

## Лекция 11

### Негармонические волны в равновесных средах с дисперсией

#### 1. Общее решение начальной задачи

- а. Основные формулы для пространственно гармонических волн*
- б. Построение общего решения в случае негармонических волн*
- в. Пространственно временная функция точечного источника*
- г. Задача об эволюции негармонического начального возмущения*

#### 2. Квазигармоническое приближение. Групповая скорость

- а. Спектральная плотность начального возмущения*
- б. Случай слабой дисперсии – групповая скорость*
- в. Скорость переноса энергии волны*
- г. Групповая скорость и метод медленных амплитуд*
- д. Групповые скорости конкретных волн*
- е. Поток энергии волн*

#### 3. Расплывание импульсов в равновесных средах с дисперсией

- а. Постановка задачи и общие соображения*
- б. Компьютерное моделирование*
- в. Метод стационарной фазы*

$$A(t, z) = A(t, k) \exp(ikz) \quad D(\hat{\omega}, k) A(t, k) = F(t, k)$$

$$A(t, k) = \frac{1}{2\pi} \int_{C(\omega)} \frac{P_{n-1}(\omega, k)}{D(\omega, k)} \exp(-i\omega t) d\omega$$

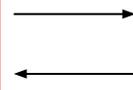
$$A(t, k) = \sum_{m=1}^n [A_m(k) \exp(-i\omega_m(k)t)]$$

$$A(t, k) = \int_0^t G(\tau, k) F(t-\tau, k) d\tau$$

$$G(t, k) = \frac{1}{2\pi} \int_{C(\omega)} \frac{1}{D(\omega, k)} \exp(-i\omega t) d\omega$$

$$D(\hat{\omega}, \hat{k}) A(t, z) = F(t, z)$$

$$A(t, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(t, k) \exp(ikz) dk$$



$$A(t, k) = \int_{-\infty}^{+\infty} A(t, z) \exp(-ikz) dz$$

$$A(t, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dk \int_{C(\omega)} \frac{P_{n-1}(\omega, k)}{D(\omega, k)} \exp(-i\omega t + ikz) d\omega$$

$$A(t, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{m=1}^n [A_m(k) \exp(-i\omega_m(k)t + ikz)] dk$$

$$A(t, z) = \int_0^t d\tau \int_{-\infty}^{+\infty} dx G(\tau, x) F(t-\tau, z-x)$$

$$D(\hat{\omega}, \hat{k}) G(t, z) = \delta(t) \delta(z)$$

$$G(t, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(t, k) \exp(ikz) dk = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dk \int_{C(\omega)} \frac{1}{D(\omega, k)} \exp(-i\omega t + ikz) d\omega$$

$$A(t, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(k) \exp(-i\omega(k)t + ikz) dk$$

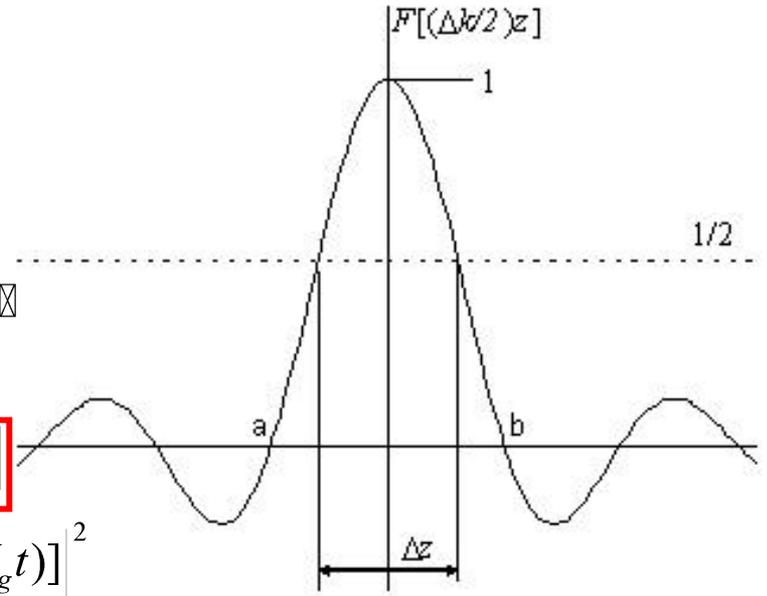
$$A(0, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(k) \exp(ikz) dk \equiv A_0(z) \quad \Delta z \cdot \Delta k \sim 2\pi \quad A(k) = \int_{-\infty}^{+\infty} A_0(z) \exp(-ikz) dz \equiv A_0(k)$$

**Примеры дельтаобразных  $A(k)$  и  $A(z)$**

$$A_0(k) = C_0 \frac{2\pi}{\Delta k} \begin{cases} 1, & k \in [k_0 - \Delta k/2, k_0 + \Delta k/2] \\ 0, & k \notin [k_0 - \Delta k/2, k_0 + \Delta k/2] \end{cases} \quad A_0(z) = F[(\Delta k/2)z] \cdot C_0 \exp(ik_0 z), \quad F(\xi) = \frac{\sin(\xi)}{\xi}$$

$$A(t, z) = C_0 \frac{1}{\Delta k} \int_{k_0 - \Delta k/2}^{k_0 + \Delta k/2} \exp[-i\omega(k)t + ikz] dk$$

$$\omega(k) = \omega(k_0) + (k - k_0) \left( \frac{d\omega}{dk} \right)_{k=k_0} + \frac{1}{2} (k - k_0)^2 \left( \frac{d^2\omega}{dk^2} \right)_{k=k_0} + \dots$$



$$A(t, z) = F[(\Delta k/2)(z - V_g t)] \cdot C_0 \exp[-i\omega(k_0)t + ik_0 z]$$

$$V_g = \left( \frac{d\omega}{dk} \right)_{k=k_0}$$

$$W(t, z) = |A(t, z)|^2 = C_0^2 \frac{\sin^2[(\Delta k/2)(z - V_g t)]}{[(\Delta k/2)(z - V_g t)]^2}$$

$$V_g = c_0 \quad V_g = \frac{c}{\sqrt{1 + \omega_{Le}^2 / k^2 c^2}} = \frac{c^2}{V_\Phi}$$

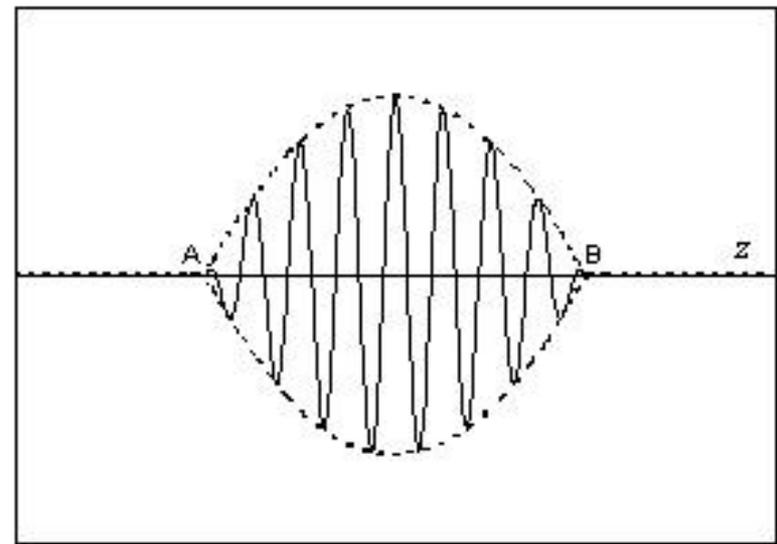
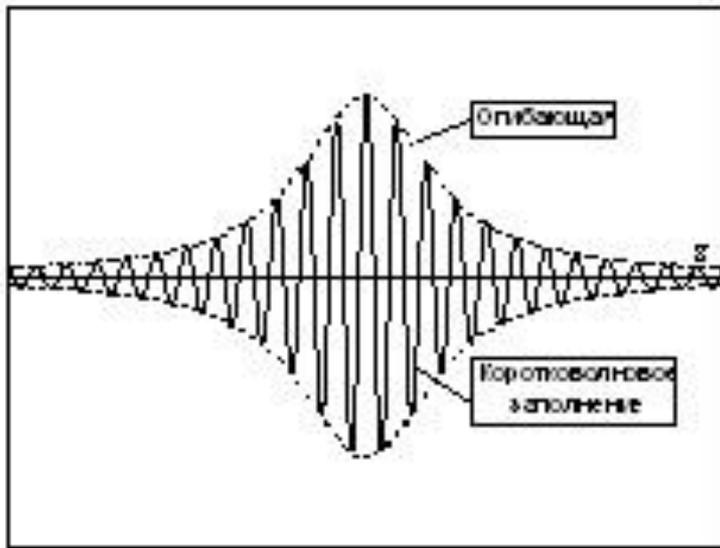
$$V_g = \frac{V_{Te}}{\sqrt{1 + \omega_{Le}^2 / (k^2 V_{Te}^2)}} = \frac{V_{\gamma e}}{V_\Phi}$$

$$V_g = 2kc^2 \frac{\Omega_e}{\omega_p^2} = 2V_\Phi$$

$$V_g = 0 \quad V_g = \frac{c_0}{\sqrt{1 + k_\perp^2 / k^2}} = \frac{c_0^2}{V_\Phi}$$

$$V_g = \frac{V_S}{(1 + k^2 r_{De}^2)^{3/2}} = \frac{V_\Phi}{(1 + k^2 r_{De}^2)}$$

$$V_g^{(B,M)} = u$$



$$A(t, z) = \Phi(t, z) \exp(-i\omega_0 t + ik_0 z)$$

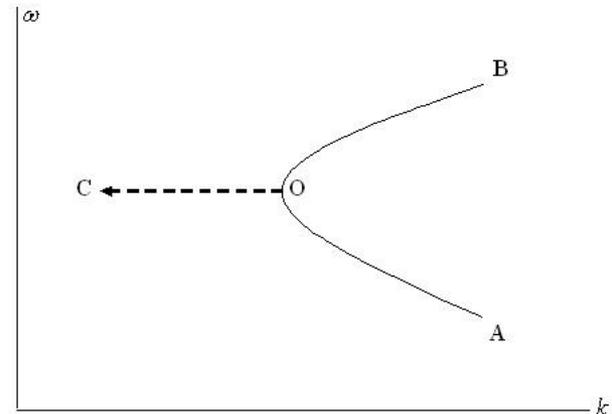
$$D\left(\omega_0 + i \frac{\partial}{\partial t}, k_0 - i \frac{\partial}{\partial z}\right) \Phi(t, z) = 0$$

$$\frac{\partial D}{\partial \omega} \frac{\partial \Phi}{\partial t} - \frac{\partial D}{\partial k} \frac{\partial \Phi}{\partial z} + i\mathcal{R} = 0$$

$$V_g = \frac{d\omega}{dk} = - \frac{\partial D / \partial k}{\partial D / \partial \omega}$$

$$\frac{\partial \Phi}{\partial t} + V_g \frac{\partial \Phi}{\partial z} = 0$$

$$\frac{\partial D}{\partial \omega} = 0 \rightarrow \omega(k) = \omega_0 \pm \left( 2 \frac{\partial D}{\partial k} / \frac{\partial^2 D}{\partial \omega^2} \right)^{1/2} \sqrt{k - k_0} + \dots$$



$$A(t, \mathbf{r}) = \tilde{A}(t, \mathbf{r}) \exp(-i\omega_0 t + i\mathbf{k}_0 \cdot \mathbf{r}) \quad D\left(\omega_0 + i\frac{\partial}{\partial t}, \mathbf{k}_0 - i\frac{\partial}{\partial \mathbf{r}}\right) \tilde{A}(t, \mathbf{r}) = 0$$

$$\frac{\partial \tilde{A}}{\partial t} + \mathbf{V}_g \frac{\partial \tilde{A}}{\partial \mathbf{r}} = 0$$

$$V_g = \frac{d\omega}{d\mathbf{k}} = -\frac{\partial D}{\partial \mathbf{k}} / \frac{\partial D}{\partial \omega}$$

$$\frac{\partial W}{\partial t} + \mathbf{V}_g \frac{\partial W}{\partial \mathbf{r}} = 0$$

$$W(t, \mathbf{r}) \sim |A(t, \mathbf{r})|^2$$

$$\mathbf{S}(t, \mathbf{r}) = \mathbf{V}_g W(t, \mathbf{r})$$

$$\frac{\partial W}{\partial t} + \text{div } \mathbf{S} = 0 \rightarrow \frac{d}{dt} \iiint_V W(t, \mathbf{r}) d\mathbf{r} = -\iint_{\Sigma} \mathbf{S} \cdot d\boldsymbol{\sigma}$$

$$\mathbf{S}_l(t, \mathbf{r}) = -\frac{\partial}{\partial \mathbf{k}} [\omega \varepsilon^l(\omega, \mathbf{k})] \frac{|\tilde{E}_l(t, \mathbf{r})|^2}{16\pi}$$

$$\mathbf{S}_{tr}(t, \mathbf{r}) = -\frac{\partial}{\partial \mathbf{k}} \left[ \omega \left( \varepsilon^{tr}(\omega, \mathbf{k}) - \frac{k^2 c^2}{\omega^2} \right) \right] \frac{|\tilde{E}_{tr}(t, \mathbf{r})|^2}{16\pi}$$

$$\mathbf{S}_{tr}(t, \mathbf{r}) = \frac{\mathbf{k} c^2}{\omega} \frac{|\tilde{E}_{tr}(t, \mathbf{r})|^2}{8\pi}$$

$$\uparrow \\ c[\mathbf{E} \times \mathbf{B}] / 4\pi$$

$$\varepsilon^l = \varepsilon^{tr} = 1 - \frac{\omega_{Le}^2}{(\omega^2 - k^2 V_{\gamma e}^2)} \rightarrow$$

$$\mathbf{S}_l(t, \mathbf{r}) = \frac{\mathbf{k}}{\omega} V_{\gamma e}^2 \frac{\omega^2}{\omega_{Le}^2} \frac{|\tilde{E}_l(t, \mathbf{r})|^2}{8\pi}$$

$$\mathbf{S}_{tr}(t, \mathbf{r}) = \frac{\mathbf{k}}{\omega} \left( c^2 + V_{\gamma e}^2 \frac{\omega_{Le}^2 \omega^2}{(\omega^2 - k^2 V_{\gamma e}^2)} \right) \frac{|\tilde{E}_{tr}(t, \mathbf{r})|^2}{8\pi}$$

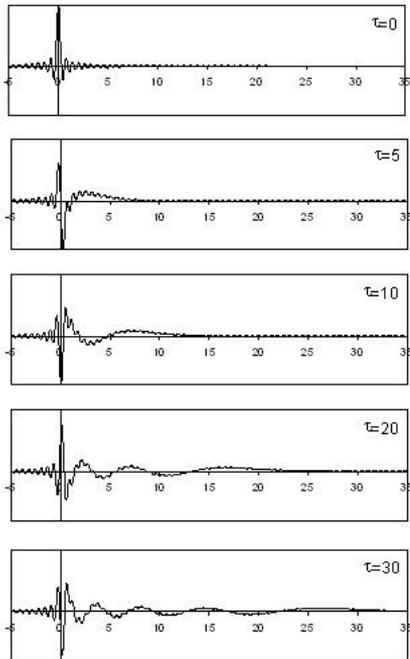
$$\Delta t \sim 2\pi \left( (\Delta k)^2 \left| \frac{d^2 \omega}{dk^2} \right| \right)^{-1}$$

$$\omega(k) = \frac{kV_0}{\sqrt{1 + k^2 V_0^2 / \Omega_0^2}} - \text{акустический тип}$$

$$\left( \frac{\partial^2}{\partial t^2} - V_0^2 \frac{\partial^2}{\partial z^2} - V_0^2 \Omega_0^{-2} \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial z^2} \right) A(t, z) = 0$$

$$A_0(k) = C_0 \frac{\pi}{\Delta k} \begin{cases} 1, & k \in [-\Delta k, \Delta k], \\ 0, & k \notin [-\Delta k, \Delta k]. \end{cases}$$

$$a(\tau, \xi) = \frac{1}{\Delta} \int_0^\Delta \cos \left( \frac{x}{\sqrt{1+x^2}} \tau - x\xi \right) dx$$

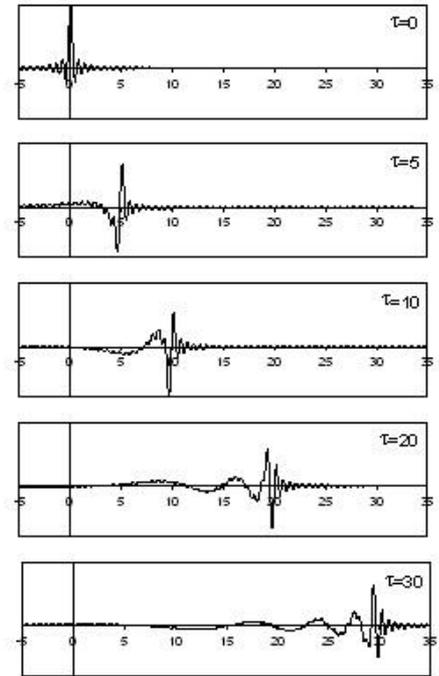


$$\omega(k) = \sqrt{\Omega_0^2 + k^2 V_0^2} - \text{оптический тип}$$

$$\left( \frac{\partial^2}{\partial t^2} - V_0^2 \frac{\partial^2}{\partial z^2} + \Omega_0^2 \right) A(t, z) = 0$$

$$\tau = \Omega_0 t, \quad \xi = \frac{\Omega_0}{V_0} z, \quad x = \frac{kV_0}{\Omega_0}, \quad \Delta = (\Delta k) \frac{V_0}{\Omega_0}.$$

$$a(\tau, \xi) = \frac{1}{\Delta} \int_0^\Delta \cos \left( \sqrt{1+x^2} \tau - x\xi \right) dx$$



$$I(\lambda) = \int_a^b P(q) \exp[i\lambda\Psi(q)] dq$$

$$x = \Psi(x) \rightarrow I(\lambda) = \int_{\Psi(a)}^{\Psi(b)} (P(x)/\Psi'(x)) \exp(i\lambda x) dx \sim \frac{1}{\lambda}$$

$$\Psi'(q_0) = 0 \quad P(q_0) \exp\left[i\lambda\left(\Psi(q_0) + \frac{1}{2}(q - q_0)^2 \Psi''(q_0)\right)\right]$$

$$I(\lambda) = \left| \frac{2\pi}{\lambda\Psi''(q_0)} \right|^{1/2} P(q_0) \exp\left(i\lambda\Psi(q_0) \pm i\frac{\pi}{4}\right)$$

$$A(t, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A_0(k) \exp[it\Psi(k)] dk,$$

$$\Psi(k) = k \frac{z}{t} - \omega(k)$$

$$\frac{d\Psi}{dk} = \frac{z}{t} - V_g(k) = 0$$

$$k_0(\xi) = \frac{\Omega_0}{V_0} \left[ \left( \frac{V_0}{\xi} \right)^{2/3} - 1 \right]^{1/2}$$

$$k_0(\xi) = \frac{\Omega_0}{V_0} \left[ \left( \frac{V_0}{\xi} \right)^2 - 1 \right]^{-1/2}$$

$$\xi = z/t$$

$$A(t, z) = \frac{A_0(k_0)}{\sqrt{2\pi t |G(k_0)|}} \exp\left[-i\omega(k_0)t + ik_0 z \pm i\frac{\pi}{4}\right], \quad G(k_0) = \left( \frac{d^2\omega}{dk^2} \right)_{k=k_0}, \quad k_0 = k_0(z/t)$$

$$\frac{dk_0}{dz} = \frac{1}{t |G(k_0)|}$$