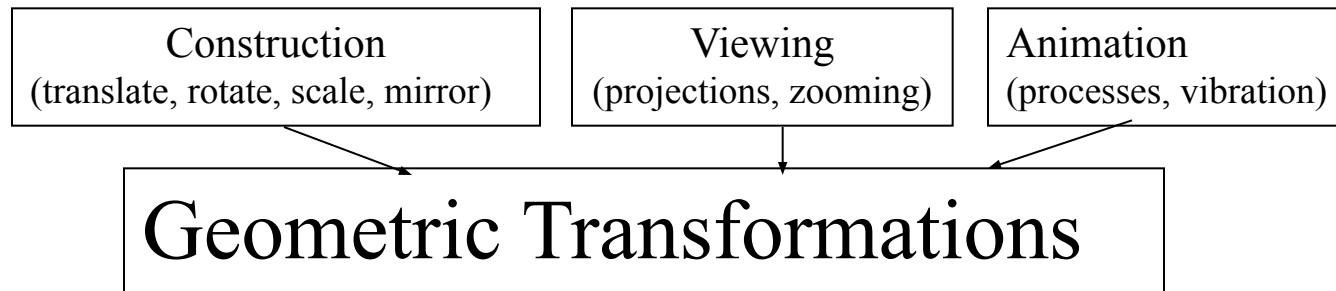


Geometric Transformations

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Spring, 2018 AUA

Intro & General Information



General Information

Transformation of a point is basic in GT. It can be formulated as follows:
Given a point P that belongs to a geometric model find the corresponding point P in the new position such that*

$$\mathbf{P}^* = f(\mathbf{P}, \text{transformation parameters})$$

- The transformation parameters should provide ONE-TO-ONE-MAPPING.
- Multiple transformations can be combined to yield a single transformation which should have the same effect as the sequential application of original ones. CONCATENATION /kənˌkɑtənˈeɪʃ(ə)n/

Equation of \mathbf{P}^* for graphics hardware should be in matrix notation:

$$\mathbf{P}^* = [\mathbf{T}]\mathbf{P},$$

where $[\mathbf{T}]$ is the transformation matrix.

Translation

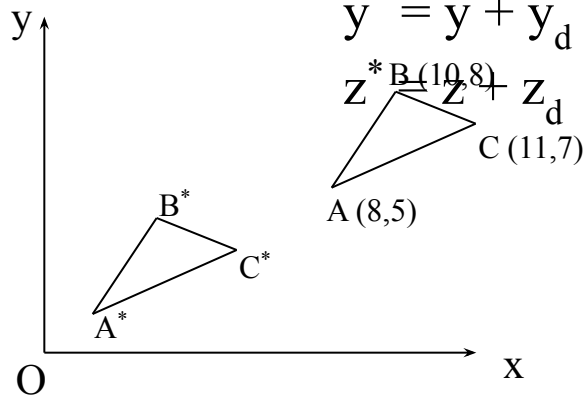
Translation is a rigid-body transformation (Euclidean) when each entity of the model remains parallel, or each point moves an equal distance in a given direction:

$\mathbf{P}^* = \mathbf{P} + \mathbf{d}$ (for both 2D and 3D). In a scalar form (for 3D):

$$x^* = x + x_d$$

$$y^* = y + y_d$$

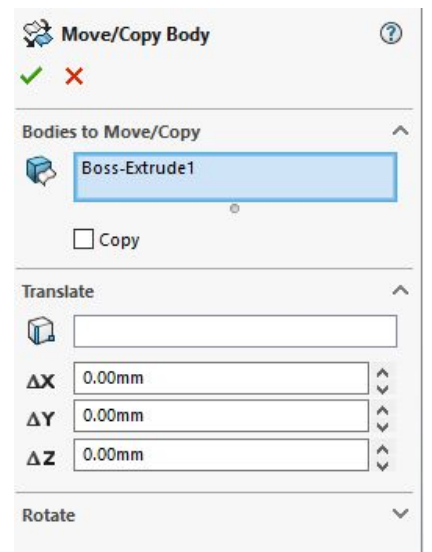
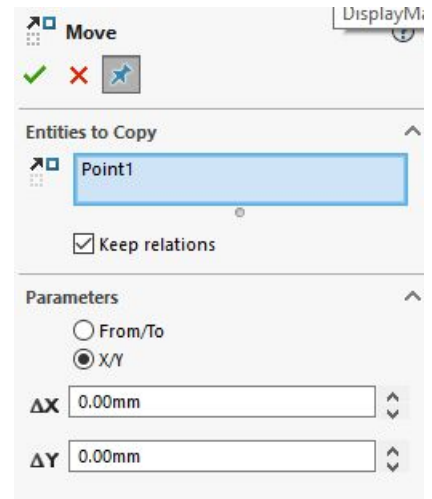
$$z^* = z + z_d$$



Question: Find the coordinates of vertices A^* , B^* , and C^* of the translated triangle.

The distance vector of translation: $\mathbf{D} = [-7 \ -4]^T$.

Verify that the lengths of the edges are unchanged.



Scaling

Scaling is used to change the size of an entity or a model.

$$\mathbf{P}^* = [\mathbf{S}]\mathbf{P}$$

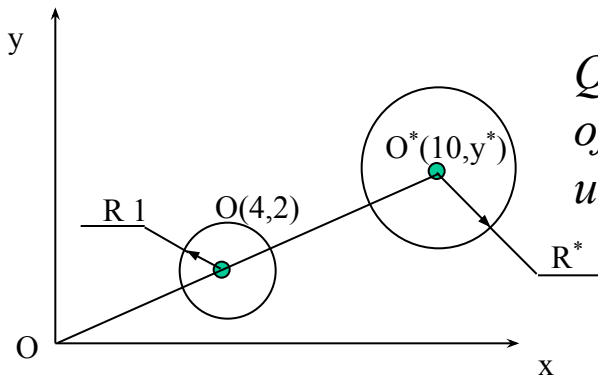
For general case
$$[\mathbf{S}] = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix},$$

where s_x , s_y , and s_z are the scaling factors in the X, Y, and Z directions respectively.

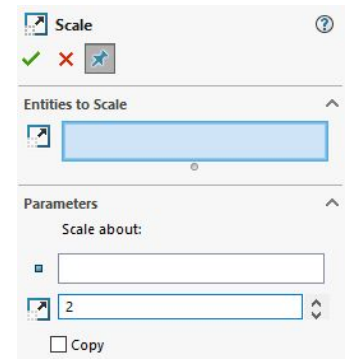
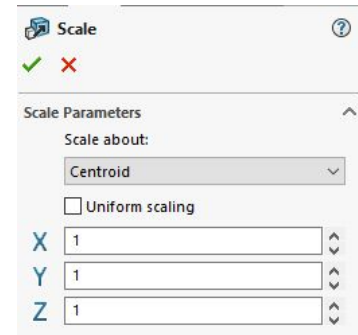
If $0 < s < 1$ - compression

If $s > 1$ - stretching

$s_x = s_y = s_z$ - uniform scaling, otherwise - non-uniform



Question: The larger circle is the scaled copy of the smaller one. Can you say that we have a uniform scaling? Why? Define y^ and R^* .*



Mirror

Plane* \Rightarrow Negate the corresponding coordinate
 Mirror through $\left\{ \begin{array}{l} \text{Line}^* \Rightarrow \text{Reflect through 2 planes intersecting at the axis} \\ \text{Point}^* \Rightarrow \text{Reflect through 3 planes intersecting at the point} \end{array} \right.$
 * plane - principal plane, line - X, Y, or Z axes, point - CS origin

$$\mathbf{P}^* = [\mathbf{M}]\mathbf{P},$$

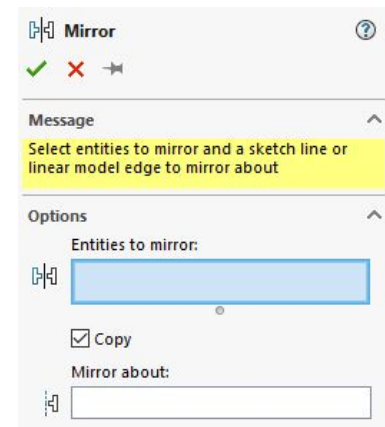
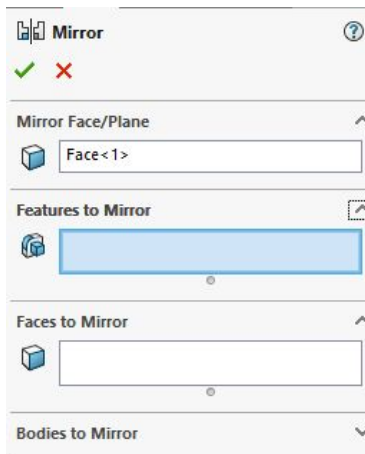
where

$$[\mathbf{M}] = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{pmatrix} \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

Question: Define the signs (in the matrix) for the reflections (mirroring) through:

- $x = 0, y = 0, z = 0$ planes
- X, Y, and Z axes
- the CS origin

*



Rotation

Rotation is a non-commutative transformation (depends on sequence).

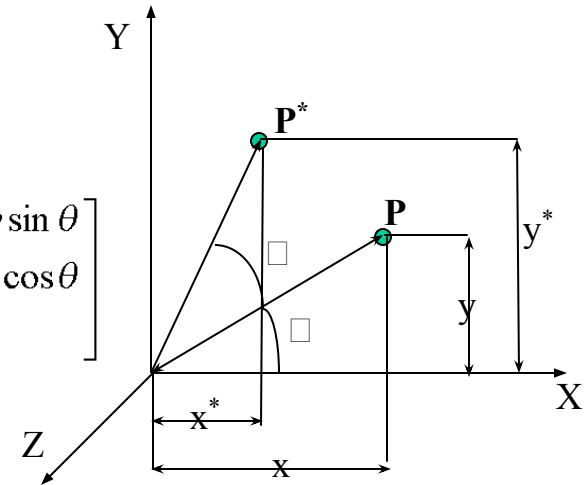
$$P^* = [R] \cdot P \quad P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \cdot \cos \alpha \\ r \cdot \sin \alpha \\ z \end{bmatrix}$$

$$P^* = \begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} = \begin{bmatrix} r \cdot \cos(\alpha + \theta) \\ r \cdot \sin(\alpha + \theta) \\ z \end{bmatrix} = \begin{bmatrix} r \cdot \cos \alpha \cos \theta - r \cdot \sin \alpha \sin \theta \\ r \cdot \cos \alpha \sin \theta + r \cdot \sin \alpha \cos \theta \\ z \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \\ z \end{bmatrix}$$

$$[R_z] = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$[R_y] = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$



Question: Let the length of a major and minor axes of an ellipse with the center on the origin of the CS be $2a$ and $2b$ respectively, and α - the angle between the major axis and the x-axis. Then, derive the expression of an ellipse in the (O,x,y) system.

Homogeneous Transformation - 1

When we scale then rotate, the transformed image is given by:

$$\mathbf{P}^* = ([R][S])\mathbf{P}$$

where $[S]$, $[R]$, $[R]$ $[S]$ are 3x3 transformation matrices. This is not the case for a translation ($\mathbf{P}^* = \mathbf{P} + \mathbf{d}$). The goal is to find a $[D]$ such that

$$\mathbf{P} + \mathbf{d} = [D]\mathbf{P}$$

in order to perform valid matrix multiplication.

This is found by using a homogeneous coordinates.

Homogeneous Transformation maps n-dimensional space into (n+1)- dim.

3D representation of the point vector - $\mathbf{P} = [x, y, z]^T$

Homogeneous rep. of the same vector - $\mathbf{P} = [xw, yw, zw, w]^T$ where $w = 1$

Homogeneous Transformation - 2

The transformation matrices in new (homogeneous) representation:

$$[D] = \begin{bmatrix} 1 & 0 & 0 & x_d \\ 0 & 1 & 0 & y_d \\ 0 & 0 & 1 & z_d \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [S] = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

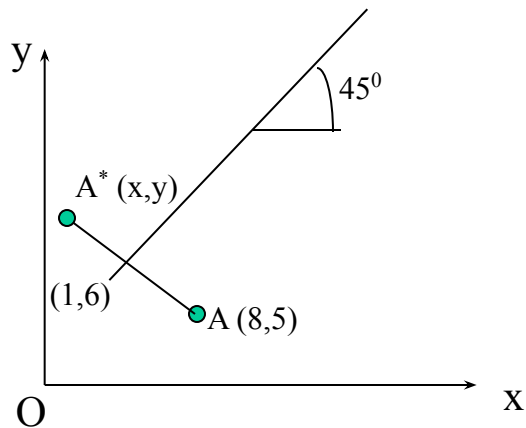
$$[M] = \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [R] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Composition of Transformations

Now we are able to include all the transformations in a single matrix. In case of composition of transformations: $\mathbf{P}^* = [T_n][T_{n-1}] \dots [T_2][T_1]\mathbf{P}$, where $[T_i]$ are different transformation matrices.

Sequence is important!

Practice: Mirror point A through the given line and find x and y.



Another example

- Scale line AB about point M by factor of 2 and then mirror new line $A'B'$ about the origin.

