7.1 Trigonometric Identities

Trigonometric Identities

We know that an **equation** is a statement that two mathematical expressions are equal. For example, the following are equations:

$$x + 2 = 5$$

$$(x + 1)^2 = x^2 + 2x + 1$$

$$\sin^2 t + \cos^2 t = 1.$$

An **identity** is an equation that is true for all values of the variable(s). The last two equations above are identities, but the first one is not, since it is not true for values of *x* other than 3.

Trigonometric Identities

A **trigonometric** identity is an identity involving trigonometric functions. We begin by listing some of the basic trigonometric identities.

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

Simplifying Trigonometric Expressions

Identities enable us to write the same expression in different ways. It is often possible to rewrite a complicated-looking expression as a much simpler one.

To simplify algebraic expressions, we used factoring, common denominators, and the Special Product Formulas.

To simplify trigonometric expressions, we use these same techniques together with the fundamental trigonometric identities.

Example 1 – *Simplifying a Trigonometric Expression*

Simplify the expression cos t + tan t sin t.

Solution:

We start by rewriting the expression in terms of sine and cosine.

$$\cos t + \tan t \sin t = \cos t + \left(\frac{\sin t}{\cos t}\right)\sin t$$ Reciprocal identity

$$= \frac{\cos^2 t + \sin^2 t}{\cos t}$$ Common denominator

$$= \frac{1}{\cos t}$$ Pythagorean identity

$$= \sec t$$ Reciprocal identity

Proving Trigonometric Identities

Proving Trigonometric Identities

Many identities follow from the fundamental identities.

In the examples that follow, we learn how to prove that a given trigonometric equation is an identity, and in the process we will see how to discover new identities.

First, it's easy to decide when a given equation is *not* an identity.

All we need to do is show that the equation does not hold for some value of the variable (or variables).

Proving Trigonometric Identities

Thus the equation

$$\sin x + \cos x = 1$$

is not an identity, because when $x = \pi/4$, we have

$$\sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \neq 1$$

To verify that a trigonometric equation is an identity, we transform one side of the equation into the other side by a series of steps, each of which is itself an identity.

Proving Trigonometric Identities

GUIDELINES FOR PROVING TRIGONOMETRIC IDENTITIES

1. **Start with one side.** Pick one side of the equation, and write it down. Your goal is to transform it into the other side. It's usually easier to start with the more complicated side.
2. **Use known identities.** Use algebra and the identities you know to change the side you started with. Bring fractional expressions to a common denominator, factor, and use the fundamental identities to simplify expressions.
3. **Convert to sines and cosines.** If you are stuck, you may find it helpful to rewrite all functions in terms of sines and cosines.

Example 2 – *Proving an Identity by Rewriting in Terms of Sine and Cosine*

Consider the equation $\cos\theta\,(\sec\theta - \cos\theta) = \sin^2\theta$.

(a) Verify algebraically that the equation is an identity.

(b) Confirm graphically that the equation is an identity.

Solution:

(a) The left-hand side looks more complicated, so we start with it and try to transform it into the right-hand side:

$$\text{LHS} = \cos\theta\,(\sec\theta - \cos\theta)$$

$$= \cos\theta\left(\frac{1}{\cos\theta} - \cos\theta\right) \qquad \text{Reciprocal identity}$$

Example 2 – *Solution*

cont'd

$= 1 - \cos^2\theta$ Expand

$= \sin^2\theta = \text{RHS}$ Pythagorean identity

(b) We graph each side of the equation to see whether the graphs coincide. From Figure 1 we see that the graphs of $y = \cos\theta\,(\sec\theta - \cos\theta)$ and $y = \sin^2\theta$ are identical.

This confirms that the equation is an identity.

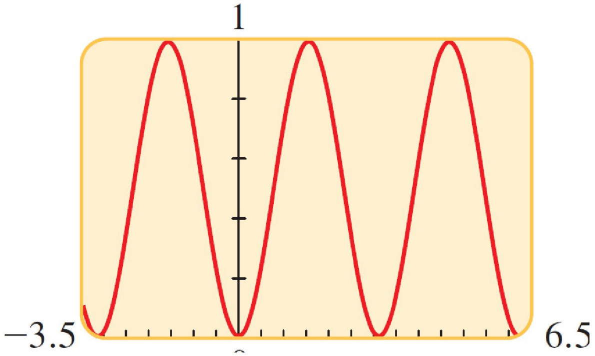


Figure 1

Proving Trigonometric Identities

In Example 2 it isn't easy to see how to change the right-hand side into the left-hand side, but it's definitely possible. Simply notice that each step is reversible.

In other words, if we start with the last expression in the proof and work backward through the steps, the right-hand side is transformed into the left-hand side.

You will probably agree, however, that it's more difficult to prove the identity this way. That's why it's often better to change the more complicated side of the identity into the simpler side.

Proving Trigonometric Identities

In Example 3 we introduce “something extra” to the problem by multiplying the numerator and the denominator by a trigonometric expression, chosen so that we can simplify the result.

Example 3 – *Proving an Identity by Introducing Something Extra*

Verify the identity $\dfrac{\cos u}{1 - \sin u} = \sec u + \tan u$.

Solution:

We start with the left-hand side and multiply the numerator and denominator by $1 + \sin u$:

$$\text{LHS} = \frac{\cos u}{1 - \sin u}$$

$$= \frac{\cos u}{1 - \sin u} \cdot \frac{1 + \sin u}{1 + \sin u}$$

Multiply numerator and denominator by $1 + \sin u$

Example 3 – *Solution*

cont'd

$$= \frac{\cos u\,(1 + \sin u)}{1 - \sin^2 u}$$ Expand denominator

$$= \frac{\cos u\,(1 + \sin u)}{\cos^2 u}$$ Pythagorean identity

$$= \frac{1 + \sin u}{\cos u}$$ Cancel common factor

$$= \frac{1}{\cos u} + \frac{\sin u}{\cos u}$$ Separate into two fractions

$= \sec u + \tan u$ Reciprocal identities

Proving Trigonometric Identities

Here is another method for proving that an equation is an identity.

If we can transform each side of the equation *separately*, by way of identities, to arrive at the same result, then the equation is an identity. Example 6 illustrates this procedure.

Example 4 – *Proving an Identity by Working with Both Sides Separately*

Verify the identity $\dfrac{1 + \cos\theta}{\cos\theta} = \dfrac{\tan^2\theta}{\sec\theta - 1}$.

Solution:

We prove the identity by changing each side separately into the same expression. (You should supply the reasons for each step.)

$$\text{LHS} = \frac{1 + \cos\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\cos\theta}{\cos\theta}$$

$$= \sec\theta + 1$$

Example 4 – *Solution*

cont'd

$$\text{RHS} = \frac{\tan^2\theta}{\sec\theta - 1} = \frac{\sec^2\theta - 1}{\sec\theta - 1} = \frac{(\sec\theta - 1)(\sec\theta + 1)}{\sec\theta - 1}$$

$$= \sec\theta + 1$$

It follows that LHS = RHS, so the equation is an identity.

Proving Trigonometric Identities

We conclude this section by describing the technique of *trigonometric substitution*, which we use to convert algebraic expressions to trigonometric ones. This is often useful in calculus, for instance, in finding the area of a circle or an ellipse.

Example 5 – *Trigonometric Substitution*

Substitute $\sin\theta$ for x in the expression $\sqrt{1 - x^2}$, and simplify. Assume that $0 \leq \theta \leq \pi/2$.

Solution:

Setting $x = \sin\theta$, we have

$$\sqrt{1 - x^2} = \sqrt{1 - \sin^2\theta}$$ Substitute $x = \sin\theta$

$$= \sqrt{\cos^2\theta}$$ Pythagorean identity

$$= \cos\theta$$ Take square root

The last equality is true because $\cos\theta \geq 0$ for the values of θ in question.

7.2 Addition and Subtraction Formulas

Addition and Subtraction Formulas

We now derive identities for trigonometric functions of sums and differences.

ADDITION AND SUBTRACTION FORMULAS

Formulas for sine:

$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$

$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Formulas for cosine:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Formulas for tangent:

$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$

$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

Example 1 – *Using the Addition and Subtraction Formulas*

Find the exact value of each expression.

(a) $\cos 75°$ **(b)** $\cos \frac{\pi}{12}$

Solution:

(a) Notice that 75° = 45° + 30°. Since we know the exact values of sine and cosine at 45° and 30°, we use the Addition Formula for Cosine to get

$$\cos 75° = \cos (45° + 30°)$$

$$= \cos 45° \cos 30° - \sin 45° \sin 30°$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}\sqrt{3} - \sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example 1 – *Solution*

cont'd

(b) Since $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$, the Subtraction Formula for Cosine gives

$$\text{cc}\frac{\pi}{12} = \text{cc}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Example 2 – *Proving a Cofunction Identity*

Prove the cofunction identity $\cos\left(\frac{\pi}{2} - u\right) = \sin u$.

Solution:

By the Subtraction Formula for Cosine we have

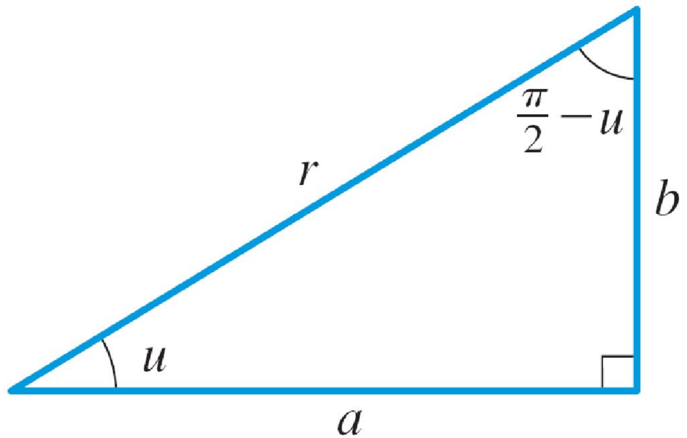
$$\cos\left(\frac{\pi}{2} - u\right) = \cos\frac{\pi}{2}\cos u + \sin\frac{\pi}{2}\sin u$$

$$= 0 \bullet \cos u + 1 \bullet \sin u$$

$$= \sin u$$

Addition and Subtraction Formulas

The cofunction identity in Example 3, as well as the other cofunction identities, can also be derived from the following figure.



$$\cos\left(\frac{\pi}{2} - u\right) = \frac{b}{r} = \sin u$$

The next example is a typical use of the Addition and Subtraction Formulas in calculus.

Example 3 – *An identity from Calculus*

If $f(x) = \sin x$, show that

$$\frac{f(x+h)-f(x)}{h} = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$

Solution:

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}$$

Definition of *f*

$$= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

Addition Formula for Sine

Example 3 – *Solution*

cont'd

$$= \frac{\sin x\,(\cos h - 1) + \cos x \sin h}{h}$$

Factor

$$= \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)$$

Separate the fraction

Evaluating Expressions Involving Inverse Trigonometric Functions

Evaluating Expressions Involving Inverse Trigonometric Functions

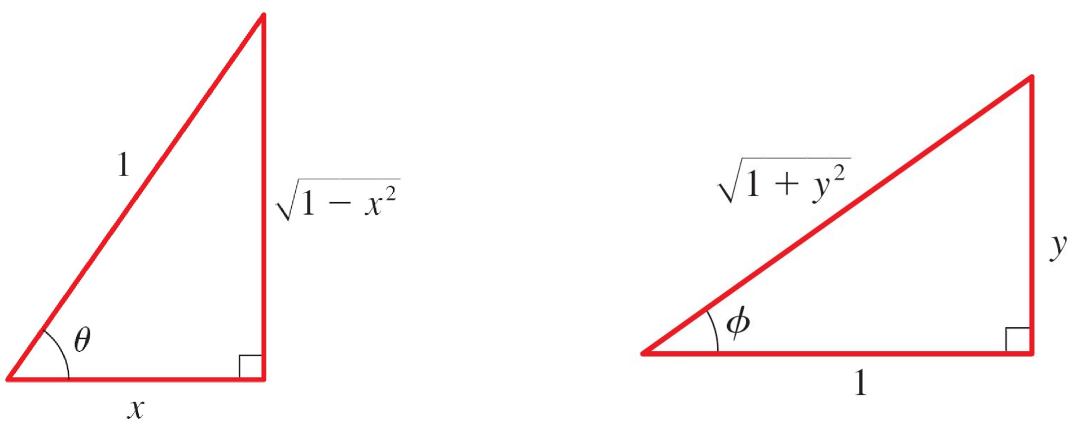
Expressions involving trigonometric functions and their inverses arise in calculus. In the next examples we illustrate how to evaluate such expressions.

Example 4 – *Simplifying an Expression Involving Inverse Trigonometric Functions*

Write $\sin(\cos^{-1} x + \tan^{-1} y)$ as an algebraic expression in x and y, where $-1 \leq x \leq 1$ and y is any real number.

Solution:

Let $\theta = \cos^{-1}x$ and $\varphi = \tan^{-1}y$. We sketch triangles with angles θ and φ such that $\cos\theta = x$ and $\tan \varphi = y$ (see Figure 2).



$\cos \theta = x$

$\tan \varphi = y$

Figure 2

Example 4 – *Solution*

cont'd

From the triangles we have

$\sin \theta = \sqrt{1 - x^2}$ $\cos \varphi = \dfrac{1}{\sqrt{1 + y^2}}$ $\sin \varphi = \dfrac{y}{\sqrt{1 + y^2}}$

From the Addition Formula for Sine we have

$$\sin(\cos^{-1} x + \tan^{-1} y) = \sin(\theta + \varphi)$$

$$= \sin \theta \cos \varphi + \cos \theta \sin \varphi$$

Addition Formula for Sine

Example 4 – *Solution*

cont'd

$$= \sqrt{1 - x^2}\,\frac{1}{\sqrt{1 + y^2}} + x\,\frac{y}{\sqrt{1 + y^2}}$$

From triangles

$$= \frac{1}{\sqrt{1 + y^2}}\left(\sqrt{1 - x^2} + xy\right)$$

Factor $\frac{1}{\sqrt{1 + y^2}}$

Expressions of the Form $A \sin x + B \cos x$

Expressions of the Form *A* sin *x* + *B* cos *x*

We can write expressions of the form *A* sin *x* + *B* cos *x* in terms of a single trigonometric function using the Addition Formula for Sine. For example, consider the expression

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$$

If we set $\varphi = \pi/3$, then $\cos \varphi = \frac{1}{2}$ and $\sin \varphi = \sqrt{3}/2$, and we can write

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \cos \varphi \sin x + \sin \varphi \cos x$$

$$= \sin(x + \varphi) = \sin\left(x + \frac{\pi}{3}\right)$$

Expressions of the Form $A \sin x + B \cos x$

We are able to do this because the coefficients $\frac{1}{2}$ and $\sqrt{3}/2$ are precisely the cosine and sine of a particular number, in this case, $\pi/3$.

We can use this same idea in general to write $A \sin x + B \cos x$ in the form $k \sin(x + \varphi)$.

We start by multiplying the numerator and denominator by $\sqrt{A^2 + B^2}$ to get

$A \sin x + B \cos x$

$$= \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}} \sin x + \frac{B}{\sqrt{A^2 + B^2}} \cos x\right)$$

Expressions of the Form $A \sin x + B \cos x$

We need a number φ with the property that

$$\cos \varphi = \frac{A}{\sqrt{A^2 + B^2}} \qquad \text{and} \qquad \sin \varphi = \frac{B}{\sqrt{A^2 + B^2}}$$

Figure 4 shows that the point (A, B) in the plane determines a number φ with precisely this property.

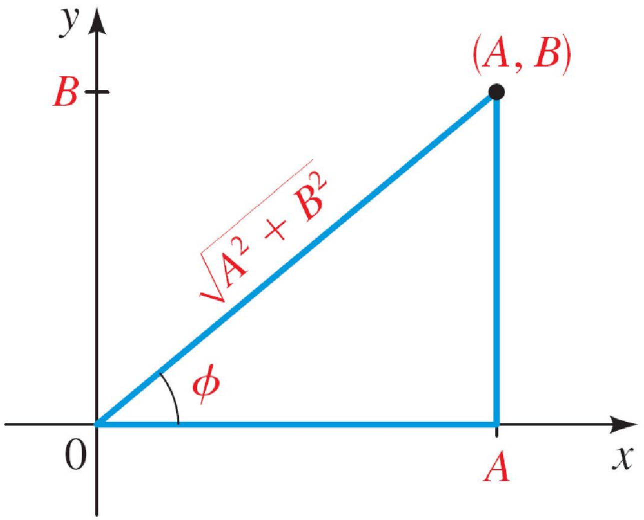


Figure 4

Expressions of the Form $A \sin x + B \cos x$

With this φ we have

$$A \sin x + B \cos x = \sqrt{A^2 + B^2}(\cos \varphi \sin x + \sin \varphi \cos x)$$

$$= \sqrt{A^2 + B^2} \sin(x + \varphi)$$

We have proved the following theorem.

SUMS OF SINES AND COSINES

If A and B are real numbers, then

$$A \sin x + B \cos x = k \sin(x + \phi)$$

where $k = \sqrt{A^2 + B^2}$ and ϕ satisfies

$$\cos \phi = \frac{A}{\sqrt{A^2 + B^2}} \qquad \text{and} \qquad \sin \phi = \frac{B}{\sqrt{A^2 + B^2}}$$

Example 5 – *A Sum of Sine and Cosine Terms*

Express 3 sin x + 4 cos x in the form k sin(x + φ).

Solution:

By the preceding theorem, $k = \sqrt{A^2 + B^2} = \sqrt{3^2 + 4^2} = 5$. The angle φ has the property that $\sin \varphi = B/k = \frac{4}{5}$ and $\cos \varphi = A/k = \frac{3}{5}$, and φ in Quadrant I (because sin φ and cos φ are both positive), so $\varphi = \sin^{-1}\frac{4}{5}$. Using a calculator, we get $\varphi \approx 53.1°$.

Thus

$$3 \sin x + 4 \cos x \approx 5 \sin (x + 53.1°)$$

Example 5 – *Graphing a Trigonometric Function*

Write the function $f(x) = -\sin 2x + \sqrt{3} \cos 2x$ in the form $k \sin(2x + \varphi)$, and use the new form to graph the function.

Solution:

Since $A = -1$ and $B = \sqrt{3}$, we have

$$k = \sqrt{A^2 + B^2}$$

$$= \sqrt{1 + 3}$$

$$= 2.$$

The angle φ satisfies $\cos \varphi = -\frac{1}{2}$ and $\sin \varphi = \sqrt{3}/2$. From the signs of these quantities we conclude that φ is in Quadrant II.

Example 5 – *Solution*

cont'd

Thus $\varphi = 2\pi/3$.

By the preceding theorem we can write

$$f(x) = -\sin 2x + \sqrt{3}\text{s } 2x$$

$$= 2\sin\left(2x + \frac{2\pi}{3}\right)$$

Using the form

$$f(x) = 2\sin 2\left(x + \frac{\pi}{3}\right)$$

Example 5 – *Solution*

cont'd

We see that the graph is a sine curve with amplitude 2, period $2\pi/2 = \pi$, and phase shift $-\pi/3$. The graph is shown in Figure 5.

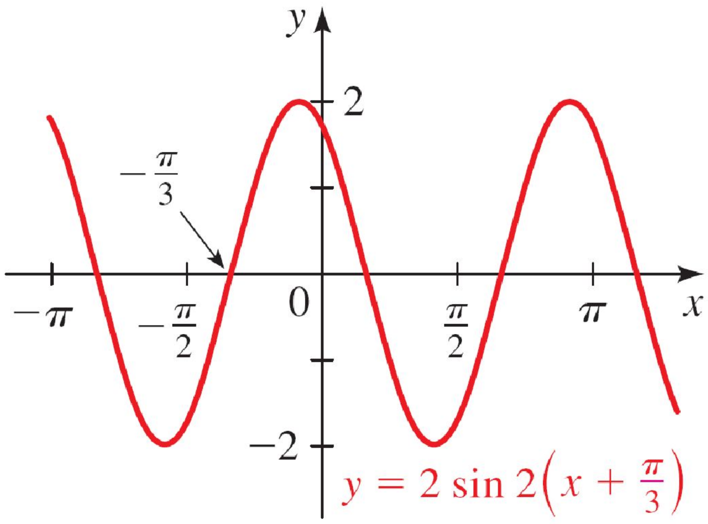


Figure 5

7.3 Double-Angle, Half-Angle, and Product-Sum Formulas

Double-Angle, Half-Angle, and Product-Sum Formulas

The identities we consider in this section are consequences of the addition formulas. The **Double-Angle Formulas** allow us to find the values of the trigonometric functions at $2x$ from their values at x.

The **Half-Angle Formulas** relate the values of the trigonometric functions at $\frac{1}{2}x$ to their values at x. The **Product-Sum Formulas** relate products of sines and cosines to sums of sines and cosines.

Double-Angle Formulas

The formulas in the following box are immediate consequences of the addition formulas.

DOUBLE-ANGLE FORMULAS

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine:

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2 \sin^2 x$$
$$= 2 \cos^2 x - 1$$

Formula for tangent:

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Example 1– *A Triple-Angle Formula*

Write cos 3x in terms of cos x.

Solution:

$$\cos 3x = \cos(2x + x)$$

$$= \cos 2x \cos x - \sin 2x \sin x$$ Addition formula

$$= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x$$ Double-Angle Formulas

$$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$$ Expand

Example 1 – *Solution*

cont'd

$= 2\cos^3 x - \cos x - 2\cos x\,(1 - \cos^2 x)$ Pythagorean identity

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$ Expand

$= 4\cos^3 x - 3\cos x$ Simplify

Double-Angle Formulas

Example 2 shows that $\cos 3x$ can be written as a polynomial of degree 3 in $\cos x$.

The identity $\cos 2x = 2\cos^2 x - 1$ shows that $\cos 2x$ is a polynomial of degree 2 in $\cos x$.

In fact, for any natural number n we can write $\cos nx$ as a polynomial in $\cos x$ of degree n.

Half-Angle Formulas

Half-Angle Formulas

The following formulas allow us to write any trigonometric expression involving even powers of sine and cosine in terms of the first power of cosine only.

This technique is important in calculus. The Half-Angle Formulas are immediate consequences of these formulas.

FORMULAS FOR LOWERING POWERS

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Example 2 – *Lowering Powers in a Trigonometric Expression*

Express $\sin^2 x \cos^2 x$ in terms of the first power of cosine.

Solution:

We use the formulas for lowering powers repeatedly.

$$\sin^2 x \cos^2 x = \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)$$

$$= \frac{1 - \cos^2 2x}{4} = \frac{1}{4} - \frac{1}{4}\cos^2 2x$$

$$= \frac{1}{4} - \frac{1}{4}\left(\frac{1 + \cos 4x}{2}\right) = \frac{1}{4} - \frac{1}{8} - \frac{\cos 4x}{8}$$

$$= \frac{1}{8} - \frac{1}{8}\cos 4x = \frac{1}{8}(1 - \cos 4x)$$

Example 2 – *Solution*

cont'd

Another way to obtain this identity is to use the Double-Angle Formula for Sine in the form $\sin x \cos x = \frac{1}{2} \sin 2x$. Thus

$$\sin^2 x \cos^2 x = \frac{1}{4} \sin^2 2x = \frac{1}{4}\left(\frac{1 - \cos 4x}{2}\right)$$

$$= \frac{1}{8}(1 - \cos 4x)$$

Half-Angle Formulas

HALF-ANGLE FORMULAS

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

The choice of the + or − sign depends on the quadrant in which $u/2$ lies.

Example 3 – *Using a Half-Angle Formula*

Find the exact value of sin 22.5°.

Solution:

Since 22.5° is half of 45°, we use the Half-Angle Formula for Sine with *u* = 45°. We choose the + sign because 22.5° is in the first quadrant:

$$\sin\frac{45^\circ}{2} = \sqrt{\frac{1 - \cos 45^\circ}{2}} \qquad \text{Half-Angle Formula}$$

$$= \sqrt{\frac{1 - \sqrt{2}/2}{2}} \qquad \cos 45^\circ = \sqrt{2}/2$$

Example 3 – *Solution*

cont'd

$$= \sqrt{\frac{2 - \sqrt{2}}{4}}$$ Common denominator

$$= \tfrac{1}{2}\sqrt{2 - \sqrt{2}}$$ Simplify

Evaluating Expressions Involving Inverse Trigonometric Functions

Evaluating Expressions Involving Inverse Trigonometric Functions

Expressions involving trigonometric functions and their inverses arise in calculus. In the next example we illustrate how to evaluate such expressions.

Example 4 – *Evaluating an Expression Involving Inverse Trigonometric Functions*

Evaluate $\sin 2\theta$, where $\cos \theta = -\frac{2}{5}$ with θ in Quadrant II.

Solution :

We first sketch the angle θ in standard position with terminal side in Quadrant II as in Figure 2.

Since $\cos \theta = x/r = -\frac{2}{5}$,
we can label a side and the hypotenuse of the triangle in Figure 2.

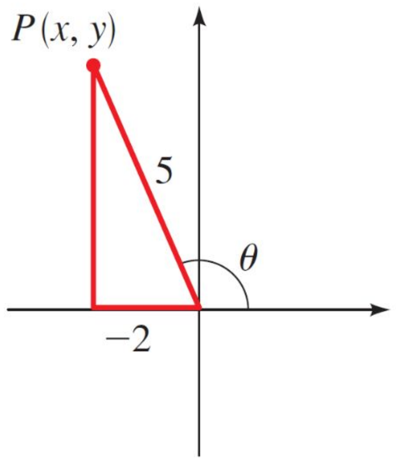


Figure 2

To find the remaining side, we use the Pythagorean Theorem.

Example 4 – *Solution*

cont'd

$x^2 + y^2 = r^2$	Pythagorean Theorem
$(-2)^2 + y^2 = 5^2$	$x = -2$, $r = 5$
$y = \pm\sqrt{21}$	Solve for y^2
$y = +\sqrt{21}$	Because $y > 0$

We can now use the Double-Angle Formula for Sine.

$\sin 2\theta = 2 \sin \theta \cos \theta$	Double-Angle Formula
$= 2\left(\frac{\sqrt{21}}{5}\right)\left(-\frac{2}{5}\right)$	From the triangle
$= -\frac{4\sqrt{21}}{25}$	Simplify

Product-Sum Formulas

It is possible to write the product sin u cos$v$ as a sum of trigonometric functions. To see this, consider the Addition and Subtraction Formulas for Sine:

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

Adding the left- and right-hand sides of these formulas gives

$$\sin(u + v) = \sin(u - v) = 2 \sin u \cos v$$

Product-Sum Formulas

Dividing by 2 gives the formula

$$\sin u \cos v = \tfrac{1}{2}[\sin(u + v) + \sin(u - v)]$$

The other three **Product-to-Sum Formulas** follow from the Addition Formulas in a similar way.

PRODUCT-TO-SUM FORMULAS

$$\sin u \cos v = \tfrac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \tfrac{1}{2}[\sin(u + v) - \sin(u - v)]$$

$$\cos u \cos v = \tfrac{1}{2}[\cos(u + v) + \cos(u - v)]$$

$$\sin u \sin v = \tfrac{1}{2}[\cos(u - v) - \cos(u + v)]$$

Product-Sum Formulas

The Product-to-Sum Formulas can also be used as Sum-to-Product Formulas. This is possible because the right-hand side of each Product-to-Sum Formula is a sum and the left side is a product. For example, if we let

$$u = \frac{x + y}{2} \qquad \text{and} \qquad v = \frac{x - y}{2}$$

in the first Product-to-Sum Formula, we get

$$\sin \frac{x + y}{2} \cos \frac{x - y}{2} = \tfrac{1}{2}(\sin x + \sin y)$$

so

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

Product-Sum Formulas

The remaining three of the following **Sum-to-Product Formulas** are obtained in a similar manner.

SUM-TO-PRODUCT FORMULAS

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

Example 5 – *Proving an Identity*

Verify the identity $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$.

Solution:

We apply the second Sum-to-Product Formula to the numerator and the third formula to the denominator.

$$\text{LHS} = \frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \frac{2\cos\dfrac{3x + x}{2}\sin\dfrac{3x - x}{2}}{2\cos\dfrac{3x + x}{2}\cos\dfrac{3x - x}{2}}$$

Sum-to-Product Formulas

Example 5 – *Solution*

cont'd

$$= \frac{2 \cos 2x \sin x}{2 \cos 2x \cos x}$$

Simplify

$$= \frac{\sin x}{\cos x}$$

Cancel

$$= \tan x$$

$$= \text{RHS}$$

7.4 Basic Trigonometric Equations

Basic Trigonometric Equations

An equation that contains trigonometric functions is called a **trigonometric equation**. For example, the following are trigonometric equations:

$\sin^2 \theta + \cos^2 \theta = 1$ $\quad$ $2 \sin\theta - 1 = 0$ $\quad$ $\tan 2\theta - 1 = 0$

The first equation is an *identity*—that is, it is true for every value of the variable θ. The other two equations are true only for certain values of θ.

To solve a trigonometric equation, we find all the values of the variable that make the equation true.

Basic Trigonometric Equations

Solving any trigonometric equation always reduces to solving a **basic trigonometric equation**—an equation of the form $T(\theta) = c$, where T is a trigonometric function and c is a constant.

In the next examples we solve such basic equations.

Example 1 – *Solving a Basic Trigonometric Equation*

Solve the equation $\sin \theta = \dfrac{1}{2}$.

Solution:

Find the solutions in one period. Because sine has period 2π, we first find the solutions in any interval of length 2π. To find these solutions, we look at the unit circle in Figure 1.

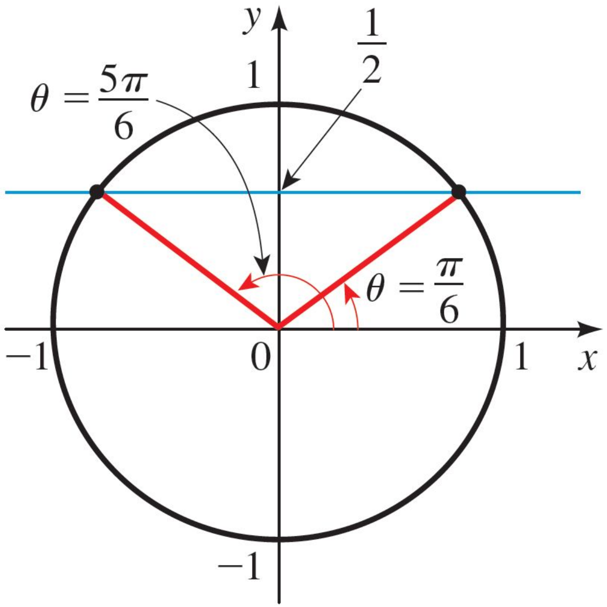


Figure 1

Example 1 – *Solution*

cont'd

We see that $\sin\theta = \frac{1}{2}$ in Quadrants I and II, so the solutions in the interval $[0, 2\pi)$ are

$$\theta = \frac{\pi}{6} \qquad \theta = \frac{5\pi}{6}$$

Find all solutions. Because the sine function repeats its values every 2π units, we get all solutions of the equation by adding integer multiples of 2π to these solutions:

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi$$

where *k* is any integer.

Example 1 – *Solution*

cont'd

Figure 2 gives a graphical representation of the solutions.

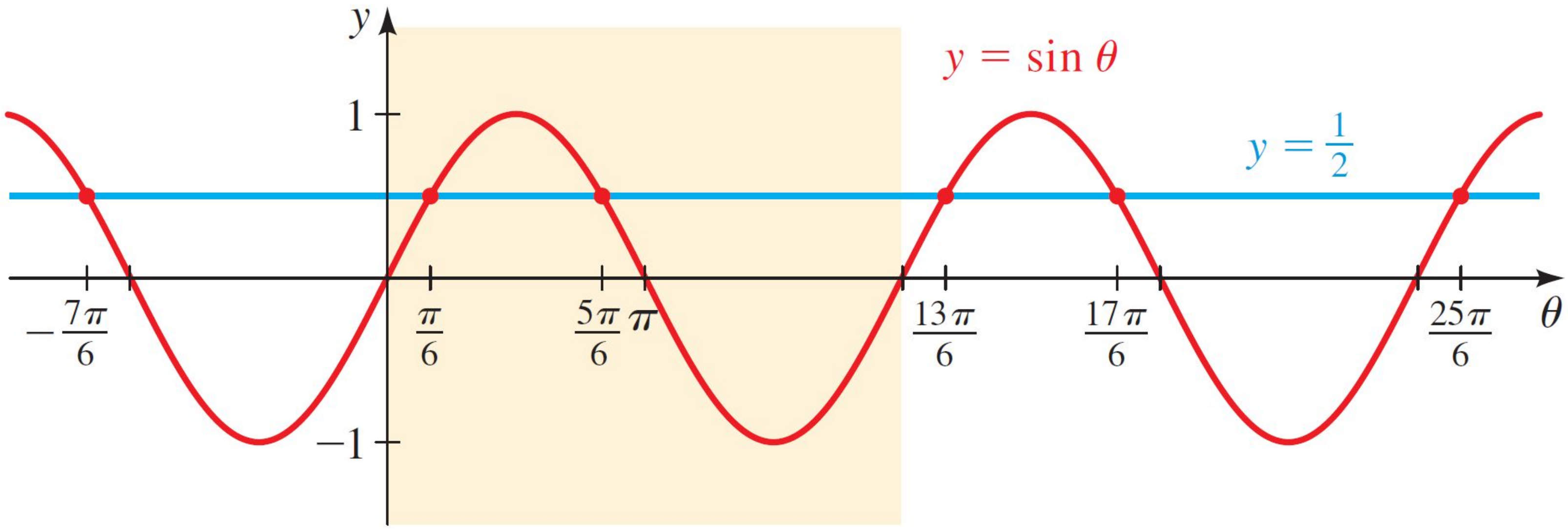


Figure 2

Example 2 – *Solving a Basic Trigonometric Equation*

Solve the equation $\tan \theta = 2$.

Solution:

Find the solutions in one period. We first find one solution by taking $\tan^{-1}$ of each side of the equation.

$$\tan \theta = 2 \qquad \text{Given equation}$$

$$\theta = \tan^{-1}(2) \qquad \text{Take } \tan^{-1} \text{ of each side}$$

$$\theta \approx 1.12 \qquad \text{Calculator (in radian mode)}$$

Example 2 – *Solution*

cont'd

By the definition of $\tan^{-1}$ the solution that we obtained is the only solution in the interval $(-\pi/2, \pi/2)$ (which is an interval of length π).

Find all solutions. Since tangent has period π, we get all solutions of the equation by adding integer multiples of π :

$$\theta \approx 1.12 + k\pi$$

where k is any integer.

Example 2 – *Solution*

cont'd

A graphical representation of the solutions is shown in Figure 6.

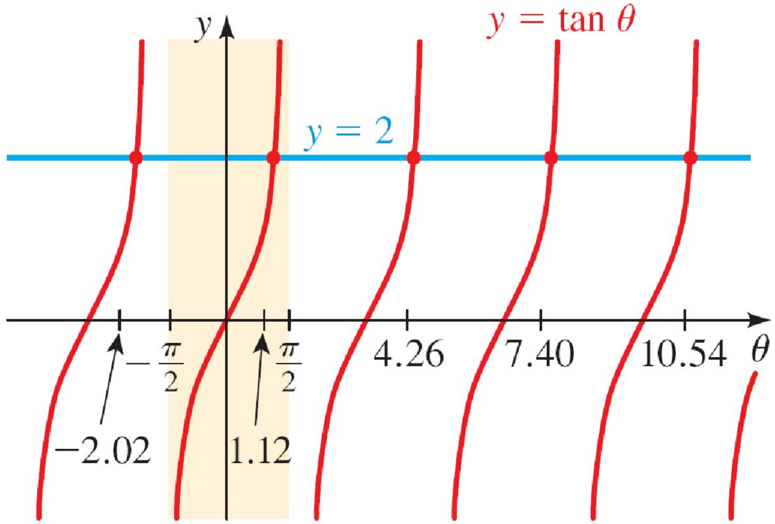


Figure 6

You can check that the solutions shown in the graph correspond to $k = -1, 0, 1, 2, 3$.

Basic Trigonometric Equations

In the next example we solve trigonometric equations that are algebraically equivalent to basic trigonometric equations.

Example 3 – *Solving Trigonometric Equations*

Find all solutions of the equation.

(a) $2 \sin \theta - 1 = 0$ **(b)** $\tan^2 \theta - 3 = 0$

Solution:

(a) We start by isolating $\sin \theta$.

$$2 \sin \theta - 1 = 0$$ Given equation

$$2 \sin \theta = 1$$ Add 1

$$\sin \theta = \frac{1}{2}$$ Divide by 2

Example 3 – *Solution*

cont'd

This last equation is the same as that in Example 1. The solutions are

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi$$

where *k* is any integer.

(b) We start by isolating tan θ.

$$\tan^2 \theta - 3 = 0 \qquad \text{Given equation}$$

$$\tan^2 \theta = 3 \qquad \text{Add 3}$$

$$\tan \theta = \pm\sqrt{3} \qquad \text{Take the square root}$$

Example 3 – *Solution*

cont'd

Because tangent has period π, we first find the solutions in any interval of length π. In the interval $(-\pi/2, \pi/2)$ the solutions are $\theta = \pi/3$ and $\theta = -\pi/3$.

To get all solutions, we add integer multiples of π to these solutions:

$$\theta = \frac{\pi}{3} + k\pi \qquad\qquad \theta = -\frac{\pi}{3} + k\pi$$

where k is any integer.

Solving Trigonometric Equations by Factoring

Solving Trigonometric Equations by Factoring

Factoring is one of the most useful techniques for solving equations, including trigonometric equations.

The idea is to move all terms to one side of the equation, factor, and then use the Zero-Product Property.

Example 4 – *A Trigonometric Equation of Quadratic Type*

Solve the equation $2\cos^2\theta - 7\cos\theta + 3 = 0$.

Solution:

We factor the left-hand side of the equation.

$$2\cos^2\theta - 7\cos\theta + 3 = 0$$ Given equation

$$(2\cos\theta - 1)(\cos\theta - 3) = 0$$ Factor

$$2\cos\theta - 1 = 0 \quad \text{or} \quad \cos\theta - 3 = 0$$ Set each factor equal to 0

$$\cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = 3$$ Solve for $\cos\theta$

Example 4 – *Solution*

cont'd

Because cosine has period 2π, we first find the solutions in the interval $[0, 2\pi)$. For the first equation the solutions are $\theta = \pi/3$ and $\theta = 5\pi/3$ (see Figure 7).

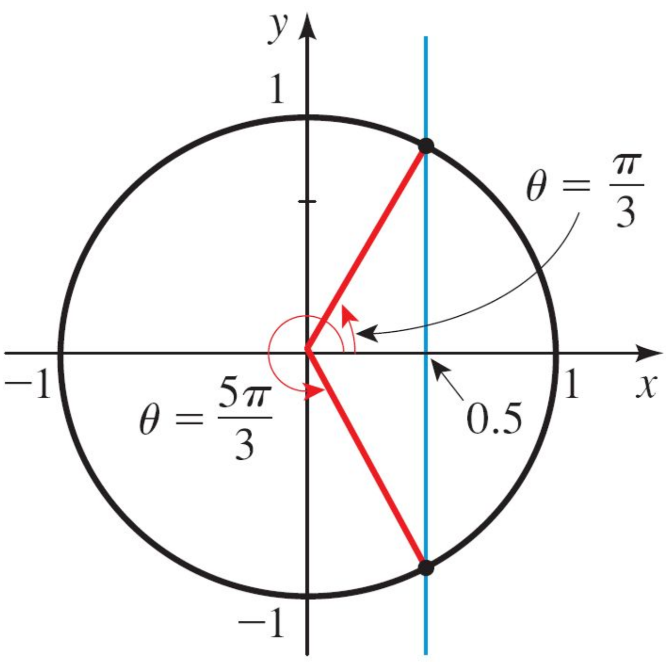


Figure 7

Example 4 – *Solution*

cont'd

The second equation has no solution because cos θ is never greater than 1.

Thus the solutions are

$$\theta = \frac{\pi}{3} + 2k\pi \qquad \theta = \frac{5\pi}{3} + 2k\pi$$

where k is any integer.

Example 5 – *Solving a Trigonometric Equation by Factoring*

Solve the equation $5 \sin \theta \cos \theta + 4 \cos \theta = 0$.

Solution:

We factor the left-hand side of the equation.

$$5 \sin \theta \cos \theta + 2 \cos \theta = 0$$ Given equation

$$\cos \theta \, (5 \sin \theta + 2) = 0$$ Factor

$$\cos \theta = 0 \quad \text{or} \quad 5 \sin \theta + 4 = 0$$ Set each factor equal to 0

$$\sin \theta = -0.8$$ Solve for $\sin \theta$

Example 5 – *Solution*

cont'd

Because sine and cosine have period 2π, we first find the solutions of these equations in an interval of length 2π.

For the first equation the solutions in the interval $[0, 2\pi)$ are $\theta = \pi/2$ and $\theta = 3\pi/2$. To solve the second equation, we take $\sin^{-1}$ of each side.

$$\sin \theta = -0.80 \qquad \text{Second equation}$$

$$\theta = \sin^{-1}(-0.80) \qquad \text{Take } \sin^{-1} \text{ of each side}$$

Example 5 – *Solution*

cont'd

$\theta \approx -0.93$ Calculator (in radian mode)

So the solutions in an interval of length 2π are $\theta = -0.93$ and $\theta = \pi + 0.93 \approx 4.07$ (see Figure 8).

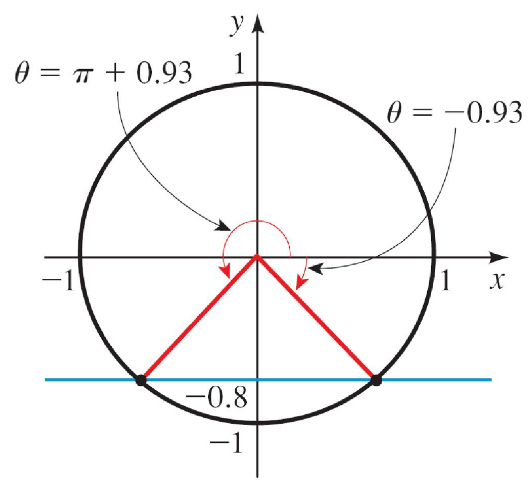


Figure 8

Example 5 – *Solution*

cont'd

We get all the solutions of the equation by adding integer multiples of 2π to these solutions.

$$\theta = \frac{\pi}{2} + 2k\pi \qquad \theta = \frac{3\pi}{2} + 2k\pi$$

$$\theta \approx -0.93 + 2k\pi \qquad \theta \approx 4.07 + 2k\pi$$

where *k* is any integer.

7.5 More Trigonometric Equations

More Trigonometric Equations

In this section we solve trigonometric equations by first using identities to simplify the equation. We also solve trigonometric equations in which the terms contain multiples of angles.

Solving Trigonometric Equations by Using Identities

Solving Trigonometric Equations by Using Identities

In the next example we use trigonometric identities to express a trigonometric equation in a form in which it can be factored.

Example 1 – *Using a Trigonometric Identity*

Solve the equation $1 + \sin\theta = 2\cos^2\theta$.

Solution:

We first need to rewrite this equation so that it contains only one trigonometric function. To do this, we use a trigonometric identity:

$$1 + \sin\theta = 2\cos^2\theta \qquad \text{Given equation}$$

$$1 + \sin\theta = 2(1 - \sin^2\theta) \qquad \text{Pythagorean identity}$$

$$2\sin^2\theta + \sin\theta - 1 = 0 \qquad \text{Put all terms on one side}$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0 \qquad \text{Factor}$$

Example 1 – *Solution*

cont'd

$2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$ — Set each factor equal to 0

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$ — Solve for $\sin\theta$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\theta = \frac{3\pi}{2}$ — Solve for θ in the interval $[0, 2\pi)$

Because sine has period 2π, we get all the solutions of the equation by adding integer multiples of 2π to these solutions.

Example 1 – *Solution*

cont'd

Thus the solutions are

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi \qquad \theta : \frac{3\pi}{2} + 2k\pi$$

where *k* is any integer.

Example 2 – *Squaring and Using an Identity*

Solve the equation $\cos\theta + 1 = \sin\theta$ in the interval $[0, 2\pi)$.

Solution:

To get an equation that involves either sine only or cosine only, we square both sides and use a Pythagorean identity.

$\cos\theta + 1 = \sin\theta$	Given equation
$\cos^2\theta + 2\cos\theta + 1 = \sin^2\theta$	Square both sides
$\cos^2\theta + 2\cos\theta + 1 = 1 - \cos^2\theta$	Pythagorean identity
$2\cos^2\theta + 2\cos\theta = 0$	Simplify

Example 2 – *Solution*

cont'd

$2 \cos \theta (\cos \theta + 1) = 0$			Factor
$2 \cos\theta = 0$	or	$\cos\theta + 1 = 0$	Set each factor equal to 0
$\cos\theta = 0$	or	$\cos\theta = -1$	Solve for $\cos\theta$
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$	or	$\theta = \pi$	Solve for θ in $[0, 2\pi)$

Because we squared both sides, we need to check for extraneous solutions. From *Check Your Answers* we see that the solutions of the given equation are $\pi/2$ and π.

Example 2 – *Solution*

cont'd

Check Your Answers:

$\theta = \frac{\pi}{2}$ $\theta = \frac{3\pi}{2}$ $\theta = \pi$

$\cos \frac{\pi}{2} + 1 = \sin \frac{\pi}{2}$ $\cos \frac{3\pi}{2} + 1 = \sin \frac{3\pi}{2}$ $\cos \pi + 1 = \sin \pi$

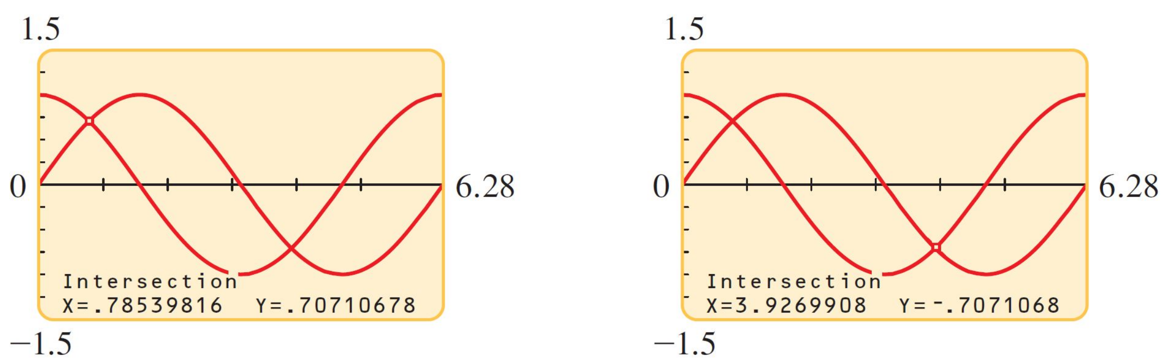
$0 + 1 = 1$ ✔ $0 + 1 \stackrel{?}{=} -1$ ✗ $-1 + 1 = 0$ ✔

Example 3 – *Finding Intersection Points*

Find the values of x for which the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ intersect.

Solution 1: Graphical

The graphs intersect where $f(x) = g(x)$. In Figure 1 we graph $y_1 = \sin x$ and $y_2 = \cos x$ on the same screen, for x between 0 and 2π.



(a) (b)

Figure 1

Example 3 – *Solution*

cont'd

Using TRACE or the intersect command on the graphing calculator, we see that the two points of intersection in this interval occur where $x \approx 0.785$ and $x \approx 3.927$.

Since sine and cosine are periodic with period 2π, the intersection points occur where

$$x \approx 0.785 + 2k\pi \qquad \text{and} \qquad x \approx 3.927 + 2k\pi$$

where k is any integer.

Example 3 – *Solution*

cont'd

Solution 2: Algebraic

To find the exact solution, we set $f(x) = g(x)$ and solve the resulting equation algebraically:

$$\sin x = \cos x \qquad \text{Equate functions}$$

Since the numbers x for which $\cos x = 0$ are not solutions of the equation, we can divide both sides by $\cos x$:

$$\frac{\sin x}{\cos x} = 1 \qquad \text{Divide by } \cos x$$

$$\tan x = 1 \qquad \text{Reciprocal identity}$$

Example 3 – *Solution*

cont'd

The only solution of this equation in the interval $(-\pi/2, \pi/2)$ is $x = \pi/4$. Since tangent has period π, we get all solutions of the equation by adding integer multiples of π:

$$x = \frac{\pi}{4} + k\pi$$

where k is any integer. The graphs intersect for these values of x.

You should use your calculator to check that, rounded to three decimals, these are the same values that we obtained in Solution 1.

Equations with Trigonometric Functions of Multiples of Angles

When solving trigonometric equations that involve functions of multiples of angles, we first solve for the multiple of the angle, then divide to solve for the angle.

Example 4 – *A Trigonometric Equation Involving a Multiple of an Angle*

Consider the equation $2 \sin 3\theta - 1 = 0$.

(a) Find all solutions of the equation.

(b) Find the solutions in the interval $[0, 2\pi)$.

Solution:

(a) We first isolate $\sin 3\theta$ and then solve for the angle 3θ.

$2 \sin 3\theta - 1 = 0$	Given equation
$2 \sin 3\theta = 1$	Add 1
$\sin 3\theta = \frac{1}{2}$	Divide by 2

Example 4 – *Solution*

cont'd

$$3\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Solve for 3θ in the interval $[0, 2\pi)$ (see Figure 2)

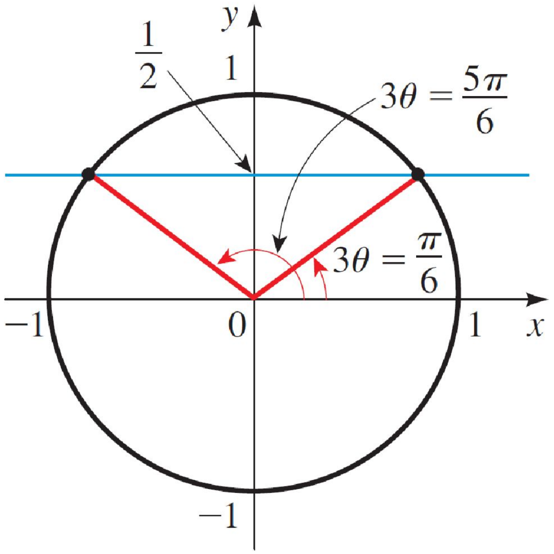


Figure 2

Example 4 – *Solution*

cont'd

To get all solutions, we add integer multiples of 2π to these solutions. So the solutions are of the form

$$3\theta = \frac{\pi}{6} + 2k\pi \qquad 3\theta = \frac{5\pi}{6} + 2k\pi$$

To solve for θ, we divide by 3 to get the solutions

$$\theta = \frac{\pi}{18} + \frac{2k\pi}{3} \qquad \theta = \frac{5\pi}{18} + \frac{2k\pi}{3}$$

where k is any integer.

Example 4 – *Solution*

cont'd

(b) The solutions from part (a) that are in the interval $[0, 2\pi)$ correspond to $k = 0$, 1, and 2. For all other values of k the corresponding values of θ lie outside this interval.

So the solutions in the interval $[0, 2\pi)$ are

$$\theta = \underbrace{\frac{\pi}{18}, \frac{5\pi}{18}}_{k=0}, \underbrace{\frac{13\pi}{18}, \frac{17\pi}{18}}_{k=1}, \underbrace{\frac{25\pi}{18}, \frac{29\pi}{18}}_{k=2}$$

Example 5 – *A Trigonometric Equation Involving a Half Angle*

Consider the equation $\sqrt{3}\tan\dfrac{\theta}{2} - 1 = 0$.

(a) Find all solutions of the equation.

(b) Find the solutions in the interval $[0, 4\pi)$.

Solution:

(a) We start by isolating $\tan\dfrac{\theta}{2}$.

$$\sqrt{3}\tan\frac{\theta}{2} - 1 = 0 \qquad \text{Given equation}$$

$$\sqrt{3}\tan\frac{\theta}{2} = 1 \qquad \text{Add 1}$$

Example 5 – *Solution*

cont'd

$$\tan\frac{\theta}{2} = \frac{1}{\sqrt{3}}$$ Divide by $\sqrt{3}$

$$\frac{\theta}{2} = \frac{\pi}{6}$$ Solve for $\frac{\theta}{2}$ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Since tangent has period π, to get all solutions, we add integer multiples of π to this solution. So the solutions are of the form

$$\frac{\theta}{2} = \frac{\pi}{6} + k\pi$$

Example 5 – *Solution*

cont'd

Multiplying by 2, we get the solutions

$$\theta = \frac{\pi}{3} + 2k\pi$$

where k is any integer.

(b) The solutions from part (a) that are in the interval $[0, 4\pi)$ correspond to $k = 0$ and $k = 1$. For all other values of k the corresponding values of x lie outside this interval. Thus the solutions in the interval $[0, 4\pi)$ are

$$x = \frac{\pi}{3}, \frac{7\pi}{3}$$