NUFYP Mathematics

3.5 Trigonometry 5

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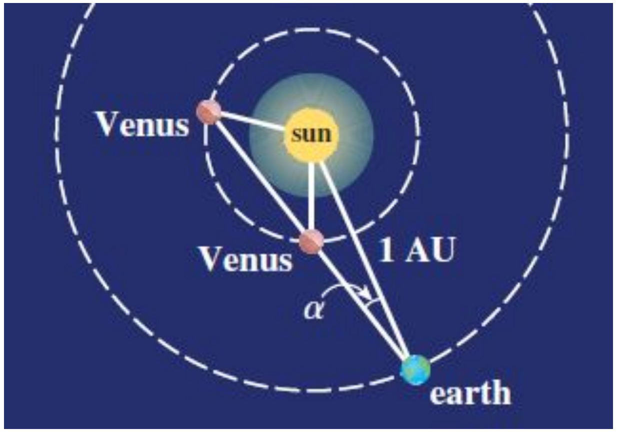
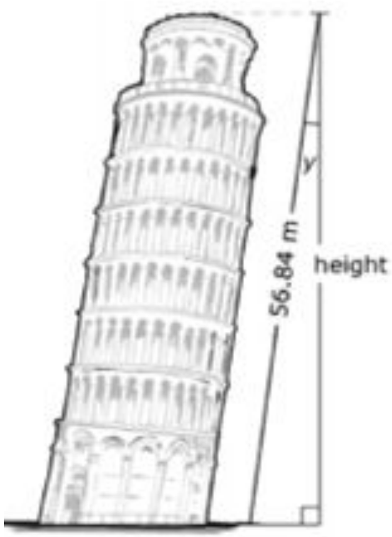
Lecture Outline

- The Law of Sines
- The Law of Cosines
- Harmonic motion

Introduction

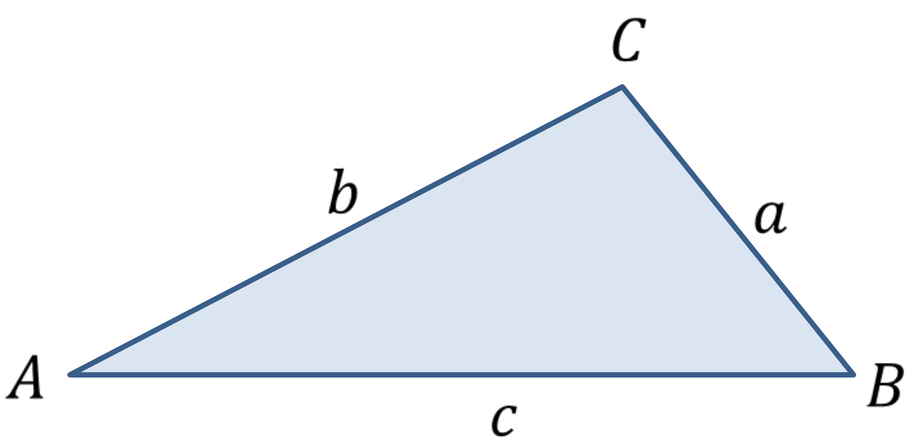
Triangles are everywhere, and we often need to find unknown angles or the side lengths. The Law of Sines and the Law of Cosines allow us to find those unknown quantities.





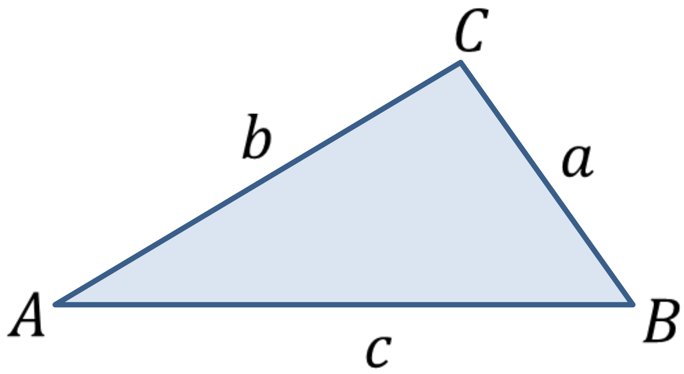
The Law of Sines

- The Law of Sines says that in any triangle the lengths of the sides are proportional to the sines of the corresponding opposite angles. That is,



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof: The Law of Sines



Area of the triangle

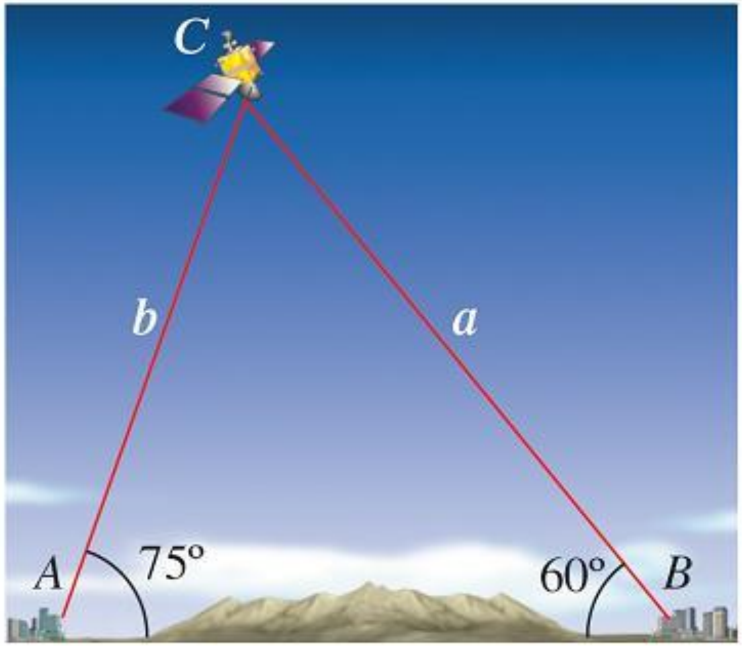
$$= \frac{1}{2} bc \sin A = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B$$

Multiplying the equation by $\frac{2}{abc}$, We have the Law of Sines.

Example 1. A satellite orbiting the earth passes directly overhead at observation stations in Phoenix and Los Angeles, 340 mi apart.
At an instant when the satellite is between these two stations, its angle of elevation is simultaneously observed to be 60° at Phoenix and 75° at Los Angeles.

How far is the satellite from Los Angeles?

Solution



Want to find "b"

We can find the angle

$\angle C = 180^o - (60^o + 75^o) = 45^o$

By using the Law of Sines,

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\rightarrow \frac{\sin 60^o}{b} = \frac{\sin 45^o}{340}$$

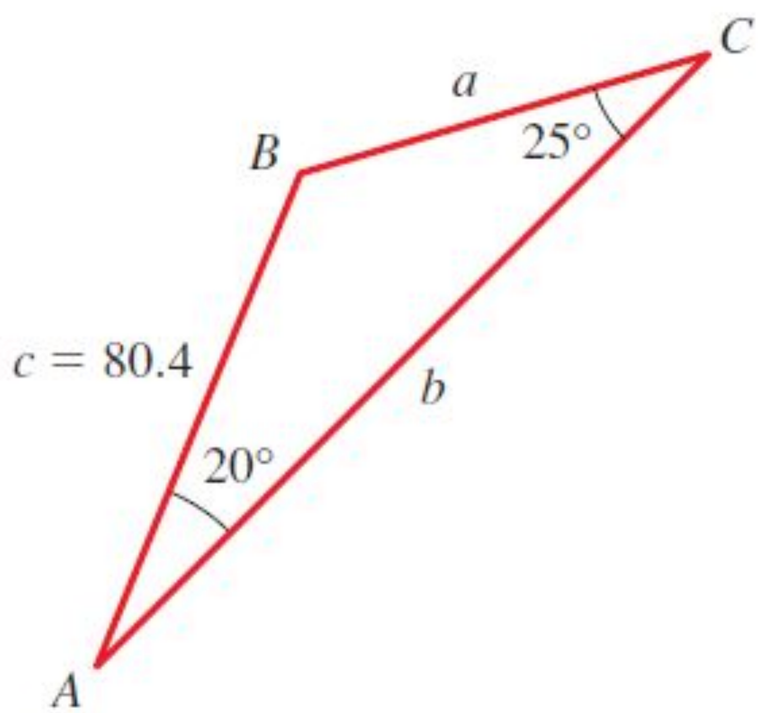
$$b = \frac{340 \sin 60^o}{\sin 45^o} = \frac{340 \times \frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}} = 170\sqrt{6} \approx 416$$

The distance of the satellite from L.A. is about 416 miles.

Your turn!

Solve the triangle in the following figure.

Here, "solve" means find all unknown angles and lengths of sides.



Solution

$$\angle B = 180^\circ - (20^\circ + 25^\circ) = 135^\circ$$

$$\frac{\sin 20^\circ}{a} = \frac{\sin 135^\circ}{b} = \frac{\sin 25^\circ}{80.4}$$

$$b = \frac{80.4 \sin 135^\circ}{\sin 25^\circ} \approx 134.5$$

$$a = \frac{80.4 \sin 135^\circ}{\sin 20^\circ} \approx 65.1$$

Example 2

- Solve the triangle if $a = 50, b = 100$, and $\angle A = 50^{o}$.

Solution

Using the law of sines, we can find the angle B first.

$$\frac{\sin 50^\circ}{50} = \frac{\sin B}{100} \rightarrow \sin B = \frac{100 \sin 50^\circ}{50} \approx 1.53$$

There is no such triangle.

The Law of Cosines

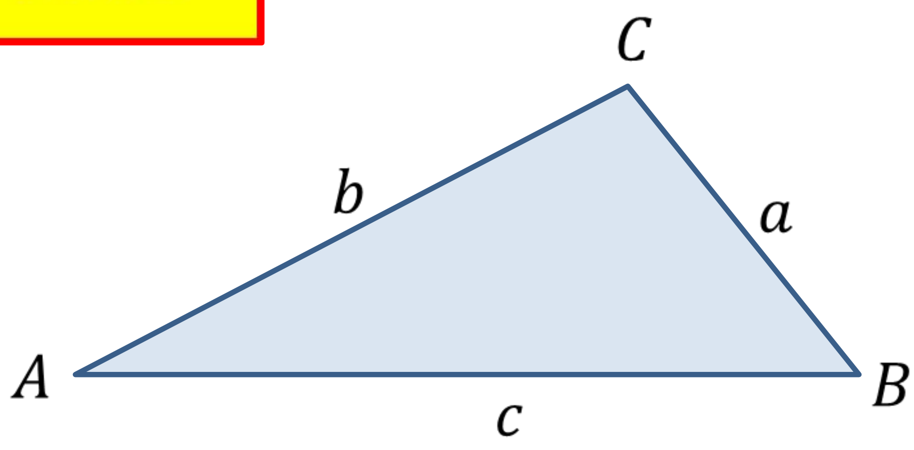
- In any triangle ABC, we have

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

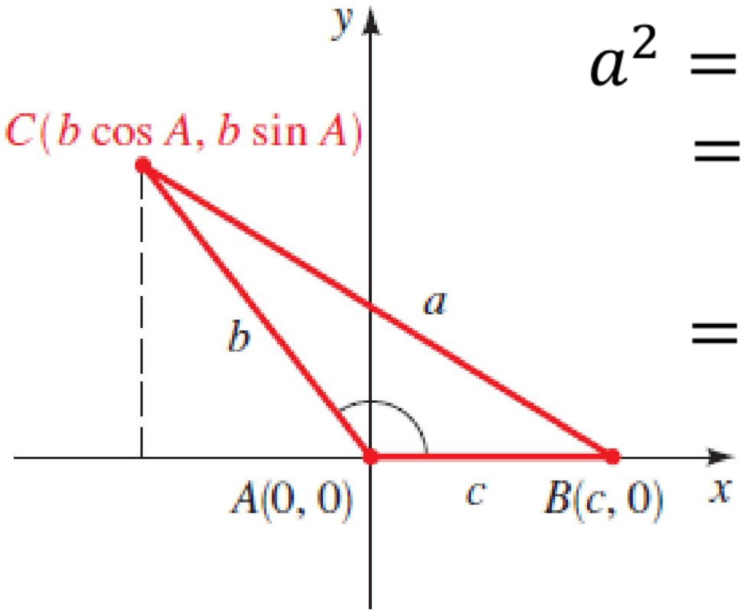
$$c^2 = a^2 + b^2 - 2ab\cos C$$

What if $A = 90°$?



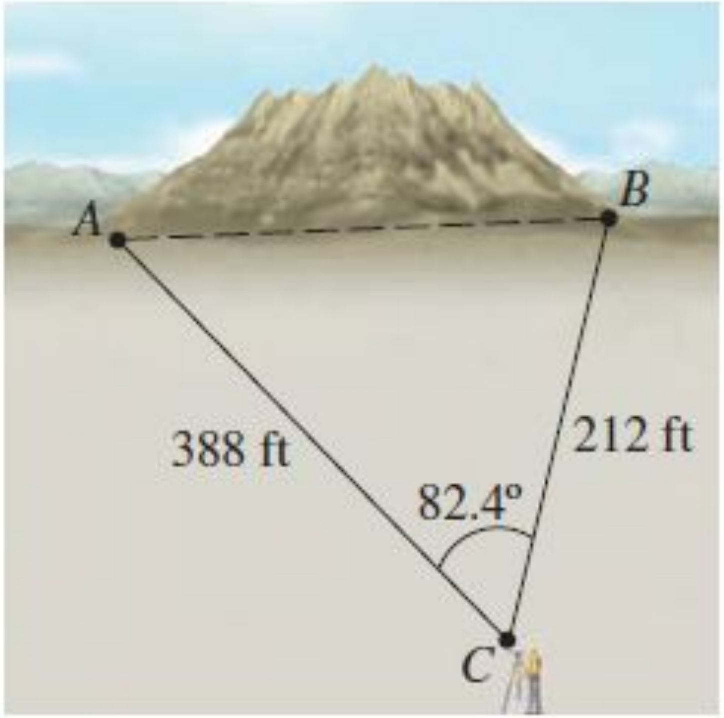
Proof: The Law of Cosines

Place the triangle so that $\angle A$ is at the origin as shown in the figure. Then, the coordinates of B and C are $(c, 0)$ and $(b\cos A\,, b\sin A)$, respectively. (What if $\angle A$ is an acute angle?) Using the distance formula,



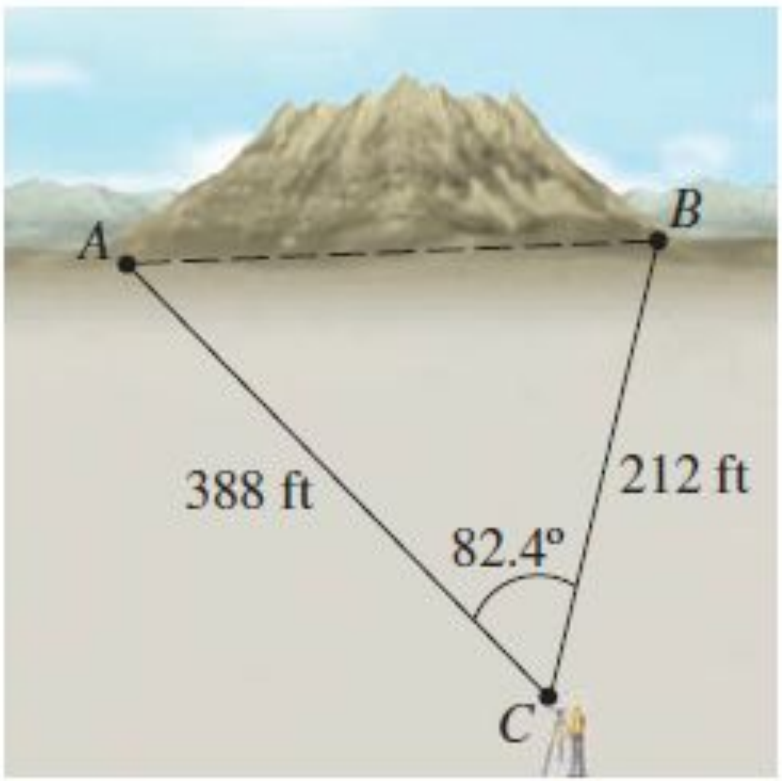
$$\begin{aligned} a^2 &= (b\cos A - c)^2 + (b\sin A - 0)^2 \\ &= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A \\ &= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A \\ &= b^2 + c^2 - 2bc\cos A \end{aligned}$$

Example 3. A tunnel is to be built through a mountain. To estimate the length of the tunnel, a surveyor makes the measurements shown in the figure below. Use the surveyor's data to approximate the length of the tunnel.



Solution

$$c^2 = 212^2 + 388^2 - 2(388)(212)\cos 82.4^o$$
$$\approx 173730.2367$$

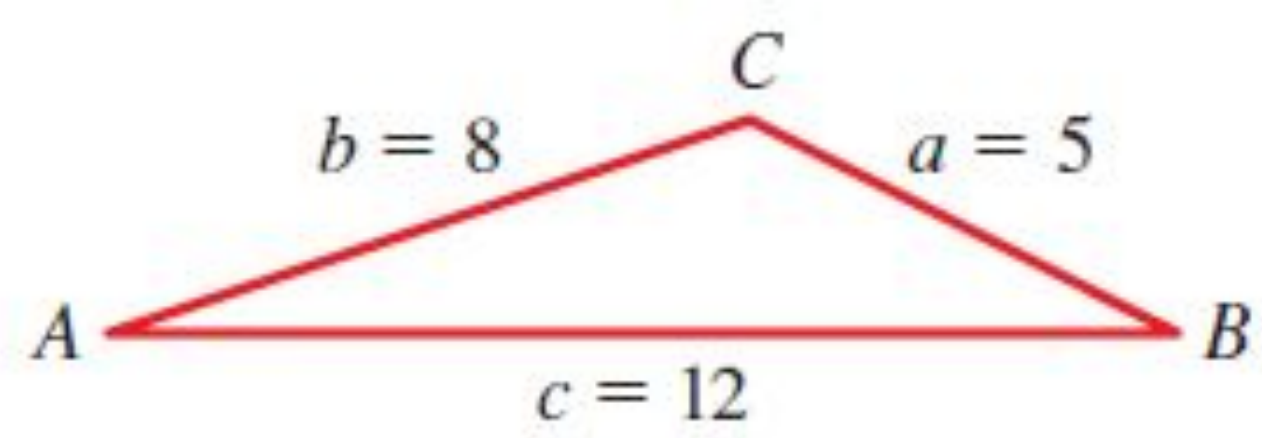


$$c \approx \sqrt{173730.2367}$$
$$\approx 416.81$$

The tunnel will be approximately 417 ft long.

Your turn!

- The sides of a triangle are $a = 5, b = 8$, and $c = 12$. Find the angles of the triangle.



Solution

- From $a^2 = b^2 + c^2 - 2bc\cos A$,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{61}{64}$$

$$\rightarrow \angle A = \cos^{-1}\left(\frac{61}{64}\right) \approx 17.61^{o}$$

- Similarly,

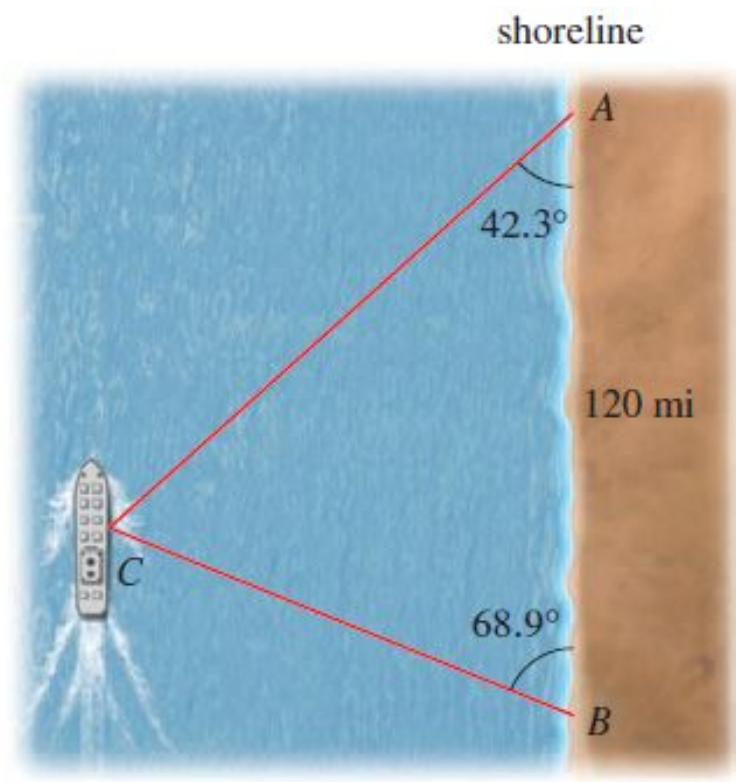
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{7}{8}$$

$$\rightarrow \angle B = \cos^{-1}\left(\frac{7}{8}\right) \approx 28.96^{o}$$

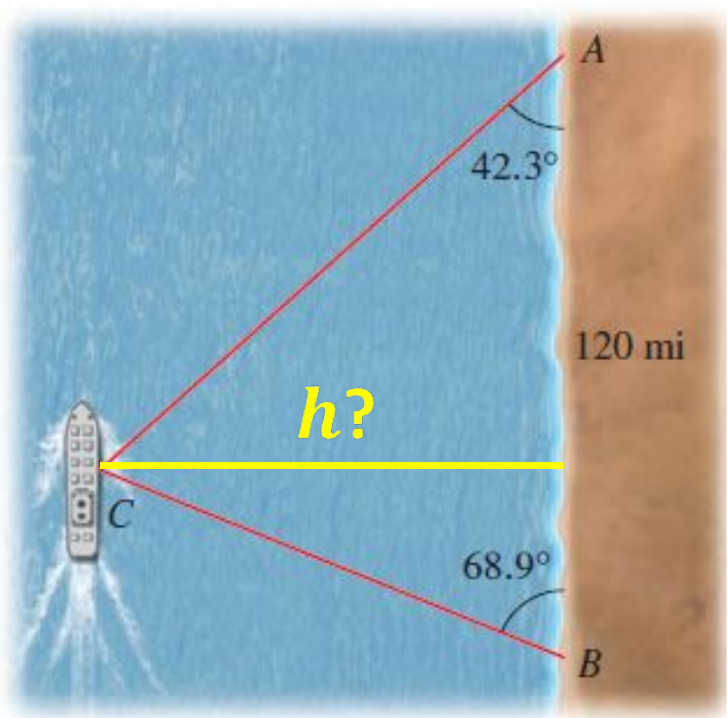
- $\angle C = 180^{o} - (17.61^{o} + 28.96^{o}) \approx 133.43^{o}$

You need to consider given information and decide whether to use the law of sines or the law of cosines.

Example 4. A boat is cruising the ocean off a straight shoreline. Points A and B are 120 mi apart on the shore. It is found that $\angle A = 42.3^{\circ}$and $\angle B = 68.9^{\circ}$. Find the shortest distance from the boat to the shore.



Solution



- We can find $\angle C = 68.8^{o}$
- Using the law of sines,

$$\frac{120}{\sin 68.8^{o}} = \frac{a}{\sin 42.3^{o}}$$

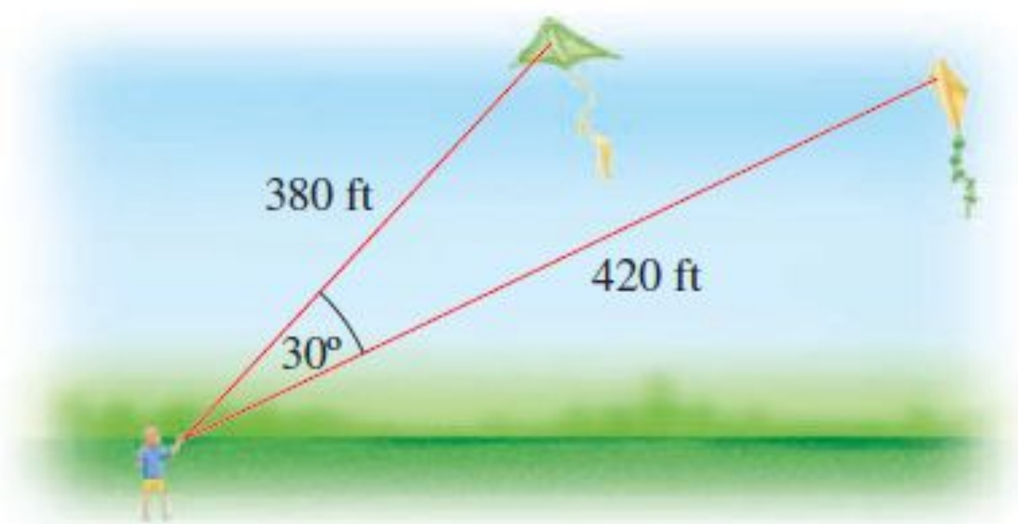
- $h = a \sin 68.9^{o}$

$$= \frac{120 \sin 42.3^{o} \sin 68.9^{o}}{\sin 68.8^{o}}$$

$\approx$ 80.8 mi

Your turn!

A boy is flying two kites at the same time. He has 380 ft of line out to one kite and 420 ft to the other. He estimates the angle between the two lines to be 30^{o}. Approximate the distance between the kites.

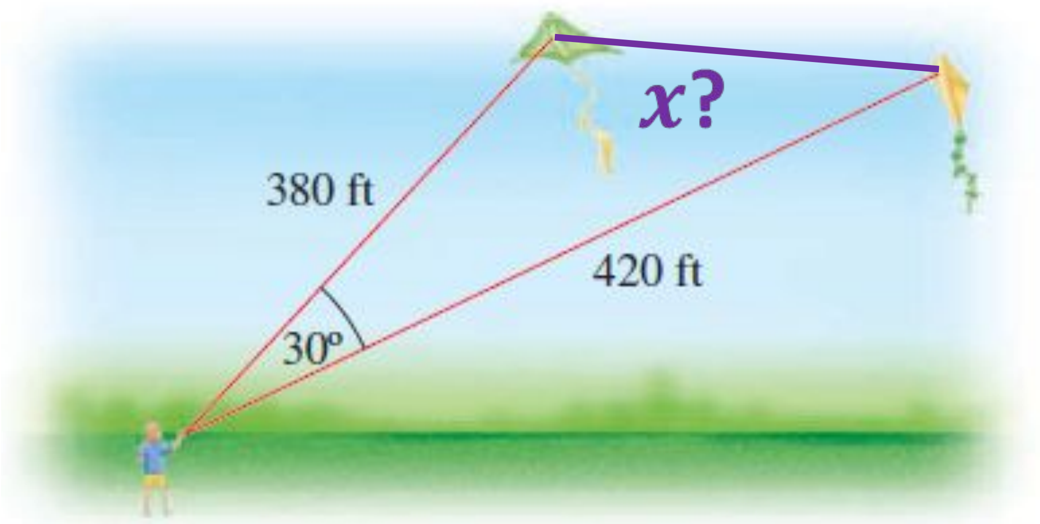


Solution

- Using the law of cosines,

$$x^2 = 380^2 + 420^2 - 2(380)(420)\cos 30^o$$

$$x \approx \sqrt{44364.69111} \approx 210.63 \text{ ft}$$



Harmonic motion

Periodic behavior – behavior that repeats over and over again – is common in nature.

Some examples are: daily variation of tide levels, changes in certain animal population, sound waves.

The trigonometric functions are ideally suited for modeling periodic behavior.

Simple harmonic motion

If the equation describing the displacement y of an object at time t is

$$y = a\sin\omega t \quad \text{or} \quad y = a\cos\omega t$$

then the object is in **simple harmonic motion**.

("simple" because the amplitude is preserved)

- Amplitude= $|a|$ Maximum displacement of the object
- Period $= \frac{2\pi}{\omega}$ Time period to complete one cycle
- Frequency= $\frac{\omega}{2\pi}$ Number of cycles per unit of time

Frequency is the reciprocal of the period.

$\omega = 2\pi\ \times$frequency is called the "angular frequency"

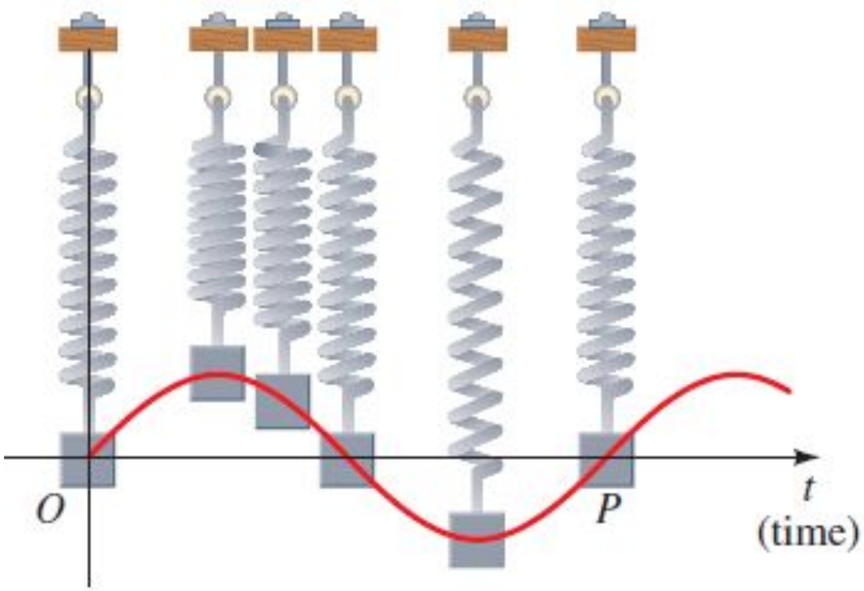
Example 5. The displacement of a mass suspended by a spring is modeled by the function

$$y = 10 \sin 4\pi t$$

where y is measured in inches and t in seconds.

(a) Find the amplitude, period, and frequency of the motion of the mass.

(b) Sketch a graph of the displacement of the mass.



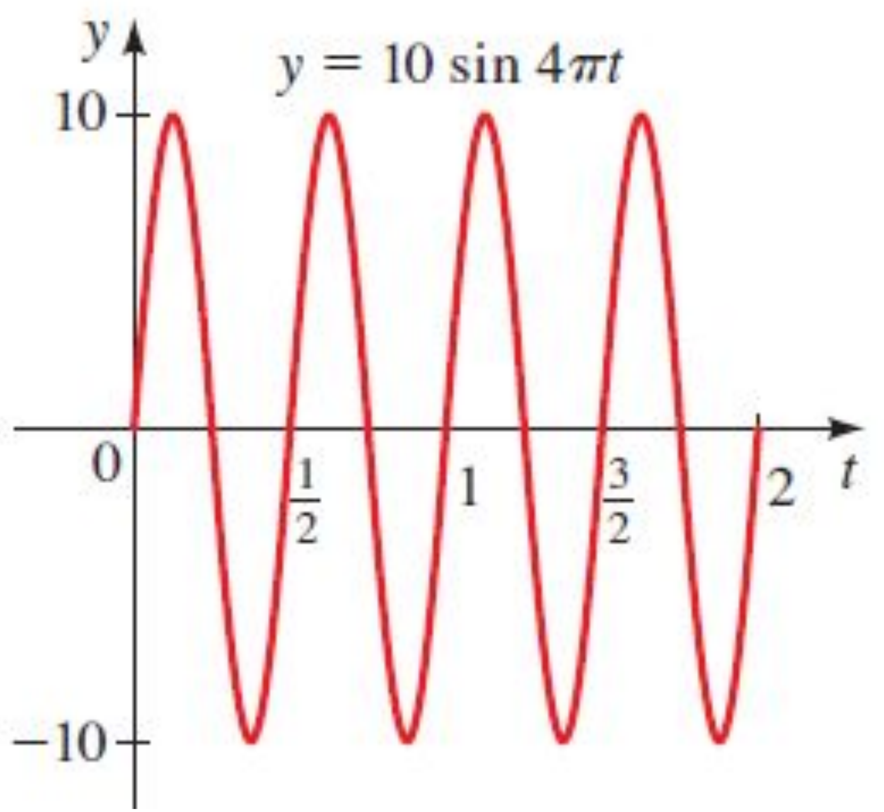
Solution

(a) amplitude = 10 in

period $= \frac{2\pi}{4\pi} = \frac{1}{2}$ s

frequency
= 2 cycles per second(Hz)

(b)



Your turn!

A tuba player plays the note E and sustains the sound for some time. For a pure E the variation in pressure from normal air pressure is given by

$$V(t) = 0.2 \sin 80\pi t$$

Where V is measured in pounds per square inch and t is measured in seconds.

(a) Find the amplitude, period, and frequency of V.

(b) Sketch a graph of V.

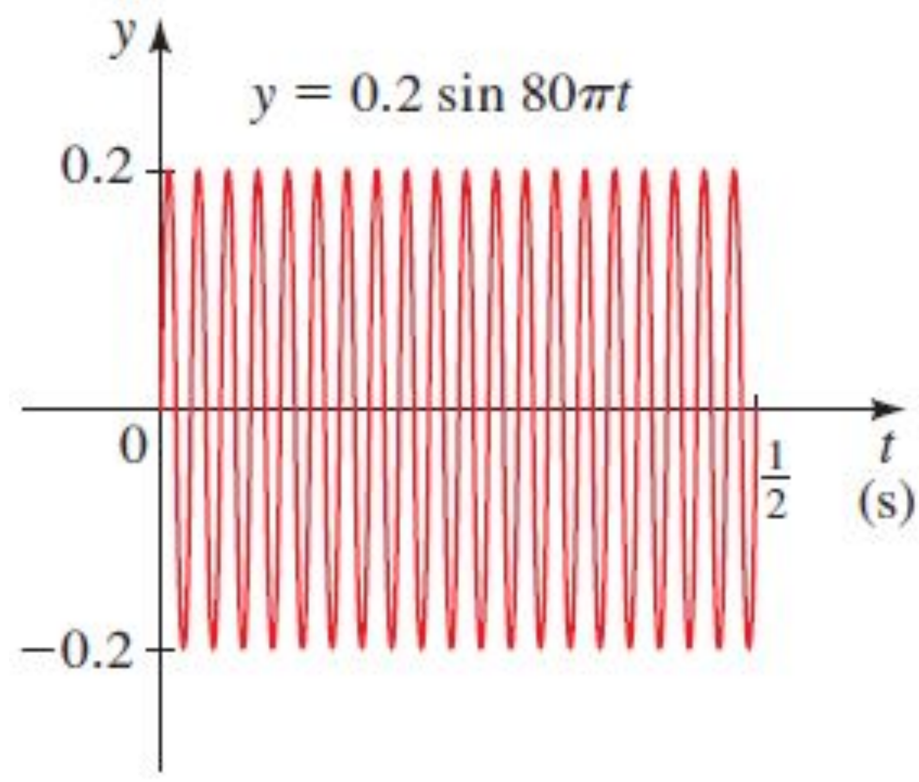
Solution

(a) amplitude $= 0.2$ lb/in^2

period $= \dfrac{2\pi}{80\pi} = \dfrac{1}{40}$ s

frequency $= 40$ Hz

(b)

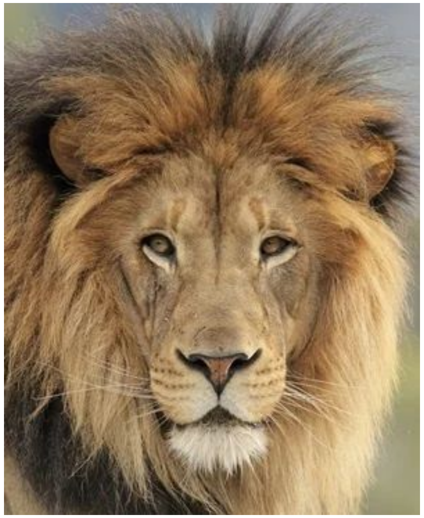


The tone of the sound depends on the frequency, and the loudness depends on the amplitude.

Going back to the preview activity, how did the sound and wave change when the frequency gets higher?

Which animal produces sound with the higher frequency?

vs.



The higher the frequency, the higher the pitch.

In the previous example (Your turn!),

(c) If the player is playing the note incorrectly and it is a little flat, how does the equation $V(t) = 0.2\sin 80\pi t$ change?

Solution

If the player is playing the note incorrectly and it is a little flat, the frequency is lower so the coefficient of t is less than 80π.

(d) If the tuba player increases the loudness of the note, how does the equation $V(t) = 0.2 \sin 80\pi t$ change?

Solution

The amplitude will increase, so the number 0.2 is replaced by a larger number.

- In general, the sine or cosine functions representing harmonic motion may be shifted horizontally or vertically.

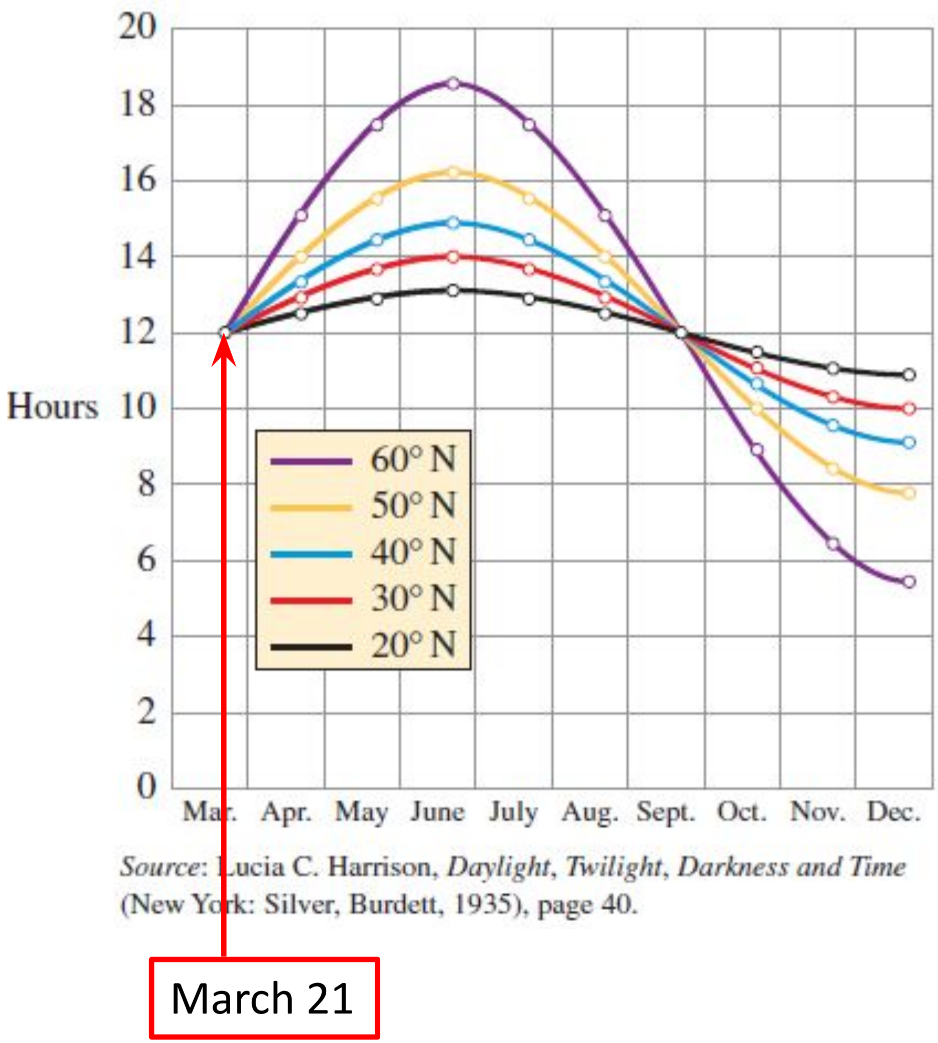
In this case, the equations take the form

$$y = a\sin\big(\omega(t-c)\big) + b$$

or

$$y = a\cos\big(\omega(t-c)\big) + b$$

Example 6



Source: Lucia C. Harrison, *Daylight, Twilight, Darkness and Time* (New York: Silver, Burdett, 1935), page 40.

The number of hours of daylight varies throughout the course of a year.

In the Northern Hemisphere, the longest day is June 21, and the shortest is December21.

The average length of daylight is 12 h, and the variation from this average depends on the latitude.

- In Philadelphia (40^{o}N latitude) the longest day of the year has 14 h 50 min of daylight, and the shortest day has 9 h 10 min of daylight.

(a) Find a function L that models the length of daylight as a function of t, the number of days from January 1 ($t = 1$ means Jan 1).

(b) An astronomer needs at least 11 hours of darkness for a long exposure astronomical photograph. On what days of the year are such long exposures possible?

Solution

(a) We need to find the coefficients a, b, c, ω

$$y = a\sin\big(\omega(t-c)\big) + b$$

- $a =$ half of the difference between the longest and the shortest

$$a = \frac{1}{2}(5\text{ h }40\text{ min}) = \frac{1}{2} \times 5.6666\ldots \approx 2.83\ldots.$$

- period= $\frac{2\pi}{\omega}$ =365 days $\rightarrow \omega \approx 0.0172$
- $b = 12$, the average length of daylight
- $c = 80$ because there are 80 days from Jan 1 to Mar 20.

Solution(continued)

(b) At least 11 hours of darkness means at most 13 hours of daylight.

$$2.83\sin\left(0.0172(t-80)\right)+12\le 13$$

$$t_1=\frac{\sin^{-1}\left(\frac{1}{2.83}\right)}{0.0172}+80\approx 100.9975$$

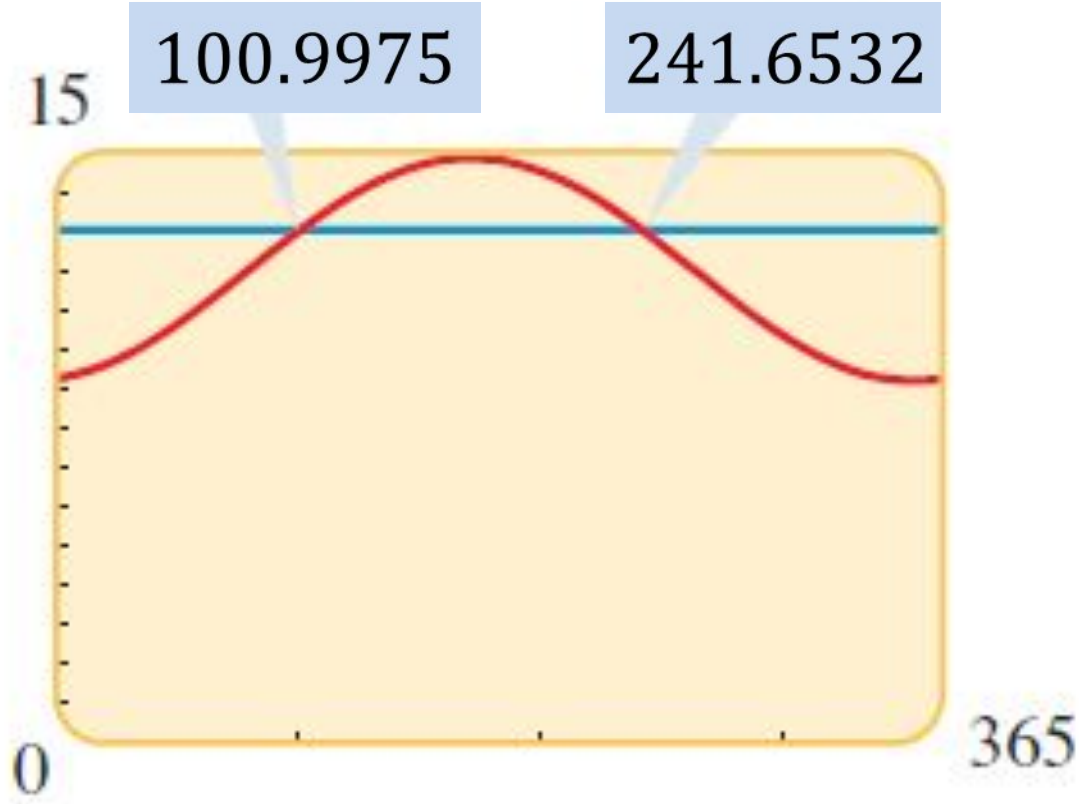
$$t_2=\frac{\pi-\sin^{-1}\left(\frac{1}{2.83}\right)}{0.0172}+80\approx 241.6532$$

Tip! Your calculator should be in radian mode.

Solution(continued)

Remember we want $y \leq 13$.
From Day 1 to Day 100 and from Day 242 to Day 365 satisfy the condition.

That is,
from Jan 1 to Apr 10
and
from Aug 30 to Dec 31,
there are fewer than
13 h of daylight.



Your turn!

A variable star is one whose brightness alternately increases and decreases. For the variable star Delta Cephei, the time between periods of maximum brightness is 5.4 days The average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude.

(a) Find a function $y = a\cos\big(\omega(t-c)\big) + b$ that models the brightness of Delta Cephei as a function of time.

(b) Sketch a graph of the brightness of Delta Cephei as a function of time.

Solution

- $$y = a\cos\big(\omega(t-c)\big) + b$$
- period$= \frac{2\pi}{\omega} = 5.4$ days, $\omega = \frac{5.4}{2\pi} \approx 1.164$
- Average brightness $b = 4$
- Amplitude $a = 0.35$
- Since there is no further information, we can take $c = 0$. Then, t is the number of days from a time when the star is at maximum brightness.

Damped harmonic motion

The amplitude of a spring in a frictionless environment will not change. The spring is in simple harmonic motion.

However, in the presence of friction, the motion of the spring eventually dies down, that is, the amplitude of the motion decreases with time.

Motion of this type is called **damped harmonic motion**.

- If the equation describing the displacement of y of an object at time t is

$$y = ke^{-ct}\sin\omega t$$

or

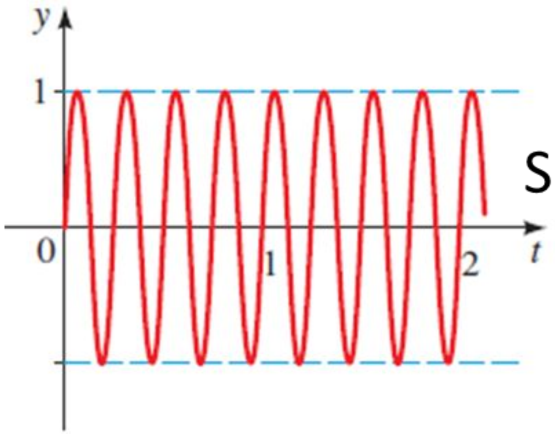
$$y = ke^{-ct}\cos\omega t$$

where $c > 0$, then the object is in damped harmonic motion.

The constant c is the damping constant, k is the initial amplitude, and $2\pi/\omega$ is the period.

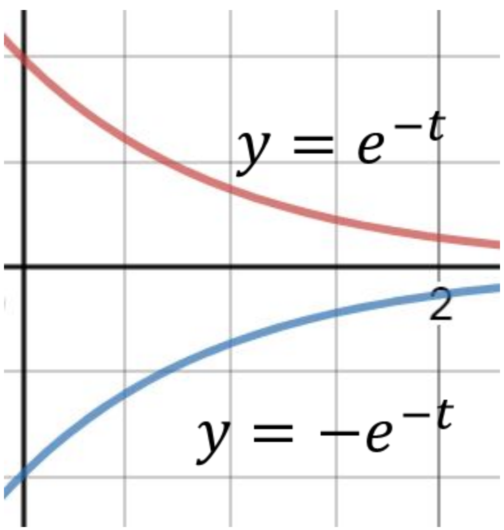
The amplitude is governed by the function $y = ke^{-ct}$.

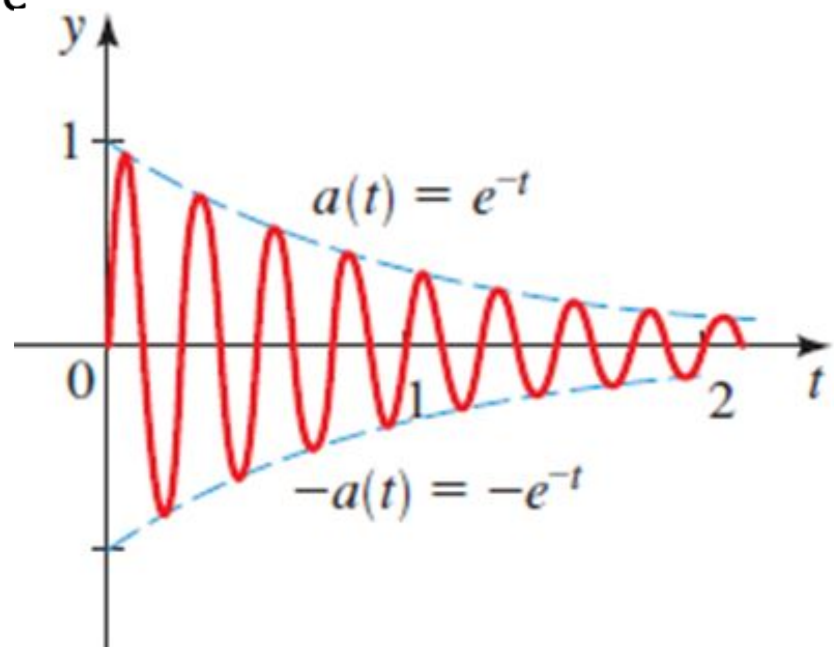
Simple harmonic vs. damped harmonic motion



$y = \sin 8\pi t$

Simple harmonic





$y = e^{-t} \sin 8\pi t$

Damped harmonic

Example 7

- Two mass-spring systems are experiencing damped harmonic motion, both at 0.5 cycles per second and both with an initial maximum displacement of 10 cm.

The first has a damping constant of 0.5 and the second has a damping constant of 0.1.

(a) Find functions of the form $g(t) = ke^{-ct}\cos\omega t$ to model the motion in each case.

(b) Graph the two functions you found in part (a). How do they differ?

Solution

- (a)For both systems,

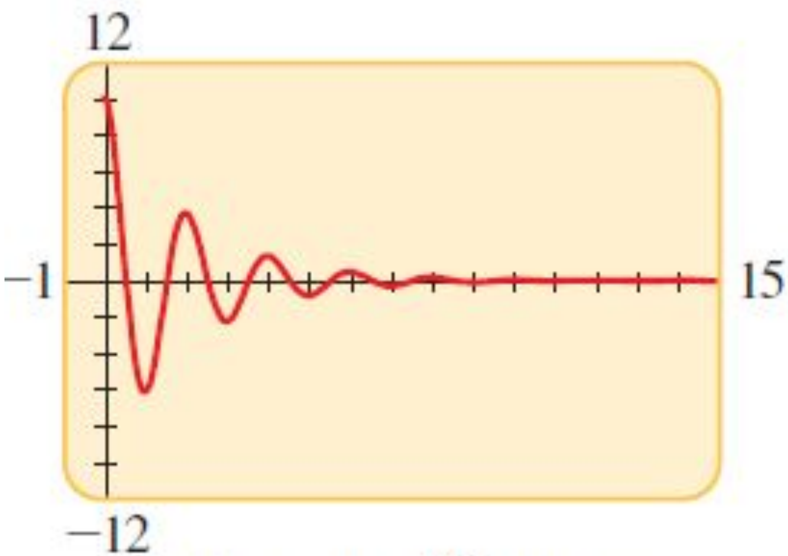
frequency$=\frac{\omega}{2\pi}=0.5 \quad \rightarrow \quad \omega=\pi$

The initial maximum displacement is $k=10$.

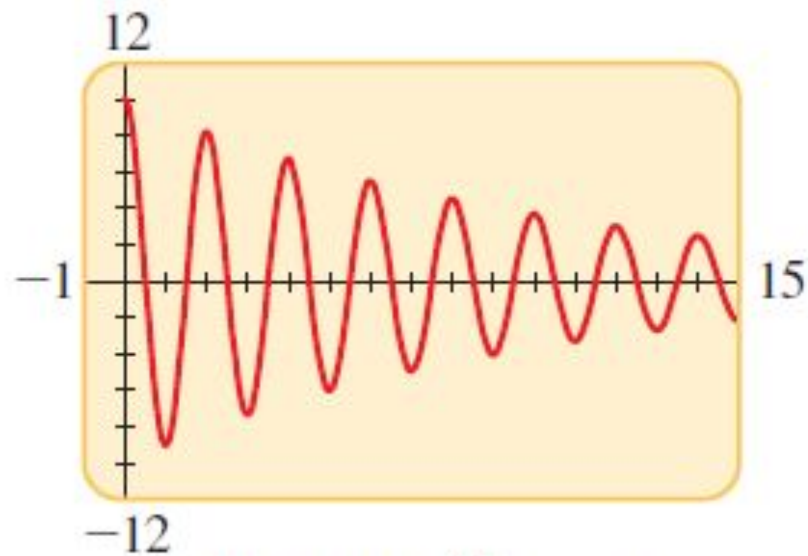
For the first system, $g_1(t)=10e^{-0.5t}\cos\pi t$

For the second system, $g_2(t)=10e^{-0.1t}\cos\pi t$

Solution(continued)



$g_1(t) = 10\,e^{-0.5t}\cos \pi t$



$g_2(t) = 10\,e^{-0.1t}\cos \pi t$

The larger the damping constant *c*, the quicker the oscillation dies down.

Example 8

A stone is dropped in a calm lake, causing waves to form. The up-and-down motion of a point on the surface of the water is modeled by damped harmonic motion. At some time the amplitude of the wave is measured, and 20 s later, it is found that the amplitude has dropped to 1/10 of this value.

Find the damping constant c.

Solution

- The amplitude is governed by the coefficient ke^{-ct} in the equations for damped harmonic motion.

$$ke^{-c(t+20)} = \frac{1}{10} ke^{-ct}$$

$$\rightarrow \quad e^{-20c} = \frac{1}{10} \quad \rightarrow \quad -20c = -\ln 10$$

$$\rightarrow \quad c = \frac{\ln 10}{20} \approx 0.12$$

Learning outcomes

3.5.1. Solve triangles by using the law of sines and the law of cosines

3.5.2. Solve problems involving simple and damped harmonic motion

Formulae

- The law of sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

- The law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$