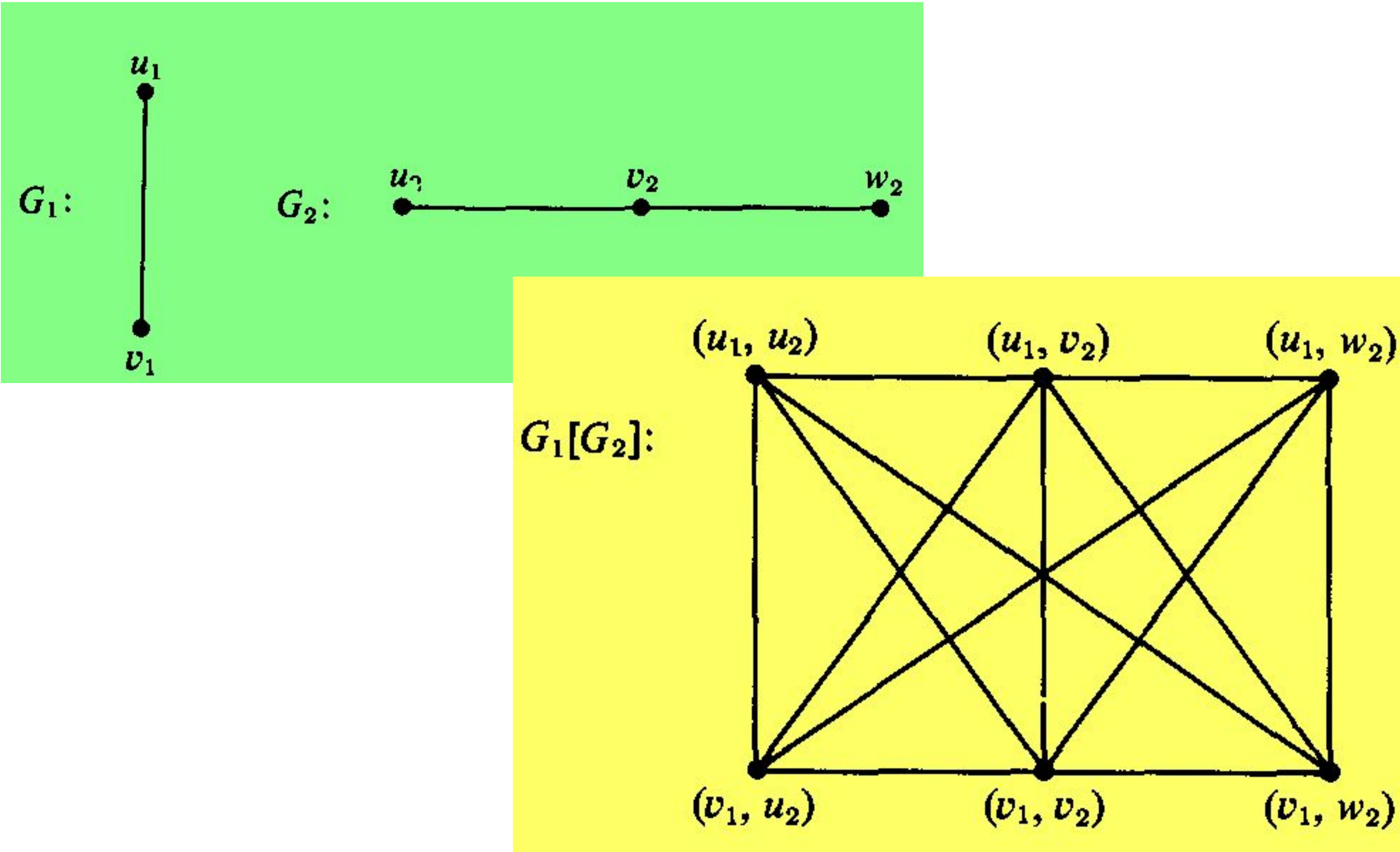


The product of two graphs.

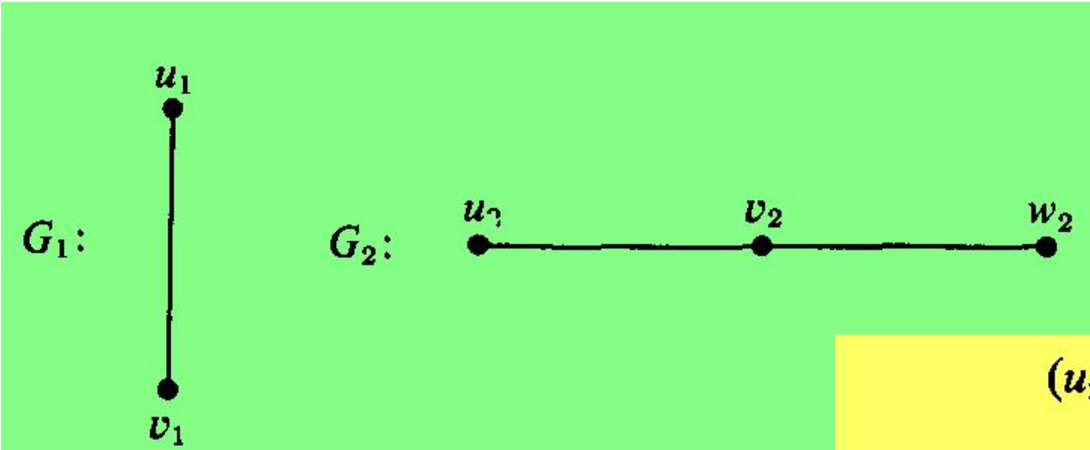
Operations on graphs

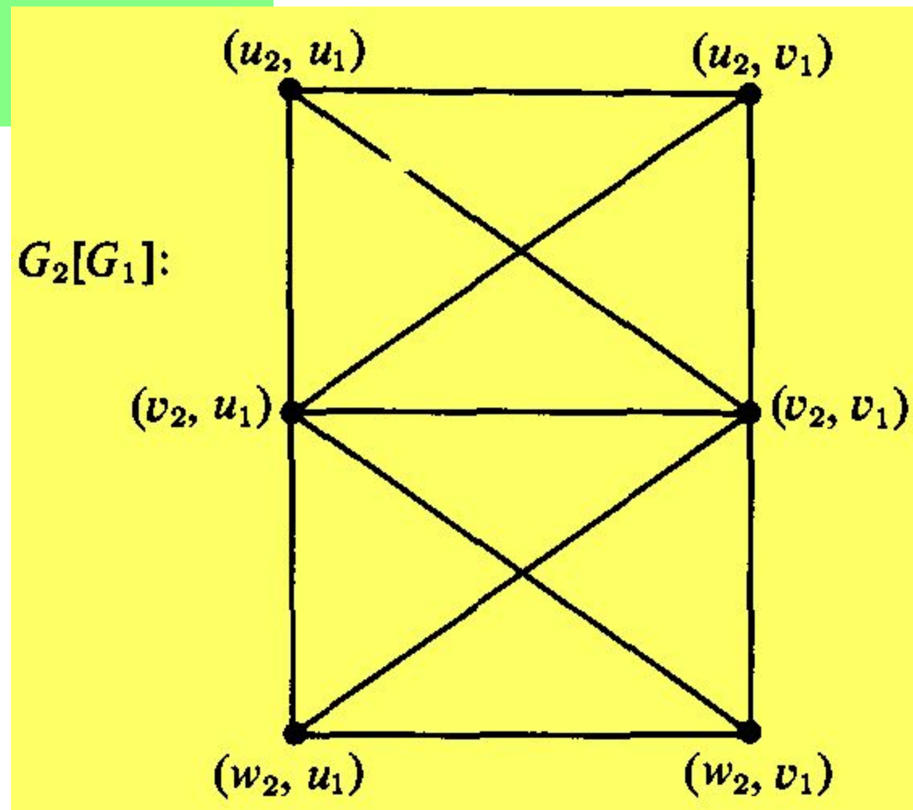
- The **composition** $G = G_1[G_2]$ also has $V_1 \times V_2$ as its point set, and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever (u_1 and v_1 are adjacent in G_1) or ($u_1 = v_1$ and u_2 and v_2 are adjacent in G_2).

Two compositions of graphs



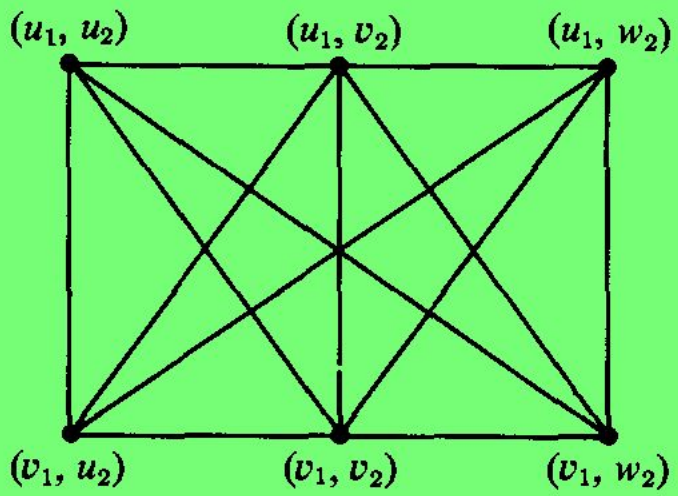
Two compositions of graphs



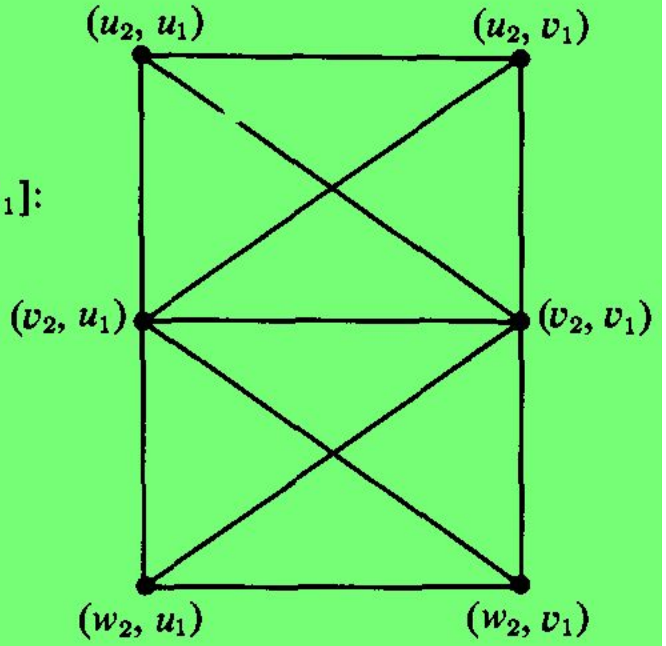


Operations on graphs

$G_1[G_2]$:



$G_2[G_1]$:



The compositions $G_1[G_2]$ and $G_2[G_1]$ are obviously not isomorphic.

Operations on graphs

- An especially important class of graphs known as cubes are most naturally expressed in terms of products.

The ***n*-cube** Q_n is defined recursively by $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$.

Thus Q_n has 2^n points which may be labeled $a_1 a_2 \ldots a_n$ where each a_i is either 0 or 1.

Two points of Q_n are adjacent if their binary representations differ at exactly one place.

Operations on graphs

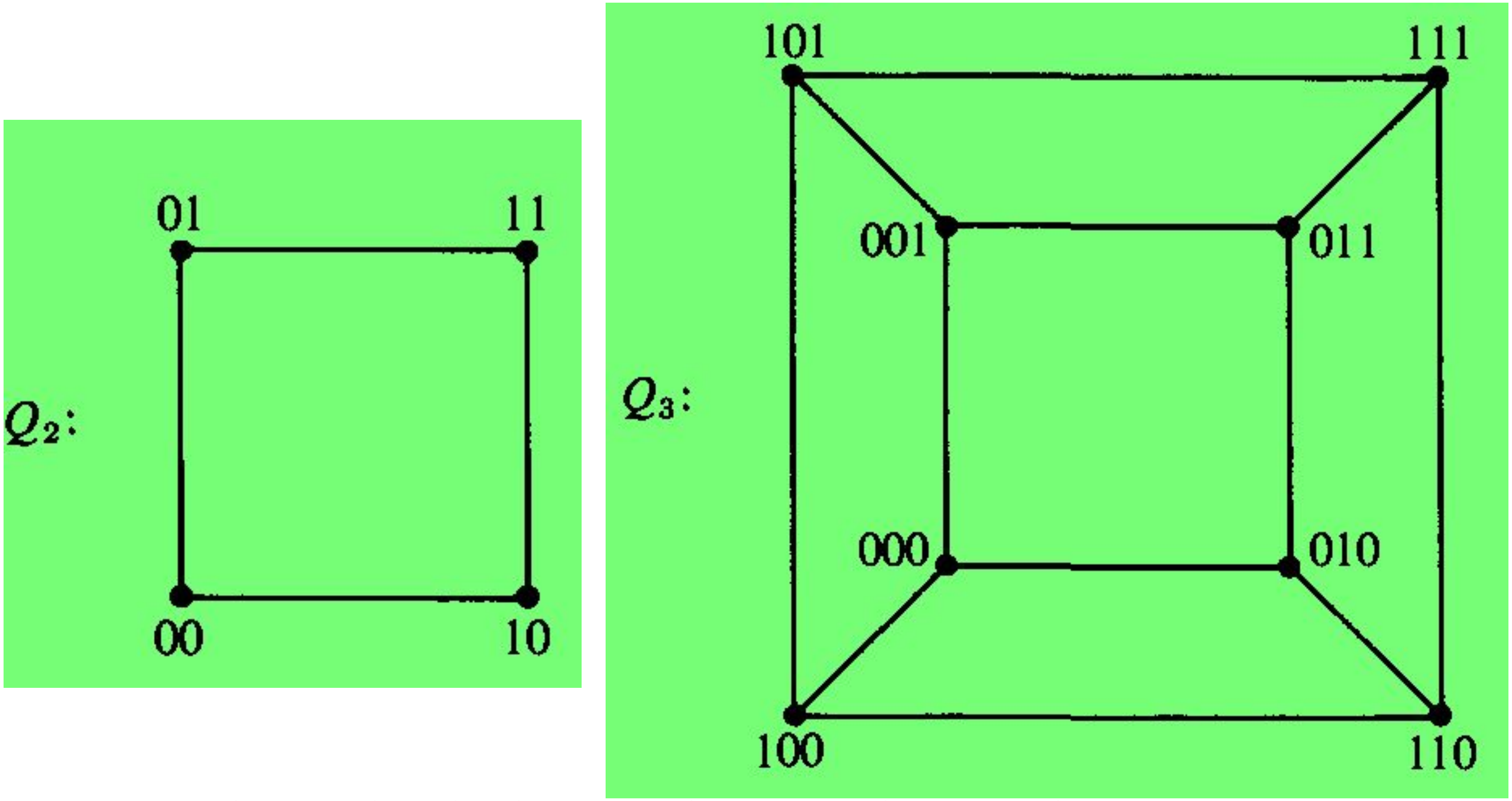


Figure shows both the 2-cube and the 3-cube, appropriately labeled.

Intersection graphs

Let S be a set and $F = \{S_1, \ldots, S_p\}$ a family of distinct nonempty subsets of S whose union is S.

The **intersection graph** of F is denoted $\Omega(F)$ and defined by $V(\Omega(F)) = F$, with S_i and S_j adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \emptyset$.

Then a graph G is **an intersection graph on** S if there exists a family F of subsets of S for which $G \cong \Omega(F)$.

Intersection graphs

Theorem 1

Every graph is an intersection graph.

<u>*Proof*</u>

For each point v_i of G, let S_i be the union of $\{v_i\}$ with the set of lines incident with v_i.

Then it is immediate that G is isomorphic with $\Omega(F)$ where $F = \{S_i\}$. ■

Metrical characteristics of graphs

- A **walk** of a graph G is an alternating sequence of points and lines $v_0, x_1, v_1, \dots, v_{n-1}, x_n, v_n$ beginning and ending with points, in which each line is incident with the two points immediately preceding and following it.

It is a **trail** if all the lines are distinct, and a **path** if all the points (and thus necessarily all the lines) are distinct.

If the walk is closed, then it is a **cycle** provided its n points are distinct and $n \geq 3$.

Metrical characteristics of graphs

- The **length** of a walk $v_0, x_1, v_1, \dots, v_{n-1}, x_n, v_n$ is n, the number of occurrences of lines in it.

The **girth** of a graph G, denoted $g(G)$, is the length of a shortest cycle (if any) in G;

the **circumference** $c(G)$ is the length of any longest cycle.

Note that these terms are undefined if G has no cycles.

Metrical characteristics of graphs

- The **distance** $d(u,v)$ between two points u and v in G is the length of a shortest path joining them if any; otherwise $d(u,v)=\infty$.

In a connected graph, distance is a metric; that is, for all points u, v, and w,

1. $d(u,v) \geq 0$, with $d(u,v)=0$ if and only if $u=v$.
2. $d(u,v)=d(v,u)$.
3. $d(u,v)+d(v,w) \geq d(u,w)$.

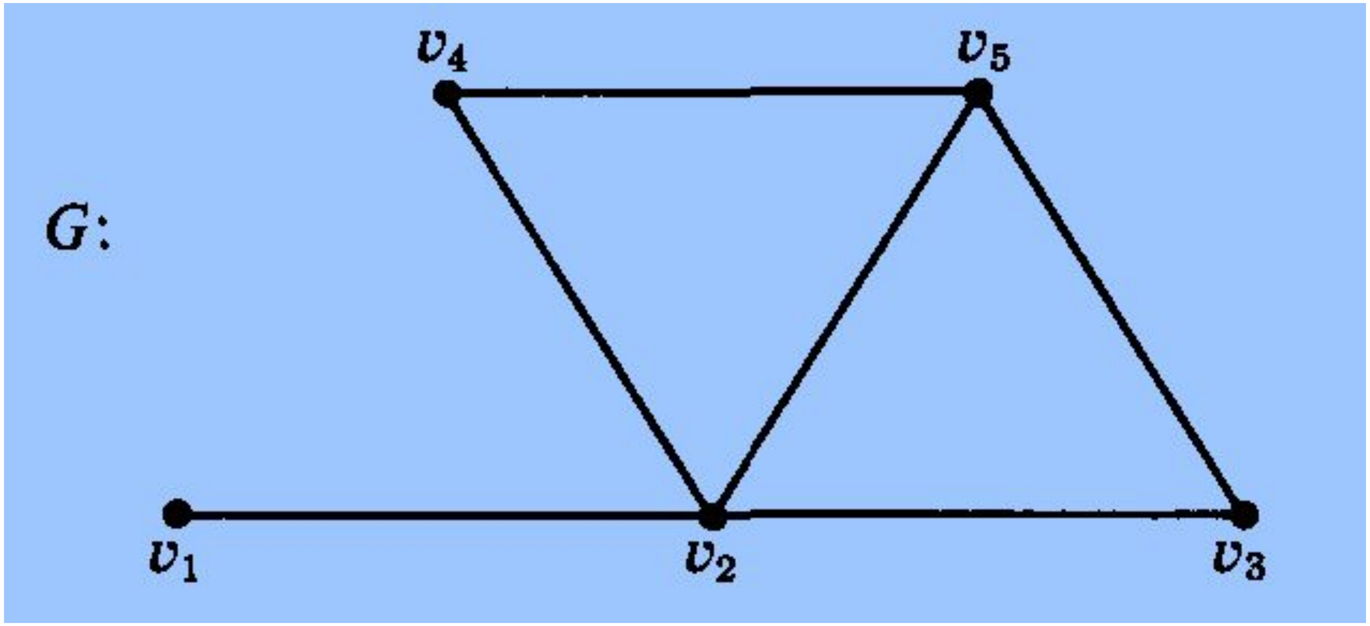
Metrical characteristics of graphs

A shortest $u - v$ path is often called a **geodesic**.

The **diameter** $d(G)$ of a connected graph G is the length of any longest geodesic.

Metrical characteristics of graphs

- The graph G of the figure has girth $g = 3$, circumference $c = 4$, and diameter $d = 2$.



A graph to illustrate walks.

König's theorem

A bigraph (or bipartite graph) G is a graph whose point set V can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2.

König's theorem

Theorem (König's theorem)

A graph is bipartite if and only if all its cycles are even.

<u>*Proof*</u>

If G is a bigraph, then its point set V can be partitioned into two sets V_1 and V_2 so that every line of G joins a point of V_1 with a point of V_2.

Thus every cycle $v_1 v_2 \dots v_n v_1$ in G necessarily has its oddly subscripted points in V_1 say, and the others in V_2, so that its length n is even.

Theorem (König's theorem)
A graph is bipartite if and only if all its cycles are even.

Proof

For the converse, we assume, without loss of generality, that G is connected (for otherwise we can consider the components of G separately).

Take any point $v_1 \in V$, and let V_1 consist of v_1 and all points at even distance from v_1 while $V_2 = V — V_1$.

Theorem (König's theorem)
A graph is bipartite if and only if all its cycles are even.

Proof

Take any point $v_1 \in V$, and let V_1 consist of v_1 and all points at even distance from v_1 while $V_2 = V — V_1$.

Since all the cycles of G are even, every line of G joins a point of V_1 with a point of V_2.

For suppose there is a line uv joining two points of V_1.

Then the union of geodesies from v_1 to v and from v_1 to u together with the line uv contains an odd cycle, a contradiction. ■