Saratov State University

APPLICATIONS OF SEMIRINGS

Postgraduate student: **I.V. Diveikin**
Academic adviser: V.B. Poplavski
Language adviser: D.A. Alexeeva

2021

What are semirings?

A semiring is an algebraic structure(R, +, •), consisting of a nonempty set *R* on which we have defined two operations, **addition** + and **multiplication** · such that the following conditions hold:

1. Addition is associative and commutative and has a neutral element: $a+(b+c)=(a+b)+c$, $a+b=b+a$, and $a+0=0+a$, for $a,b,c\in R$.
2. Multiplication is associative and has a neutral element: $a(bc)=(ab)c$, $a1=1a=a$ for $a,b,c\in R$.
3. Multiplication is distributive with respect to addition: $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$, for $a,b,c\in R$.
4. There is a neutral element regarding multiplication: $a0=0a=0$, for $a,b,c\in R$

Julius Wilhelm Richard Dedekind

Harry Schultz Vandiver

Automata theory
Tropical semiring

The *tropical semiring* is the semiring $(\mathbb{R} \cup \{\infty\}, \min, +)$ with the operations:

- $a \oplus b = \min(a, b); a, b \in \mathbb{R} \cup \{\infty\}$
- $a \otimes b = a + b; a, b \in \mathbb{R} \cup \{\infty\}$

Optimization theory

Schedule algebra is a semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ with the operations:

- $a \oplus b = \max\{a, b\}$
- $a \otimes b = a + b$

Optimization Algebra is a semiring $(\mathbb{R} \cup \{+\infty\}, \oplus, \otimes)$ with the operations:

- $a \oplus b = \min\{a, b\}$
- $a \otimes b = a + b$

Algebras of formal processes

Algebra of communicating processes consists of a finite set *R* of **atomic actions** among which there is a designated action δ (= "deadlock"). On the set *R* we define two operations, addition (usually called **choice**) and multiplication (usually called **communication merge**) in such a manner that δ is the neutral element with respect to addition and that *R*, together with these operations, is a semiring.

Combinatorial optimization

For a given positive integer *n* and a given set *S* of elements of $N^n \subset \mathbb{R}^n$ *and* a vector $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$, we want to find $t_v = \min\{v \cdot y \mid y \in S\}$, where · is the usual dot product in $\mathbb{R}^n$.

Traveling Salesman Problem

n is be the number of edges in the given graph, the set *S* is the set of all possible paths, where $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in S$ means that there is a path in which, for each $1 \le h \le n$, the edge h appears c_h times, $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, where a_h is the cost of traversing edge h.

If we consider this calculation in the optimization algebra, we see that $t_v = p(a_1,\ldots,a_n)$, where

$$p(X_1,\ldots,X_n) = \sum \left\{ X_1^{c_1} \otimes \ldots \otimes X_n^{c_n} \,\middle|\, \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in S \right\} \qquad (1)$$

In other words, the problem is reduced to the evaluation of a polynomial in several indeterminates over a semifield.