Calculus 1

Lecture 6 – **Integrals**

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Lecture overview

- Trigonometric integrals.
- Trigonometric substitutions.
- Rational Functions and partial Fractions

Trigonometric Integrals

Products of powers of Sines and cosines

$$\int \sin^m x \cos^n x \, dx$$

Case 1 If **m is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If **n is odd** in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If **both m and n are even** in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x)(\cos^2 x)(-d(\cos x))$$

$$= \int (1 - u^2)(u^2)(-du)$$

$$= \int (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

Evaluate

$$\int \cos^5 x \, dx.$$

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \, d(\sin x) \qquad \cos x \, dx = d(\sin x)$$

$$= \int (1 - u^2)^2 \, du \qquad\qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4) \, du \qquad\qquad \text{Square } 1 - u^2.$$

$$= u - \frac{2}{3} u^3 + \frac{1}{5} u^5 + C = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C$$

Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \qquad \textcolor{blue}{m \text{ and } n \text{ both even}}$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) \, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x) \, dx \right]$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x) \, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x \right).$$ \qquad \textcolor{blue}{\text{Omit constant of integration until final result.}}

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x) \cos 2x \, dx \qquad \textcolor{blue}{u = \sin 2x, \ du = 2\cos 2x \, dx}$$

$$= \frac{1}{2}\int (1 - u^2) \, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x \right). \qquad \textcolor{blue}{\text{Again omit } C.}$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x \right) + C.$$

Integrals of Powers of $\tan x$ and $\sec x$

Evaluate

$$\int \tan^4 x \, dx.$$

$$\int \tan^4 x \, dx = \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x \, dx$$

and have

$$\int u^2 \, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \qquad \int \sin mx \cos nx \, dx, \qquad \text{and} \qquad \int \cos mx \cos nx \, dx$$

$$\sin mx \sin nx = \frac{1}{2} \left[\cos (m - n)x - \cos (m + n)x \right],$$

$$\sin mx \cos nx = \frac{1}{2} \left[\sin (m - n)x + \sin (m + n)x \right],$$

$$\cos mx \cos nx = \frac{1}{2} \left[\cos (m - n)x + \cos (m + n)x \right].$$

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right],$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right],$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right].$$

Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

From Equation (4) with $m = 3$ and $n = 5$, we get

$$\int \sin 3x \cos 5x \, dx = \frac{1}{2}\int \left[\sin(-2x) + \sin 8x\right] dx$$
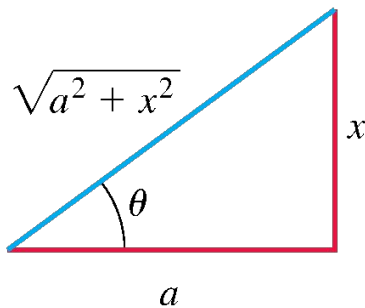
$$= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx$$

$$= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.$$

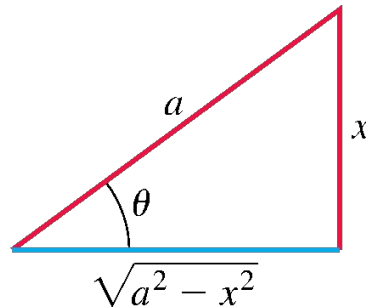# Trigonometric Substitutions

$$\int \frac{dx}{\sqrt{4 + x^2}} \qquad \int \frac{x^2 \, dx}{\sqrt{9 - x^2}} \qquad \int \frac{dx}{\sqrt{25x^2 - 4}}$$
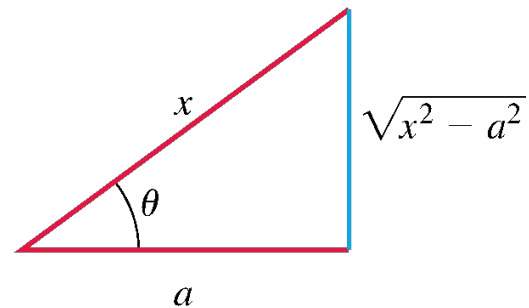
$$x = a \tan \theta \qquad\qquad x = a \sin \theta \qquad\qquad x = a \sec \theta$$

$$\sqrt{a^2 + x^2} = a|\sec \theta| \qquad \sqrt{a^2 - x^2} = a|\cos \theta| \qquad \sqrt{x^2 - a^2} = a|\tan \theta|$$

Procedure for a Trigonometric Substitution

1. Write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.

2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.

3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.

4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}$$

$$x = 2 \tan \theta, \qquad dx = 2 \sec^2 \theta \, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta.$$

$$\int \frac{dx}{\sqrt{4 + x^2}} = \int \frac{2 \sec^2 \theta \, d\theta}{\sqrt{4 \sec^2 \theta}} = \int \frac{\sec^2 \theta \, d\theta}{|\sec \theta|}$$

$$= \int \sec \theta \, d\theta$$

$$= \ln \left| \sec \theta + \tan \theta \right| + C$$

$$= \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C.$$

Evaluate

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}$$

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1 - \sin^2\theta) = 9\cos^2\theta.$$

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta\,d\theta$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. Example 3 of this section illustrates such a case.

- We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Method of Partial Fractions When $f(x)/g(x)$ Is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1 x + C_1}{(x^2 + px + q)} + \frac{B_2 x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_n x + C_n}{(x^2 + px + q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Use partial fractions to evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \, dx$$

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

$$x^2 + 4x + 1 = A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1)$$
$$= A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1)$$
$$= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C).$$

Coefficient of x^2: $A + B + C = 1$
Coefficient of x^1: $4A + 2B = 4$
Coefficient of x^0: $3A - 3B - C = 1$

$A = 3/4, B = 1/2,$ and $C = -1/4$

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \, dx = \int \left[\frac{3}{4} \frac{1}{x - 1} + \frac{1}{2} \frac{1}{x + 1} - \frac{1}{4} \frac{1}{x + 3} \right] dx$$

$$= \frac{3}{4} \ln |x - 1| + \frac{1}{2} \ln |x + 1| - \frac{1}{4} \ln |x + 3| + K$$

EXAMPLE 2   Use partial fractions to evaluate

$$\int \frac{6x + 7}{(x + 2)^2} \, dx.$$

Solution   First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$ Two terms because $(x + 2)$ is squared

$$6x + 7 = A(x + 2) + B$$ Multiply both sides by $(x + 2)^2$.

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\int \frac{6x + 7}{(x + 2)^2} \, dx = \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2} \right) dx$$

$$= 6 \int \frac{dx}{x + 2} - 5 \int (x + 2)^{-2} \, dx$$

$$= 6 \ln |x + 2| + 5(x + 2)^{-1} + C.$$

Use partial fractions to evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx$$

$$
\begin{array}{r}
2x \\
x^2 - 2x - 3 \overline{)2x^3 - 4x^2 - x - 3} \\
\underline{2x^3 - 4x^2 - 6x - 3} \\
5x - 3
\end{array}
$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx = \int 2x\, dx + \int \frac{5x - 3}{x^2 - 2x - 3}\, dx$$

EXAMPLE 5   Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2 + 1)^2}.$$

Solution   The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}.$$

Multiplying by $x(x^2 + 1)^2$, we have

$$
\begin{aligned}
1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\
&= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\
&= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A.
\end{aligned}
$$

If we equate coefficients, we get the system

$$A + B = 0, \qquad C = 0, \qquad 2A + B + D = 0, \qquad C + E = 0, \qquad A = 1.$$

Solving this system gives $A = 1, B = -1, C = 0, D = -1,$ and $E = 0$. Thus,

$$
\begin{aligned}
\int \frac{dx}{x(x^2 + 1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2 + 1} + \frac{-x}{(x^2 + 1)^2} \right] dx \\[2mm]
&= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2 + 1} - \int \frac{x\,dx}{(x^2 + 1)^2} \\[2mm]
&= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad \begin{aligned} &u = x^2 + 1, \\ &du = 2x\,dx \end{aligned} \\[2mm]
&= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\[2mm]
&= \ln|x| - \frac{1}{2}\ln(x^2 + 1) + \frac{1}{2(x^2 + 1)} + K \\[2mm]
&= \ln \frac{|x|}{\sqrt{x^2 + 1}} + \frac{1}{2(x^2 + 1)} + K.
\end{aligned}
$$

Other Ways to Determine the Coefficients

EXAMPLE 8   Find A, B, and C in the expression

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}$$

by assigning numerical values to x.

Solution   Clear fractions to get

$$x^2 + 1 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$x = 1: \qquad (1)^2 + 1 = A(-1)(-2) + B(0) + C(0)$$
$$2 = 2A$$
$$A = 1$$
$$x = 2: \qquad (2)^2 + 1 = A(0) + B(1)(-1) + C(0)$$
$$5 = -B$$
$$B = -5$$
$$x = 3: \qquad (3)^2 + 1 = A(0) + B(0) + C(2)(1)$$
$$10 = 2C$$
$$C = 5.$$

Conclusion:

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{1}{x - 1} - \frac{5}{x - 2} + \frac{5}{x - 3}.$$