

Linear Algebra

Lecture 4

The Inverse of a Matrix.

Characterizations of Invertible Matrices.

Partitioned Matrices. Matrix factorizations.

Subspaces of \mathbb{R}^n . Dimension and Rank.

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Learning Objectives:

1. The meaning and characterizations of invertible matrix.
2. Finding the LU factorization of the matrix.
3. Subspaces, Column Spaces and Null Spaces.

2.2 The Inverse of a Matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

$$\det A = ad - bc$$

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

EXAMPLE 2 Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

SOLUTION Since $\det A = 3(6) - 4(5) = -2 \neq 0$, A is invertible, and

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 6/(-2) & -4/(-2) \\ -5/(-2) & 3/(-2) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

THEOREM 5

If A is an invertible $n \times n$ matrix, then for each \mathbf{b} in \mathbb{R}^n , the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

EXAMPLE 4 Use the inverse of the matrix A in Example 2 to solve the system

$$3x_1 + 4x_2 = 3$$

$$5x_1 + 6x_2 = 7$$

SOLUTION This system is equivalent to $A\mathbf{x} = \mathbf{b}$, so

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

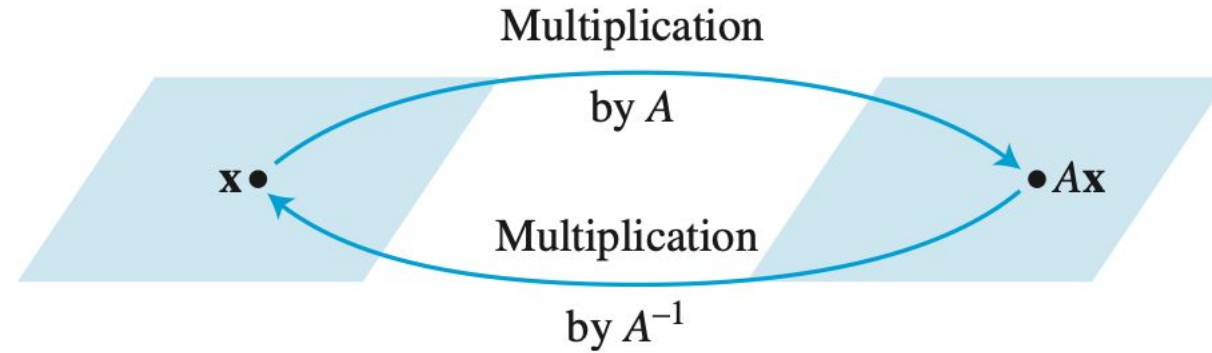


FIGURE 2 A^{-1} transforms $A\mathbf{x}$ back to \mathbf{x} .

THEOREM 6

- a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

- b. If A and B are $n \times n$ invertible matrices, then so is AB , and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

- c. If A is an invertible matrix, then so is A^T , and the inverse of A^T is the transpose of A^{-1} . That is,

$$(A^T)^{-1} = (A^{-1})^T$$

An Algorithm for Finding A^{-1}

EXAMPLE 7 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

SOLUTION

$$\begin{aligned} [A \ I] &= \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix} \end{aligned}$$

Theorem 7 shows, since $A \sim I$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

It is a good idea to check the final answer:

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is not necessary to check that $A^{-1}A = I$ since A is invertible.

2.3. Characterizations of Invertible Matrices.

THEOREM 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A , the statements are either all true or all false.

- a. A is an invertible matrix.
- b. A is row equivalent to the $n \times n$ identity matrix.
- c. A has n pivot positions.
- d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- e. The columns of A form a linearly independent set.
- f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
- g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n .
- h. The columns of A span \mathbb{R}^n .
- i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps \mathbb{R}^n onto \mathbb{R}^n .
- j. There is an $n \times n$ matrix C such that $CA = I$.
- k. There is an $n \times n$ matrix D such that $AD = I$.
- l. A^T is an invertible matrix.

Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.

EXAMPLE 1 Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

SOLUTION

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

So A has three pivot positions and hence is invertible, by the Invertible Matrix Theorem, statement (c). ■

2.4. Partitioned Matrices.

A block matrix or a **partitioned matrix** is a partition of a matrix into rectangular smaller matrices called blocks.

EXAMPLE 1 The matrix

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array} \right]$$

can also be written as the 2×3 **partitioned** (or **block**) **matrix**

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

whose entries are the *blocks* (or *submatrices*)

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -4 \end{bmatrix}$$

EXAMPLE 3 Let

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ 0 & -4 & -2 & 7 & -1 \end{array} \right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \left[\begin{array}{c|c} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array} \right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

The 5 columns of A are partitioned into a set of 3 columns and then a set of 2 columns. The 5 rows of B are partitioned in the same way—into a set of 3 rows and then a set of 2 rows. We say that the partitions of A and B are **conformable for block multiplication**. It can be shown that the ordinary product AB can be written as

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 2 \\ 2 & 1 \end{bmatrix}$$

It is important for each smaller product in the expression for AB to be written with the submatrix from A on the left, since matrix multiplication is not commutative. For instance,

$$A_{11}B_1 = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix}$$

$$A_{12}B_2 = \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix}$$

Hence the top block in AB is

$$A_{11}B_1 + A_{12}B_2 = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix} + \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 2 \end{bmatrix}$$



2.5. Matrix Factorizations.

A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices.

The LU Factorization

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

L U

FIGURE 1 An LU factorization.

$$Ly=b$$

$$Ux=y$$

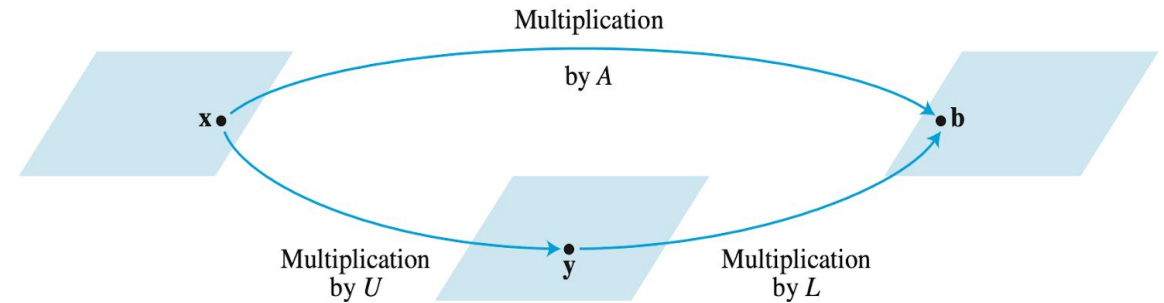


FIGURE 2 Factorization of the mapping $x \mapsto Ax$.

ALGORITHM FOR AN LU FACTORIZATION

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the *same sequence of row operations* reduces L to I .

1)

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

2)

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \xrightarrow{\div 2} \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \xrightarrow{\div 3} \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \xrightarrow{\div 2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 5 \end{bmatrix} \xrightarrow{\div 5} \begin{bmatrix} 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

2.8. Subspaces of \mathbb{R}^n .

A **subspace** of \mathbb{R}^n is any set H in \mathbb{R}^n that has three properties:

- a. The zero vector is in H .
- b. For each \mathbf{u} and \mathbf{v} in H , the sum $\mathbf{u} + \mathbf{v}$ is in H .
- c. For each \mathbf{u} in H and each scalar c , the vector $c\mathbf{u}$ is in H .

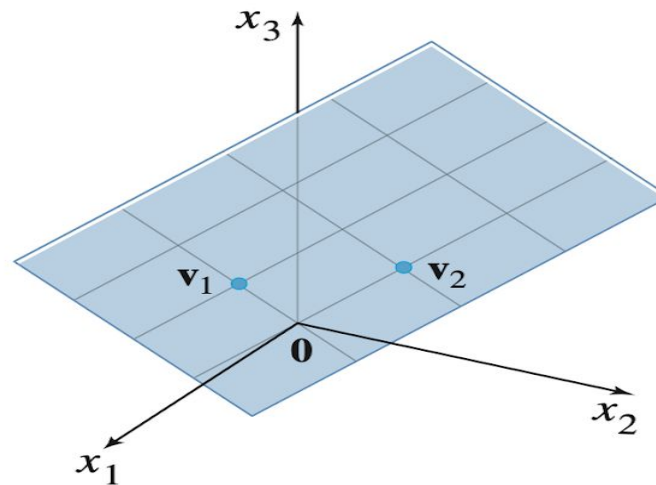


FIGURE 1

Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ as a plane through the origin.

EXAMPLE 2 A line L *not* through the origin is *not* a subspace, because it does not contain the origin, as required. Also, Figure 2 shows that L is not closed under addition or scalar multiplication. ■

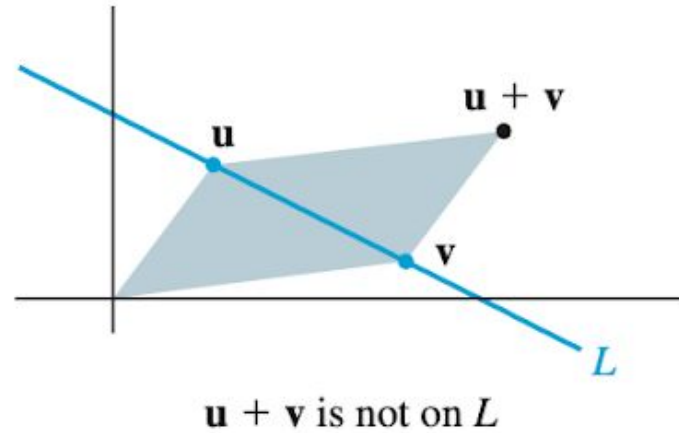
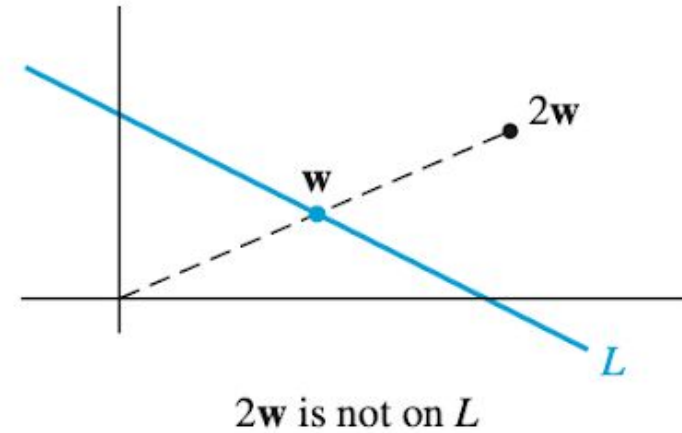


FIGURE 2



Column Space and Null Space of a Matrix

DEFINITION

The **column space** of a matrix A is the set $\text{Col } A$ of all linear combinations of the columns of A .

If $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$, with the columns in \mathbb{R}^m , then $\text{Col } A$ is the same as $\text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Example 4 shows that the **column space of an $m \times n$ matrix is a subspace of \mathbb{R}^m** . Note that $\text{Col } A$ equals \mathbb{R}^m only when the columns of A span \mathbb{R}^m . Otherwise, $\text{Col } A$ is only part of \mathbb{R}^m .

DEFINITION

The **null space** of a matrix A is the set $\text{Nul } A$ of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

DEFINITION

A **basis** for a subspace H of \mathbb{R}^n is a linearly independent set in H that spans H .

EXAMPLE 6 Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

SOLUTION First, write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{array}{l} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \\ 0 = 0 \end{array}$$

The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2 , x_4 , and x_5 free.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$
 $\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}$

$$= x_2 \mathbf{u} + x_4 \mathbf{v} + x_5 \mathbf{w} \tag{1}$$

Equation (1) shows that $\text{Nul } A$ coincides with the set of all linear combinations of \mathbf{u} , \mathbf{v} , and \mathbf{w} . That is, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ generates $\text{Nul } A$. In fact, this construction of \mathbf{u} , \mathbf{v} , and \mathbf{w} automatically makes them linearly independent, because equation (1) shows that $\mathbf{0} = x_2 \mathbf{u} + x_4 \mathbf{v} + x_5 \mathbf{w}$ only if the weights x_2 , x_4 , and x_5 are all zero. (Examine entries 2, 4, and 5 in the vector $x_2 \mathbf{u} + x_4 \mathbf{v} + x_5 \mathbf{w}$.) So $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a *basis* for $\text{Nul } A$. ■

2.9. Dimension and Rank.

DEFINITION

Suppose the set $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ is a basis for a subspace H . For each \mathbf{x} in H , the **coordinates of \mathbf{x} relative to the basis \mathcal{B}** are the weights c_1, \dots, c_p such that $\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_p \mathbf{b}_p$, and the vector in \mathbb{R}^p

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the **coordinate vector of \mathbf{x} (relative to \mathcal{B})** or the **\mathcal{B} -coordinate vector of \mathbf{x}** .¹

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then \mathcal{B} is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ because \mathbf{v}_1 and \mathbf{v}_2 are linearly independent. Determine if \mathbf{x} is in H , and if it is, find the coordinate vector of \mathbf{x} relative to \mathcal{B} .

SOLUTION If \mathbf{x} is in H , then the following vector equation is consistent:

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

The scalars c_1 and c_2 , if they exist, are the \mathcal{B} -coordinates of \mathbf{x} . Row operations show that

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $c_1 = 2$, $c_2 = 3$, and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The basis \mathcal{B} determines a “coordinate system” on H , which can be visualized by the grid shown in Figure 1. ■

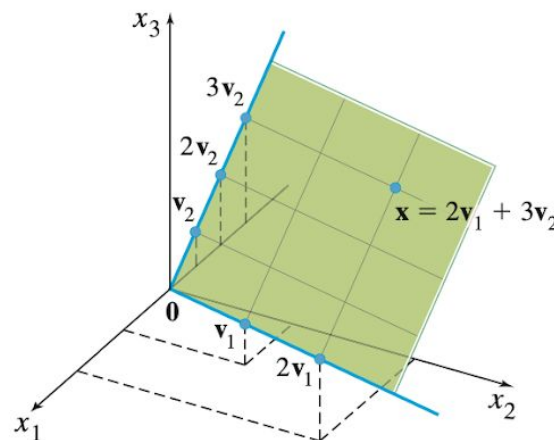


FIGURE 1 A coordinate system on a plane H in \mathbb{R}^3 .

DEFINITION

The **dimension** of a nonzero subspace H , denoted by $\dim H$, is the number of vectors in any basis for H . The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.²

DEFINITION


The **rank** of a matrix A , denoted by $\text{rank } A$, is the dimension of the column space of A .

EXAMPLE 3 Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

SOLUTION Reduce A to echelon form:

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns 

The matrix A has 3 pivot columns, so $\text{rank } A = 3$.

THEOREM 14

The Rank Theorem

If a matrix A has n columns, then $\text{rank } A + \dim \text{Nul } A = n$.

THEOREM 15

The Basis Theorem

Let H be a p -dimensional subspace of \mathbb{R}^n . Any linearly independent set of exactly p elements in H is automatically a basis for H . Also, any set of p elements of H that spans H is automatically a basis for H .

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

- m. The columns of A form a basis of \mathbb{R}^n .
- n. $\text{Col } A = \mathbb{R}^n$
- o. $\dim \text{Col } A = n$
- p. $\text{rank } A = n$
- q. $\text{Nul } A = \{\mathbf{0}\}$
- r. $\dim \text{Nul } A = 0$