



# Geodetic Control Network

## Lecture 4.

Computations on the ellipsoid

## Computations on the ellipsoid

- Ellipsoidal curves (normal sections, curve of alignment, the geodesic);
- The computation of ellipsoidal triangles;

# The ellipsoidal curves

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**The normal section and the reverse normal section**

**The curve of alignment**

**The geodesic**



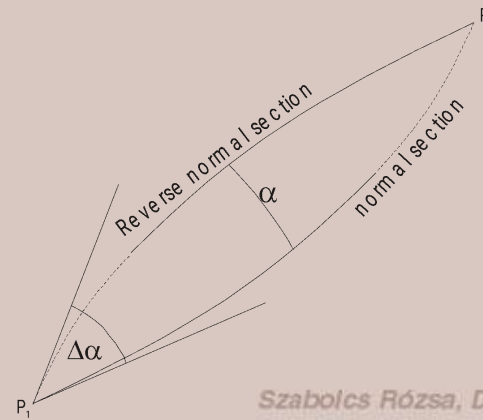
# The normal section

## The normal section and the reverse normal section

At each ellipsoidal point an ellipsoidal normal is defined (which is orthogonal to the surface of the ellipsoid). The intersection of those planes, which contain the ellipsoidal normal, with the ellipsoidal surface is called a normal section. At each point an infinite number of normal sections exist.

Normal sections are usually ellipses. When the point is located on the Equator, than a circular normal section can also be formed.

When a certain normal section is defined between two points on the ellipsoid ( $P_1P_2$ ), than it must be noted that it differs from the normal section between  $P_2P_1$ , since the ellipsoidal normals have a skewness. The latter section is called the **reverse normal section**.



# The normal section

The normal section and the reverse normal section has an angle  $\Delta\alpha$ :

$$\Delta\alpha = \frac{1}{4} \rho'' \frac{s^2}{a^2} e^2 \cos^2 \varphi_1 \sin 2\alpha$$

The maximal distance between the normal section and the reverse normal section is:

$$d = \frac{1}{16} \frac{s^3}{N_1^2} e^2 \cos^2 \varphi_1 \sin 2\alpha$$

Where  $N_1$  is the radius of the curvature in the prime vertical.

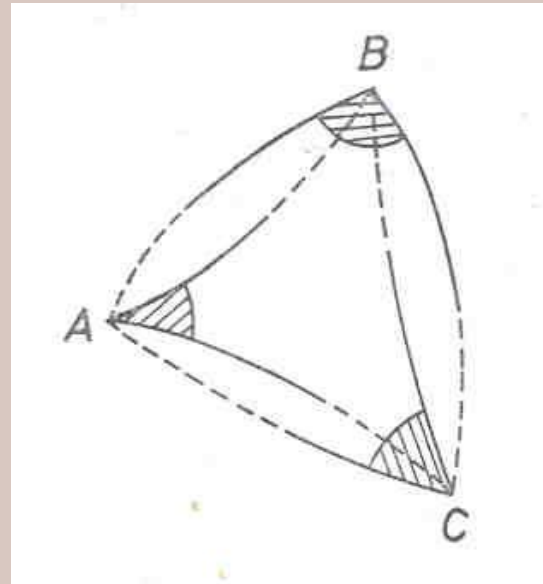
# The normal section

## Notes on the normal section:

- when both of the points are located on the same meridian, then the normal section and the reverse normal section coincide.
- when both of the points are located on the Equator, then both the normal and the reverse normal section coincides with the Equator.
- when both of the points are located on the same parallel curve (same latitude), then the normal section lies not on the parallel curve, but on the opposite side of the equator.

# Disadvantage of the normal section

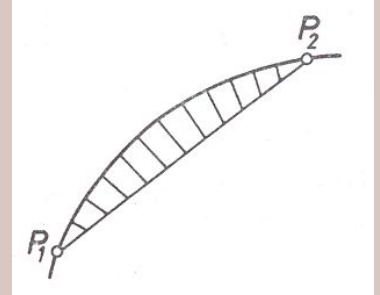
- When we want measure the angles of a triangle and connect the nodes of the triangle with normal sections, then the observed angles are not consequent. -> a different ellipsoidal curve should be used for the representation



# The curve of alignment

**It is usually used in the Anglo-Saxon region.**

Let's suppose that  $P_1$  and  $P_2$  are two ellipsoidal points. Let's connect the  $P_1$  and  $P_2$  and form a chord inside the ellipsoid. By drawing the ellipsoidal normal from each point of the  $P_1P_2$  chord, the intersections of these normals form the curve of alignment.



When in any point of the CoA an ellipsoidal normal is drawn, and a vertical plane is created, which contains  $P_1$ , then  $P_2$  lies in this vertical plane as well, since the plane contains two points of the chord, thus it must contain all the points on the chord.

**Thus the CoA can be defined as the sum of those points, in which the normal sections pointing to  $P_1$  and  $P_2$  has the azimuth difference of  $180^\circ$ .**

The CoA connects to the normal sections of  $P_1$  and  $P_2$  tangential -> this is the main advantage, since the angular observations are equal to the angles between the curves of alignment.



# The curve of alignment

## Notes on the CoA:

- When  $P_1$  and  $P_2$  are on the same meridian, then the CoA is the meridian itself;
- When  $P_1$  and  $P_2$  are on the Equator, then the CoA is the Equator.
- When  $P_1$  and  $P_2$  are on the same parallel curve, then the CoA is located from both the parallel curve and the normal section in the opposite side of the Equator.

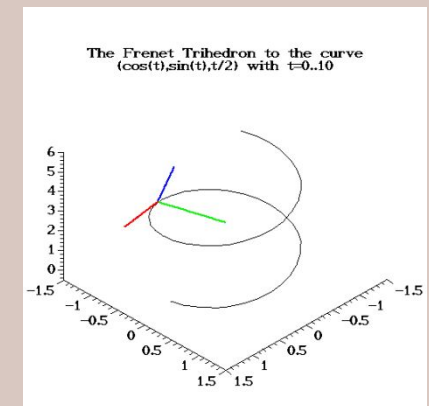
# The geodesic

The general solution to define the sides of the ellipsoidal triangles is application of the geodesic.

To define the geodesic, first we need to brush up our knowledge on the Frenet trihedron.

For any points on any curve in the 3D space three mutually orthogonal vectors can be created (*the Frenet trihedron*), which are:

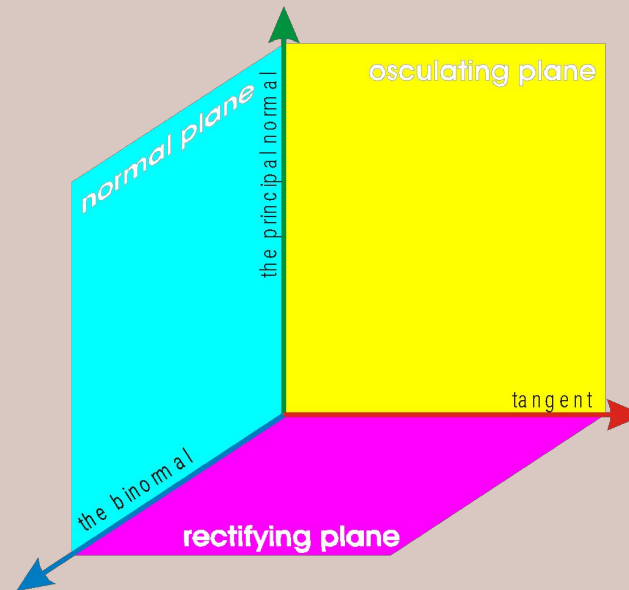
- the **tangent**;
- the **principal normal** (perpendicular to the tangent, and is aligned with the radius of curvature of the curve);
- the **binormal** (perpendicular to both the tangent and the principal normal);



# The geodesic

The Frenet-frame contains **three different planes** (formed by a pair of the three vectors):

- normal plane (the plane of the principal normal and the binormal);
- osculating plane (the plane of the principal normal and the tangent);
- rectifying plane (the plane of the tangent and the binormal).



# The geodesic

**The geodesic:** is an ellipsoidal curve, where at each point of the curve the principal normal of the curve coincides with the normal of the ellipsoid.

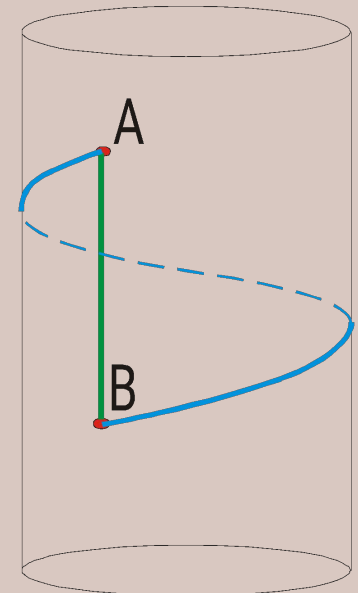
Or: In each point the osculating plane of the curve is a normal plane of the ellipsoidal surface.

Or: The rectifying plane of the curve coincides the tangential plane of the ellipsoidal surface.

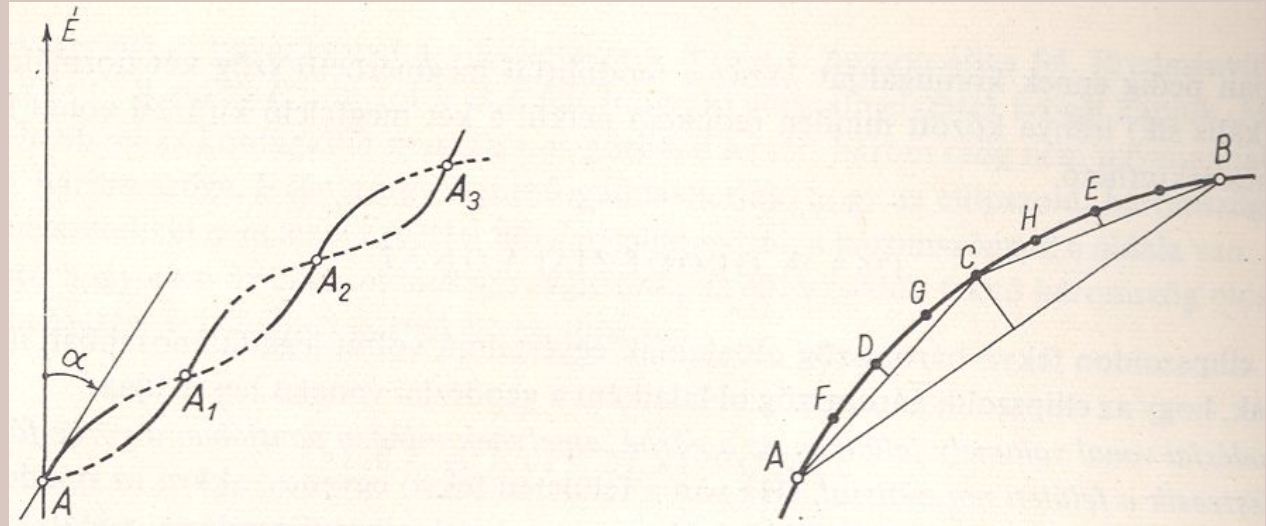
Or: The geodesic is a specific curve among the curves on the surface, that has the shortest path between the two points. This is a sufficient criteria, but NOT a required one (helix against the straight line between two points on the same element of a cylinder).

Example:

- straight lines on the surfaces are geodesics (e.g. cylinder)



# Graphical derivation of geodesic on the ell.



# Properties of the geodesic



# The geodesic on surfaces of rotation

On surfaces of rotation the following equation can be derived for the geodesic (Clairaut-equation):

$$r \sin \alpha = C$$

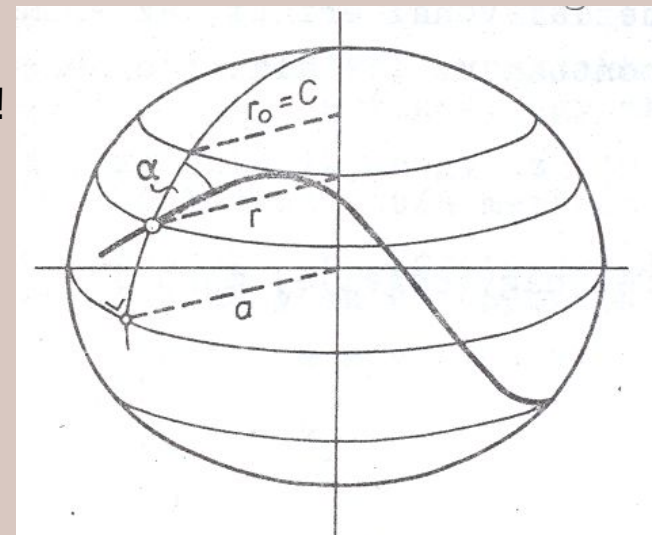
## Where

$r$  is the radius of the parallel circle;

$\alpha$  is the azimuth of the curve at the point;

$C$  is constant.

A required but not sufficient criteria!



# The solution of ellipsoidal triangles

**Solution: the computation of the length of the sides of the triangle from angular observations and one distance observation!**

The ellipsoid is usually approximated by a sphere.

When the study area is relatively small, then the Gauss-theorem is used, which states that an infinitesimal part of the ellipsoid can be approximated by an infinitesimal part of a sphere. The radius of such a sphere is the mean-radius of the ellipsoid in the centroid of the infinitesimal part:

$$R = \sqrt{MN}$$

where

$M$  is the curvature in the meridian direction

$N$  is the curvature of the prime vertical



# The excess angle of the ellipsoidal triangle

Ellipsoidal triangles are approximated by spherical triangles.

The spherical excess angle:

$$\varepsilon'' = \rho'' \frac{F}{R^2} \approx 0,005F$$

When the triangles are small (planar approximation):

$$F = \frac{1}{2}bc \sin \alpha$$

Where  $b, c$  are the sides of the triangle and  $\alpha$  is the angle between them.

Since (sine-theorem)

$$b = \frac{c \sin \beta}{\sin \gamma}$$

# The excess angle of the ellipsoidal triangle

The excess angle:

$$\varepsilon'' = \rho'' \frac{F}{R^2} = \rho'' \frac{bc \sin \alpha}{2R^2} = \rho'' \frac{c^2 \sin \alpha \sin \beta}{2R^2 \sin \gamma}$$

In case of large triangles (bigger than 5000 km<sup>2</sup>) the ellipsoidal excess angle should be computed:

$$\varepsilon'' = \rho'' \frac{F}{R^2} \left( 1 + \frac{a^2 + b^2 + c^2}{24R^2} \right)$$

# The Legendre method

The ellipsoidal triangles are approximated by spherical triangles (Gauss-theorem) that has the same angles as the ellipsoidal ones. Legendre proved that unless the triangle is big, the triangles can be solved with the planar approximation, when the spherical angles are decreased by one third of the excess angle.

$$\alpha' = \alpha - \frac{\varepsilon}{3}$$

$$\beta' = \beta - \frac{\varepsilon}{3}$$

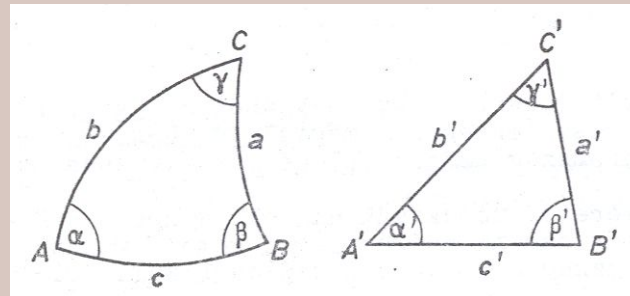
$$\gamma' = \gamma - \frac{\varepsilon}{3}$$



$$a' = a$$

$$b' = b$$

$$c' = c$$



# The Legendre method

When the spherical (ellipsoidal) triangle is not small enough, then

$$\alpha - \alpha' = \frac{\varepsilon}{3} \left( 1 + \frac{m^2 - a^2}{20R^2} \right)$$

$$\beta - \beta' = \frac{\varepsilon}{3} \left( 1 + \frac{m^2 - b^2}{20R^2} \right)$$

$$\gamma - \gamma' = \frac{\varepsilon}{3} \left( 1 + \frac{m^2 - c^2}{20R^2} \right)$$

where

$$m^2 = \frac{a^2 + b^2 + c^2}{3}$$

Note: the difference reaches the level of 0.001" in the distance of 200km only!

# The Soldner method

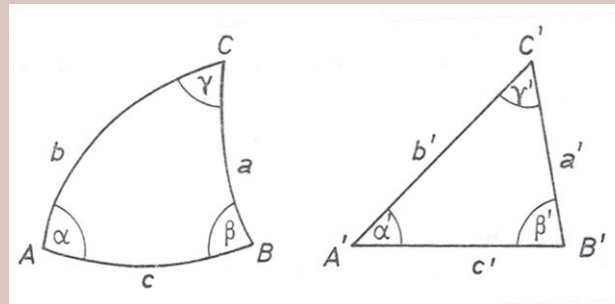
$$a' = a - \frac{a^3}{6R^2}$$
$$b' = b - \frac{b^3}{6R^2}$$
$$c' = c - \frac{c^3}{6R^2}$$

**Additive constants**

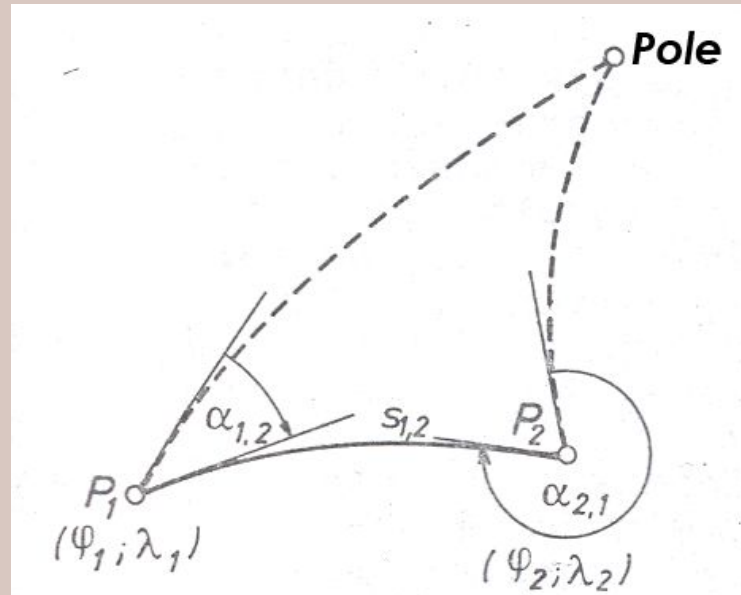
$$\alpha' = \alpha$$

$$\beta' = \beta$$

$$\gamma' = \gamma$$



# The computation of coordinates on the ell.



## 1st and 2nd fundamental task

- solving polar ellipsoidal triangles
- $P_1P_2$  curve is usually a geodesic, in English literature mostly the normal section is used and the results are corrected for the geodesic

# The computation of coordinates on the ell.

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## Various solution depending on the distance:

- up to 200 km (polynomial solutions)
- up to 1000 km (usually based on normal sections) used by long baseline distance observations
- up to 20000 km (air navigation) based on the Clairaut equation

**We'll focus on the first one!**

# 1st fundamental task

## Legendre's polynomial method

Known  $P_1 (\phi_1, \lambda_1)$  and  $\alpha_{1,2}$ , then the  $\phi_2, \lambda_2$  coordinates and the  $\alpha_{2,1}$  azimuth depends on the ellipsoidal distance only.

$$\varphi_2 = f_1(s)$$

$$\lambda_2 = f_2(s)$$

$$\alpha_{2,1} = f_3(s)$$

These function can be written in MacLaurin series:

$$\varphi_2 = f_1(s)_0 + \left( \frac{\partial f_1}{\partial s} \right)_0 s + \left( \frac{\partial^2 f_1}{\partial s^2} \right)_0 \frac{s^2}{2} + \left( \frac{\partial^3 f_1}{\partial s^3} \right)_0 \frac{s^3}{6} + \dots$$

$$\lambda_2 = f_2(s)_0 + \left( \frac{\partial f_2}{\partial s} \right)_0 s + \left( \frac{\partial^2 f_2}{\partial s^2} \right)_0 \frac{s^2}{2} + \left( \frac{\partial^3 f_2}{\partial s^3} \right)_0 \frac{s^3}{6} + \dots$$

$$\alpha_{2,1} = 180^\circ + f_3(s)_0 + \left( \frac{\partial f_3}{\partial s} \right)_0 s + \left( \frac{\partial^2 f_3}{\partial s^2} \right)_0 \frac{s^2}{2} + \left( \frac{\partial^3 f_3}{\partial s^3} \right)_0 \frac{s^3}{6} + \dots$$



# Legendre's polynomial method

$$\varphi_2 = f_1(s)_0 + \left(\frac{\partial f_1}{\partial s}\right)_0 s + \left(\frac{\partial^2 f_1}{\partial s^2}\right)_0 \frac{s^2}{2} + \left(\frac{\partial^3 f_1}{\partial s^3}\right)_0 \frac{s^3}{6} + \dots$$

$$\lambda_2 = f_2(s)_0 + \left(\frac{\partial f_2}{\partial s}\right)_0 s + \left(\frac{\partial^2 f_2}{\partial s^2}\right)_0 \frac{s^2}{2} + \left(\frac{\partial^3 f_2}{\partial s^3}\right)_0 \frac{s^3}{6} + \dots$$

$$\alpha_{2,1} = 180^\circ + f_3(s)_0 + \left(\frac{\partial f_3}{\partial s}\right)_0 s + \left(\frac{\partial^2 f_3}{\partial s^2}\right)_0 \frac{s^2}{2} + \left(\frac{\partial^3 f_3}{\partial s^3}\right)_0 \frac{s^3}{6} + \dots$$

$$f_1(s)_0 = \varphi_1$$

$$f_2(s)_0 = \lambda_1$$

$$f_3(s)_0 = \alpha_{12}$$

$$\begin{aligned} \frac{\partial f_1}{\partial s} &= \frac{d\varphi}{ds} = \frac{\cos \alpha}{M} \\ \frac{\partial f_2}{\partial s} &= \frac{d\lambda}{ds} = \frac{\sin \alpha}{N \cos \varphi} \\ \frac{\partial f_3}{\partial s} &= \frac{d\alpha}{ds} = \frac{\sin \alpha \tan \varphi}{N} \end{aligned}$$

**The differential equation system  
of the geodesic**

# Legendre's polynomial method

Practical computations using the Legendre's method:

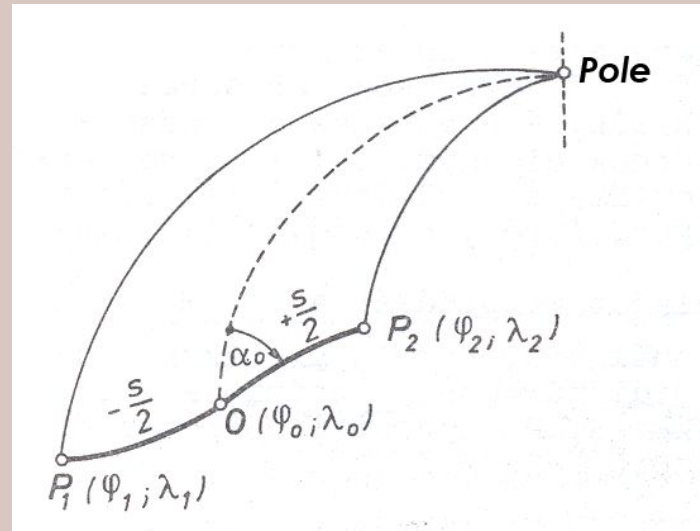
- slow convergence of the series ( $s=100, n=5; s=30, n=4$ );
- Schoeps (1960) published tables containing the polynomial coefficients depending on the position;
- Gerstbach (1974) published computational formulae for calculators and computers ( $s < 60\text{km}, 45^\circ < \phi < 54.5^\circ$ , accuracy is  $0.0001''$ )

**Gerstbach formulae will be used during the practicals!**

# Gauss's method of mean latitudes

**Principle:** The Legendre series are applied to the middle of the geodesic  $\rightarrow$  shorter arc  $\rightarrow$  faster convergence  $\rightarrow$   $n$  is less

**Problem:**  $\phi_0, \lambda_0, \alpha_0$  is not known, since  $\phi_2, \lambda_2, \alpha_{21}$  is the purpose of the computations! Iterative computations!



# 2nd fundamental task

## Gauss's method of mean latitudes

Now it is a useful method, since the coordinates of both endpoints are known.

Writing the Legendre polynomials for  $P_1$ :

$$\varphi_1 = f_1(s)_0 + \left(\frac{\partial f_1}{\partial s}\right)_0 \left(-\frac{s}{2}\right) + \frac{1}{2} \left(\frac{\partial^2 f_1}{\partial s^2}\right)_0 \left(-\frac{s}{2}\right)^2 + \frac{1}{6} \left(\frac{\partial^3 f_1}{\partial s^3}\right)_0 \left(-\frac{s}{2}\right)^3 + \dots$$

$$\lambda_1 = f_2(s)_0 + \left(\frac{\partial f_2}{\partial s}\right)_0 \left(-\frac{s}{2}\right) + \frac{1}{2} \left(\frac{\partial^2 f_2}{\partial s^2}\right)_0 \left(-\frac{s}{2}\right)^2 + \frac{1}{6} \left(\frac{\partial^3 f_2}{\partial s^3}\right)_0 \left(-\frac{s}{2}\right)^3 + \dots$$

$$\alpha_{1,2} = f_3(s)_0 + \left(\frac{\partial f_3}{\partial s}\right)_0 \left(-\frac{s}{2}\right) + \frac{1}{2} \left(\frac{\partial^2 f_3}{\partial s^2}\right)_0 \left(-\frac{s}{2}\right)^2 + \frac{1}{6} \left(\frac{\partial^3 f_3}{\partial s^3}\right)_0 \left(-\frac{s}{2}\right)^3 + \dots$$

## 2nd fundamental task

Writing the Legendre polynomials for  $P_2$ :

$$\varphi_2 = f_1(s)_0 + \left(\frac{\partial f_1}{\partial s}\right)_0 \left(\frac{s}{2}\right) + \frac{1}{2} \left(\frac{\partial^2 f_1}{\partial s^2}\right)_0 \left(\frac{s}{2}\right)^2 + \frac{1}{6} \left(\frac{\partial^3 f_1}{\partial s^3}\right)_0 \left(\frac{s}{2}\right)^3 + \dots$$

$$\lambda_2 = f_2(s)_0 + \left(\frac{\partial f_2}{\partial s}\right)_0 \left(\frac{s}{2}\right) + \frac{1}{2} \left(\frac{\partial^2 f_2}{\partial s^2}\right)_0 \left(\frac{s}{2}\right)^2 + \frac{1}{6} \left(\frac{\partial^3 f_2}{\partial s^3}\right)_0 \left(\frac{s}{2}\right)^3 + \dots$$

$$\alpha_{2,1} = 180^\circ + f_3(s)_0 + \left(\frac{\partial f_3}{\partial s}\right)_0 \left(\frac{s}{2}\right) + \frac{1}{2} \left(\frac{\partial^2 f_3}{\partial s^2}\right)_0 \left(\frac{s}{2}\right)^2 + \frac{1}{6} \left(\frac{\partial^3 f_3}{\partial s^3}\right)_0 \left(\frac{s}{2}\right)^3 + \dots$$

Where:

$$f_1(s)_0 = \varphi_0$$

$$f_2(s)_0 = \lambda_0$$

$$f_3(s)_0 = \alpha_{01} - 180^\circ = \alpha_{02} = \alpha_0$$

$$\varphi_0 = \frac{\varphi_1 + \varphi_2}{2}$$

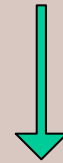
## 2nd fundamental task

Let's compute the difference of the Legendre polynomials between  $P_2$  and  $P_1$ :

$$\Delta\varphi = \varphi_2 - \varphi_1$$

$$\Delta\lambda = \lambda_2 - \lambda_1$$

$$\Delta\alpha = \alpha_{20} - \alpha_{10} - 180^\circ = \alpha_{21} - \alpha_{12} - 180^\circ$$



$$\Delta\varphi = \left(\frac{\partial f_1}{\partial s}\right)_0 s + \frac{1}{24} \left(\frac{\partial^3 f_1}{\partial s^3}\right)_0 s^3 + \dots$$

$$\Delta\lambda = \left(\frac{\partial f_2}{\partial s}\right)_0 s + \frac{1}{24} \left(\frac{\partial^3 f_2}{\partial s^3}\right)_0 s^3 + \dots$$

$$\Delta\alpha = \left(\frac{\partial f_3}{\partial s}\right)_0 s + \frac{1}{24} \left(\frac{\partial^3 f_3}{\partial s^3}\right)_0 s^3 + \dots$$

## 2nd fundamental task

$$\Delta\varphi = \left(\frac{\partial f_1}{\partial s}\right)_0 s + \frac{1}{24} \left(\frac{\partial^3 f_1}{\partial s^3}\right)_0 s^3 + \dots$$

$$\Delta\lambda = \left(\frac{\partial f_2}{\partial s}\right)_0 s + \frac{1}{24} \left(\frac{\partial^3 f_2}{\partial s^3}\right)_0 s^3 + \dots$$

$$\Delta\alpha = \left(\frac{\partial f_3}{\partial s}\right)_0 s + \frac{1}{24} \left(\frac{\partial^3 f_3}{\partial s^3}\right)_0 s^3 + \dots$$

The first two equations can be solved for:

$$s \cos \alpha_0 = X$$

$$s \sin \alpha_0 = Y$$

Dividing the two results  $\alpha_0$  can be computed:

$$\tan \alpha_0 = \frac{Y}{X}, \quad \text{and}$$

$$s = \frac{X}{\cos \alpha_0} = \frac{Y}{\sin \alpha_0}$$

## 2nd fundamental task

Finally the  $\Delta\alpha$  can be computed from the third series, and:

$$\alpha_{12} = \alpha_0 - \frac{\Delta\alpha}{2}$$

$$\alpha_{21} = \alpha_0 + \frac{\Delta\alpha}{2} \pm 180^\circ$$

**Gertsbach (1974) published computational formulae for this approach.**



**Thank You for Your Attention!**

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