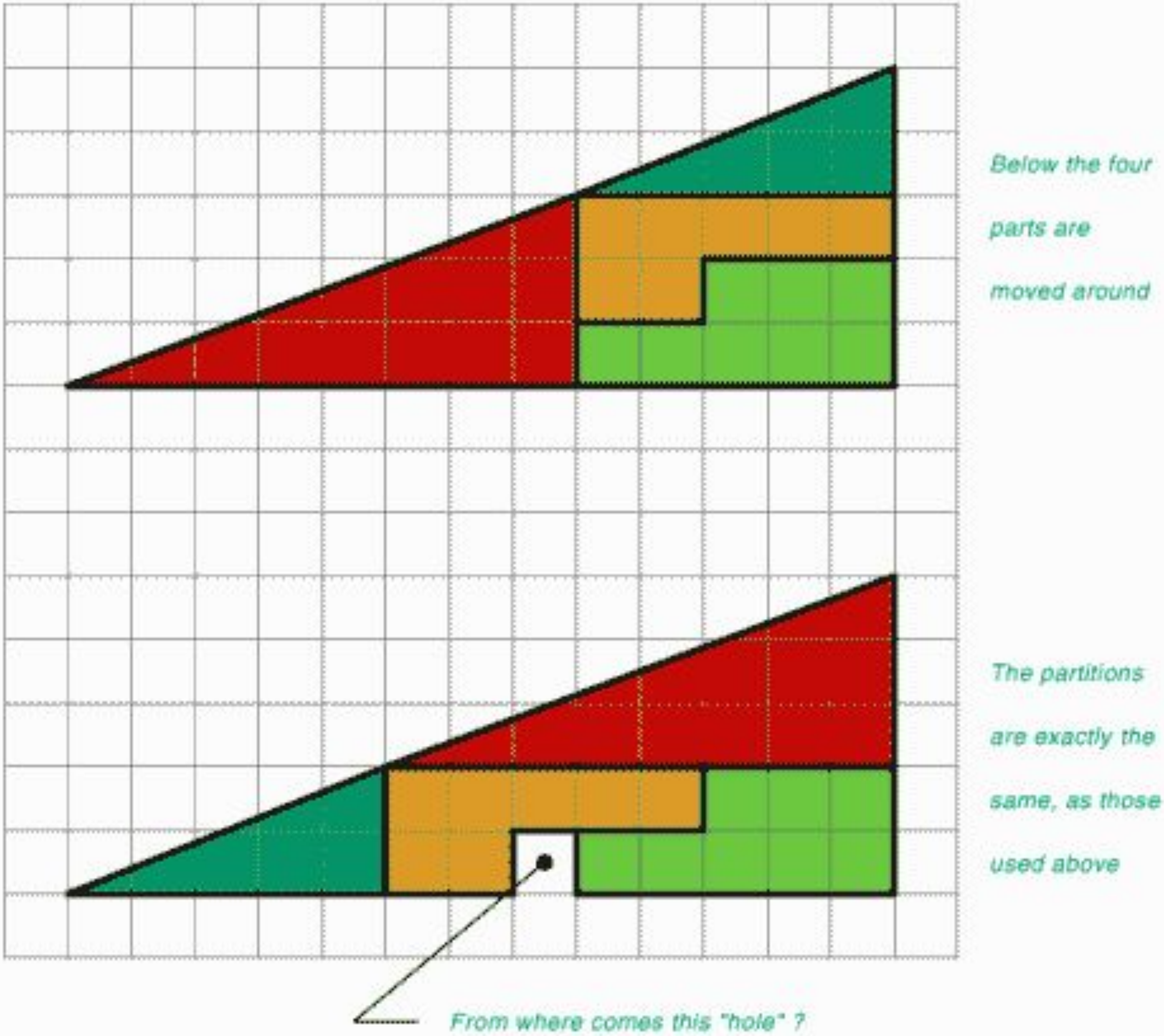

Below the four
parts are
moved around
The partitions
are exactly the
same, as those
used above
From where comes this "hole" ?

Mathematics for Economists II
Week #12

Review

Predator-prey relationships

Solving simultaneous first order differential equations

Steady states and their stability

Phase portraits of systems

Week 1:

Sets: Unions, intersections, Venn Diagrams

Functions: Definition, notation, types

Slopes

Linear programming

Week 2:

Derivative as a limit

Power rule, chain rule, product rule

$f'(x)$, slope, $f''(x)$, concavity

exponential functions, natural logs, TVM

Weeks 3 and 4:

Solve simultaneous equations

Writing equations in matrix form: $\underline{\mathbf{Ax}} = \underline{\mathbf{B}}$

Rules for adding, subtracting, multiplying matrices

Identity matrix: $I_{ij}=1$ if $i=j$, and 0 if $i \neq j$

Rank (How many "leading ones"?)

Row operations

Transposed matrices

Calculating determinants

Testing for invertibility and calculating $\underline{\mathbf{A}}^{-1}$

Cramer's Rule

Eigenvalues

Weeks 3 and 4:

Eigenvalues and Eigenvectors

Weeks 3 and 4:

Example:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Weeks 3 and 4:

Example:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$(2 - i)(2 - i) - 1 = 0$

$i^2 - 4i + 4 - 1 = 0$

$i^2 - 4i + 3 = 0$

$(i - 3)(i - 1) = 0$, so $i = 1, 3$

Eigenvalues and Eigenvectors

Now... the vectors.

Example:

i = 1, 3

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix}$**, leading to x = -y**

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$**, leading to x = y**

Eigenvectors are v_1 = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$**, v_2 =** $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Week 5:

Taylor/Maclaurin series

Week 6:

Quadratics in matrix form

Positive/negative definite, positive/negative semidefinite, indefinite matrices

Week 8:

Unconstrained optimization

The Hessian matrix

Determining max/min from the Hessian definiteness

Week 9:

The Lagrange multiplier

Using multiple multipliers for multiple constraints

Week 10:

Kuhn-Tucker inequality constrained optimization

Jacobian matrix and bordered Hessian

Week 11:

Ordinary differential equations

Integrating factors

Separable differential equations

Mathematics for Economists II
Week #12

Review

Predator-prey relationships

Solving simultaneous first order differential equations

Steady states and their stability

Phase portraits of systems

Predator-Prey Relationships

Assume that some prey x grows in population according to the first order differential equation:

$\dot{x}/x = A - By$, or $\dot{x} = x(A - By)$

where y is the population of some predator.

Predator-Prey Relationships

Assume that some prey x grows in population according to the first order differential equation:

$\dot{x}/x = A - By$, or $\dot{x} = x(A - By)$

where y is the population of some predator.

The predator's population grows according to another first order differential equation:

$\dot{y}/y = -C + Dx$, or $\dot{y} = y(-C + Dx)$

Predator-Prey Relationships

Assume that some prey x grows in population according to the first order differential equation:

$\dot{x}/x = A - By$, or $\dot{x} = x(A - By)$

where y is the population of some predator.

The predator's population grows according to another first order differential equation:

$\dot{y}/y = -C + Dx$, or $\dot{y} = y(-C + Dx)$

What is the behavior of this system? What can we determine? Let's look at Excel! :-)

Mathematics for Economists II
Week #12

Review

Predator-prey relationships

Solving simultaneous first order differential equations

Steady states and their stability

Phase portraits of systems

Solving simultaneous equations

Suppose that we want solutions for both variables (meaning explicit functions in terms of t).

How could we do that?

We'll skip the process, and focus on a shortcut.

Given: $\dot{x} = Ax + By$

$\dot{y} = Cx + Dy,$

we find the eigenvalues of:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Solving simultaneous equations

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Next, we take the eigenvalues i and calculate the eigenvectors with **|A|v** = **iv**.

Finally, we use these for the general solution:

$$x(t) = c_1 v_{1,1} e^{i_1 t} + c_2 v_{1,2} e^{i_2 t}$$

$$\mathrm{y}(t) = c_1 v_{2,1} e^{i_1 t} + c_2 v_{2,2} e^{i_2 t}$$

(This is just my notation; see SB Chapter 25)

Solving simultaneous equations - Example

$$\dot{x} = 2x + y$$
$$\dot{y} = -12x - 5y$$

First step?

Solving simultaneous equations - Example

$$\dot{x} = 2x + y$$
$$\dot{y} = -12x - 5y$$

1. Make matrix A.

$$\begin{bmatrix} 2 & 1 \\ -12 & -5 \end{bmatrix}$$

Next step?

Solving simultaneous equations - Example

$$\dot{x} = 2x + y$$
$$\dot{y} = -12x - 5y$$

2. Find the eigenvalues.

$$\begin{bmatrix} 2 & 1 \\ -12 & -5 \end{bmatrix}$$

Solving simultaneous equations - Example

$$\dot{x} = \quad 2x + \ y$$
$$\dot{y} = -12x - 5y$$

2. Find the eigenvalues.

$$\begin{bmatrix} 2 & 1 \\ -12 & -5 \end{bmatrix}$$

(2 - i)(-5 - i) + 12 = 0

10 + 5i - 2i + i^2 + 12 = 0

i2 + 3i + 2 = 2

(i + 1)(i + 2) = 0, so i = -1, -2.

Next step?

Solving simultaneous equations - Example

$$\dot{x} = \quad 2x + \ \ y$$
$$\dot{y} = -12x - 5y$$

3. Find the eigenvectors that go with each eigenvalue.

i = -1, -2

$$\begin{bmatrix} 2 & 1 \\ -12 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} x \\ y \end{bmatrix}$$, leading to ______________

$$\begin{bmatrix} 2 & 1 \\ -12 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -2 \begin{bmatrix} x \\ y \end{bmatrix}$$, leading to ______________

Next step?

Solving simultaneous equations - Example

$$\dot{x} = \quad 2x + \ \ y$$
$$\dot{y} = -12x - 5y$$

3. Find the eigenvectors that go with each eigenvalue.

i = -1, -2

$\begin{bmatrix} 2 & 1 \\ -12 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} x \\ y \end{bmatrix}$, leading to 3x = -y, so $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ -12 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -2 \begin{bmatrix} x \\ y \end{bmatrix}$, leading to 4x = -y, so $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$

Next step?

Solving simultaneous equations - Example

$$\dot{x} = \quad 2x + \ y$$
$$\dot{y} = -12x - 5y$$

4. Plug the values into the solution equations.

$$x(t) = c_1 v_{1,1} e^{i_1 t} + c_2 v_{1,2} e^{i_2 t}$$
$$\mathrm{y}(t) = c_1 v_{2,1} e^{i_1 t} + c_2 v_{2,2} e^{i_2 t}$$

So...

$$x(t) = c_1 e^{-t} + c_2 e^{-2t}$$
$$\mathrm{y}(t) = -3c_1 e^{-t} + -4c_2 e^{-2t}$$

DONE! In Excel, this is 25pt1.

Mathematics for Economists II
Week #12

Review

Predator-prey relationships

Solving simultaneous first order differential equations

Steady states and their stability

Phase portraits of systems

Steady States

How would we determine a steady state?
What's true at that steady state?
Again, the derivatives are zero!
So, let's look again at the predator/prey equations.

$$\dot{x} = x(80 - y)$$
$$\dot{y} = y(-300 + 2x)$$

Setting the changes equal to zero, we have one steady state at (0,0) (no predators, no prey).

What else?
x = C/D, and y = A/B. Let's look again!

Steady States

How about that example we solved?

$$\dot{x} = \quad 2x + \ y$$
$$\dot{y} = -12x - 5y$$

Solve these simultaneously for no change in x and y; what's the steady state?

Steady States

How about this one?

$$\dot{x} = 2 + y$$
$$\dot{y} = 1 + x - y$$

Solve these simultaneously for no change in x and y; what's the steady state?

Stability of Steady States (linear)

Now, what happens if we start a little bit away from the steady state. Do we move toward it, or away?

Let's look at what happens with a previous example:

$$\begin{aligned}\dot{x} &= 2x + y \\ \dot{y} &= -12x - 5y\end{aligned}$$

12PhasePortraitAndFieldDirectionsOfTwoDimensionalLinearSystems.cdf

Stability of Steady States (linear)

Now, what happens if we start a little bit away from the steady state. Do we move toward it, or away?

Let's look at what happens with a previous example:

$$\dot{x} = 2x + y$$
$$\dot{y} = -12x - 5y$$

We spiral in! But why? How can we tell?

Stability of Steady States (linear)

Now, what happens if we start a little bit away from the steady state. Do we move toward it, or away?

Let's look at what happens with a previous example:

$$\dot{x} = 2x + y$$
$$\dot{y} = -12x - 5y$$

We spiral in! But why? How can we tell?

Eigenvalues! :-)

Stability of Steady States (linear)

If eigenvalues are all negative, the system is stable. If there's a positive eigenvalue, it will pull us away from the steady state.

Test the stability of these two:

$$\dot{x} = \quad x - 4y$$
$$\dot{y} = -x + y$$

Stable, or unstable?

$$\dot{x} = -x$$
$$\dot{y} = \quad x - y$$

Stable, or unstable?

Stability of Steady States (linear)

If eigenvalues are all negative, the system is stable. If there's a positive eigenvalue, it will pull us away from the steady state.

Test the stability of these two:

$$\dot{x} = \quad x - 4y$$
$$\dot{y} = -x + y$$

i = -1, 3; UNSTABLE

$$\dot{x} = -x$$
$$\dot{y} = \quad x - y$$

i = -1; STABLE

Stability of nonlinear systems

What about nonlinear system, such as the predator/prey example?

$$\dot{x} = x(80 - y)$$
$$\dot{y} = y(-300 + 2x)$$

We have already seen that we spiral out. But why? How can we tell?

Eigenvalues of the Jacobian!

We don't do this here... but Example 25.2 in SB will show you how to do this if you need it.

Mathematics for Economists II
Week #12

Review

Predator-prey relationships

Solving simultaneous first order differential equations

Steady states and their stability

Phase portraits of systems

Phase portraits

Let's look at some pictures of past examples:

$$\dot{x} = \quad x + 2y$$
$$\dot{y} = -x + y$$

i = -1, 3; UNSTABLE

$$\dot{x} = -x$$
$$\dot{y} = \quad x - y$$

i = -1; STABLE

12PhasePortraitAndFieldDirectionsOfTwoDimensionalLinearSystems.cdf

Phase portraits

Let's look at some pictures of past examples:

$$\dot{x} = \quad x + 2y$$

$$\dot{y} = -x + y$$

$$\mathbf{i = \text{-}1, 3; UNSTABLE}$$

$$\dot{x} = -x$$

$$\dot{y} = \quad x - y$$

$$\mathbf{i = \text{-}1;} \ STABLE$$

Now let's draw our own!

How do we find the "stable arms"? How can we draw arrows?

Phase portraits

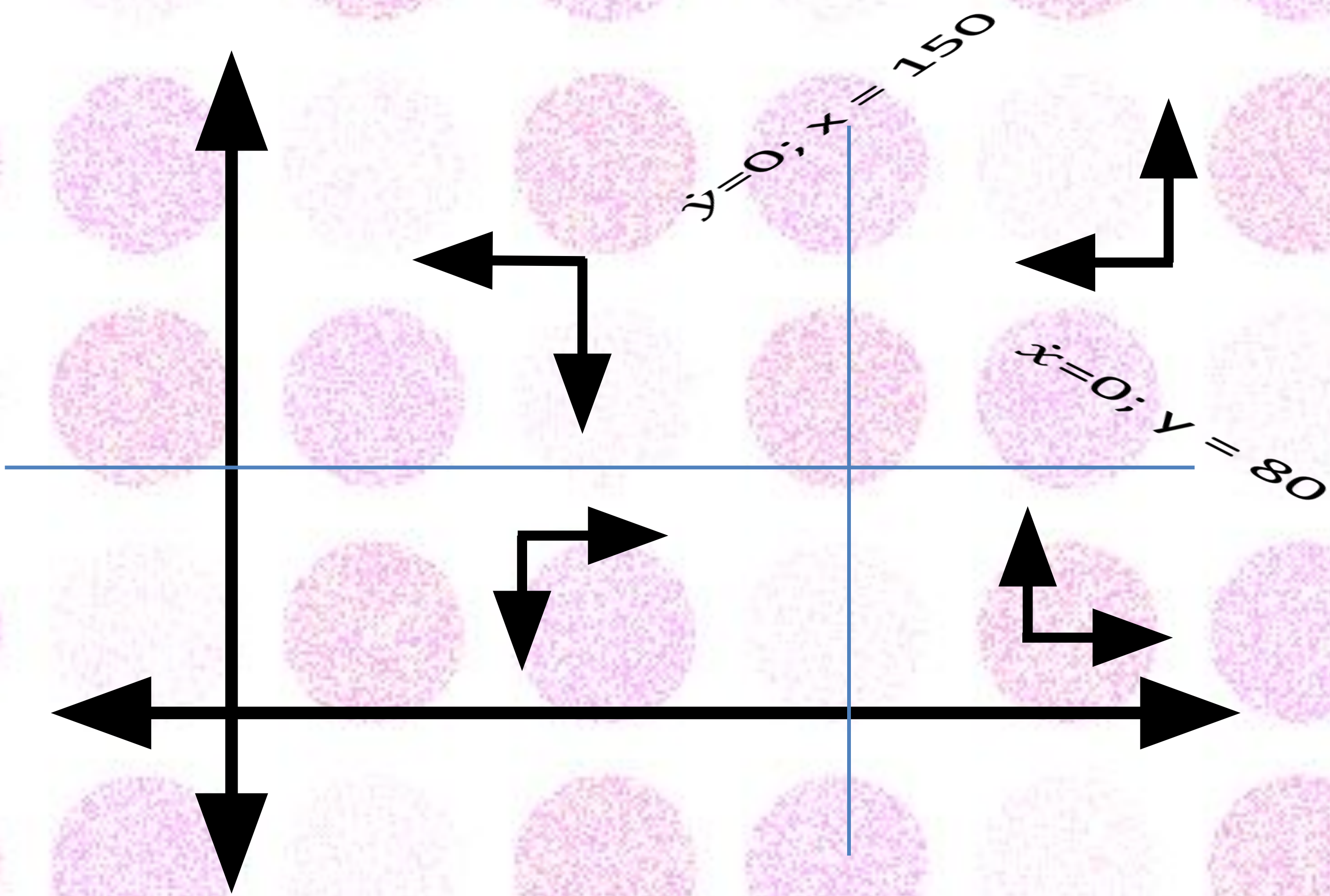
How about that nonlinear example of the predator/prey model?

$$\dot{x} = x(80 - y)$$
$$\dot{y} = y(-300 + 2x)$$

How do we find the "stable arms"?
How can we draw arrows?

Phase portraits

$$\dot{x} = x(80 - y)$$
$$\dot{y} = y(-300 + 2x)$$



Mathematics for Economists II
Week #12

COMPLETE!

Homework10.docx due by next Monday

Next week: FINAL EXAM!

Dobře se bav! :-)