Introduction to Vectors

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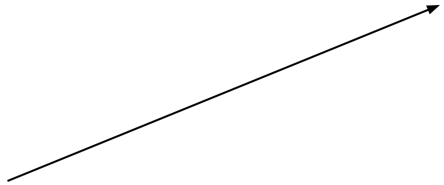
What are Vectors?

- Vectors are pairs of a direction and a magnitude. We usually represent a vector with an arrow:



- The direction of the arrow is the direction of the vector, the length is the magnitude.

Vectors in R^n

$n = 1$ R^1-space = set of all real numbers

(R^1-space can be represented geometrically by the x-axis)

$n = 2$ R^2-space = set of all ordered pair of real numbers (x_1, x_2)

(R^2-space can be represented geometrically by the xy-plane)

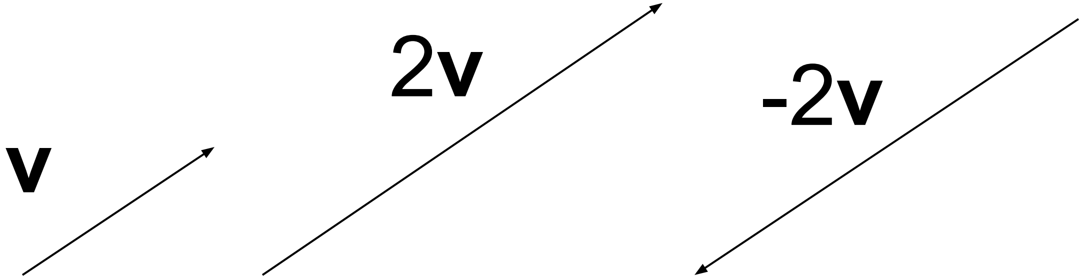
$n = 3$ R^3-space = set of all ordered triple of real numbers (x_1, x_2, x_3)

(R^3-space can be represented geometrically by the xyz-space)

$n = 4$ R^4-space = set of all ordered quadruple of real numbers (x_1, x_2, x_3, x_4)

Multiples of Vectors

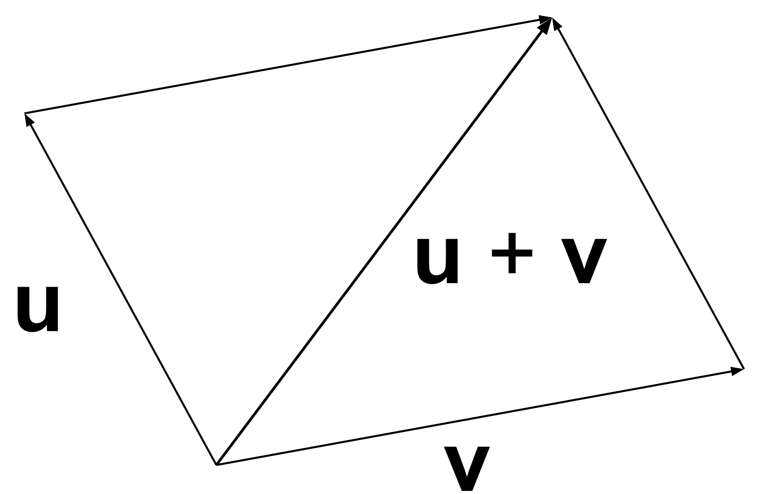
Given a real number *c*, we can multiply a vector by *c* by multiplying its magnitude by *c*:



Notice that multiplying a vector by a negative real number reverses the direction.

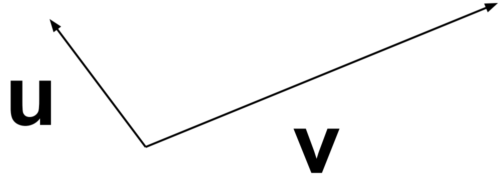
Adding Vectors

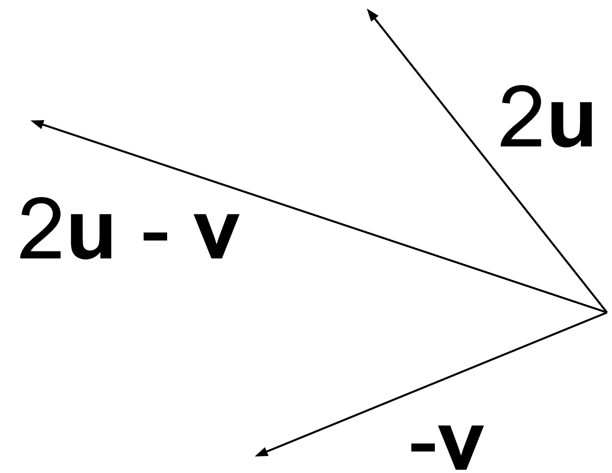
Two vectors can be added using the **Parallelogram Law**



Combinations

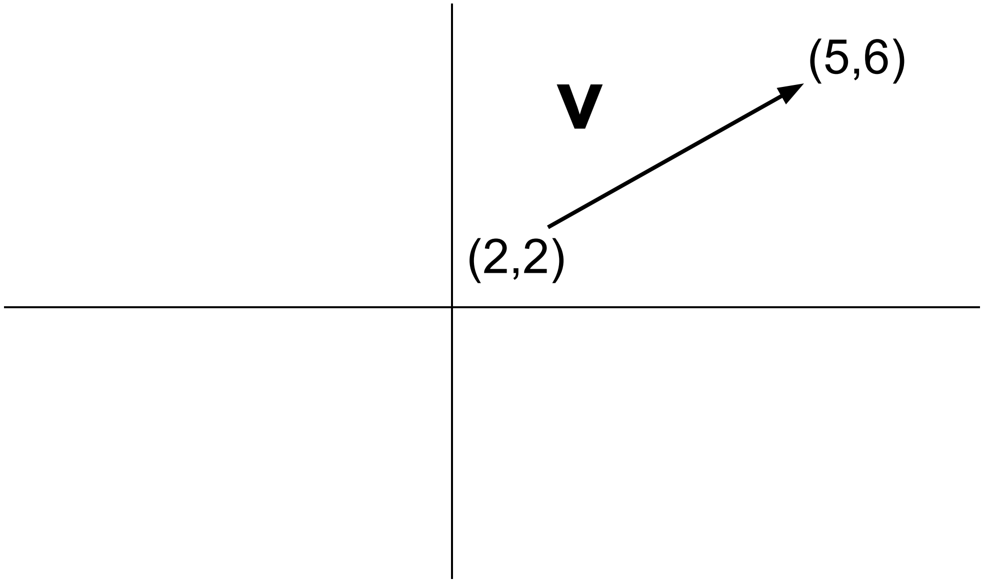
These operations can be combined.





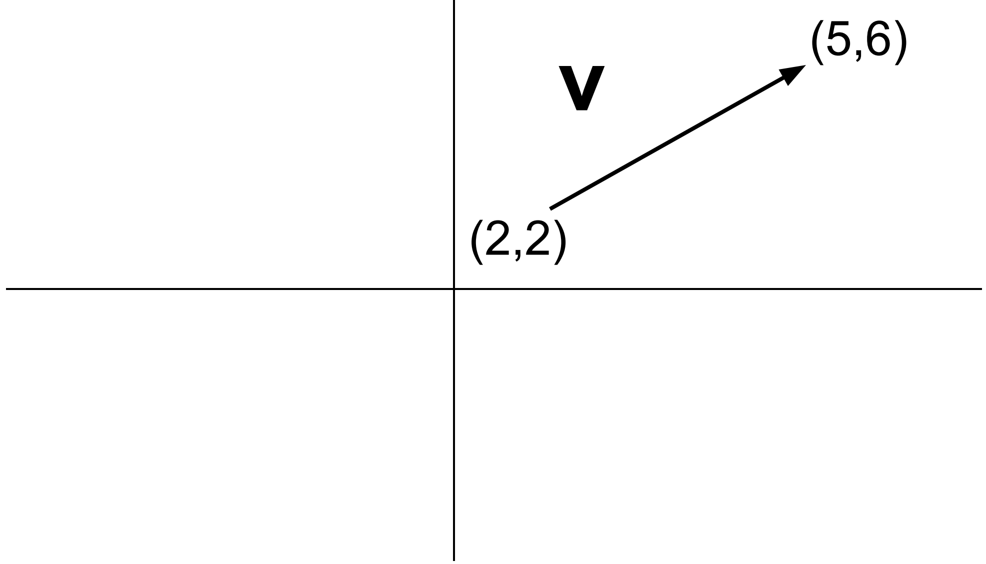
Components

To do computations with vectors, we place them in the plane and find their **components**.



Components

The initial point is the tail, the head is the terminal point. The components are obtained by subtracting coordinates of the initial point from those of the terminal point.

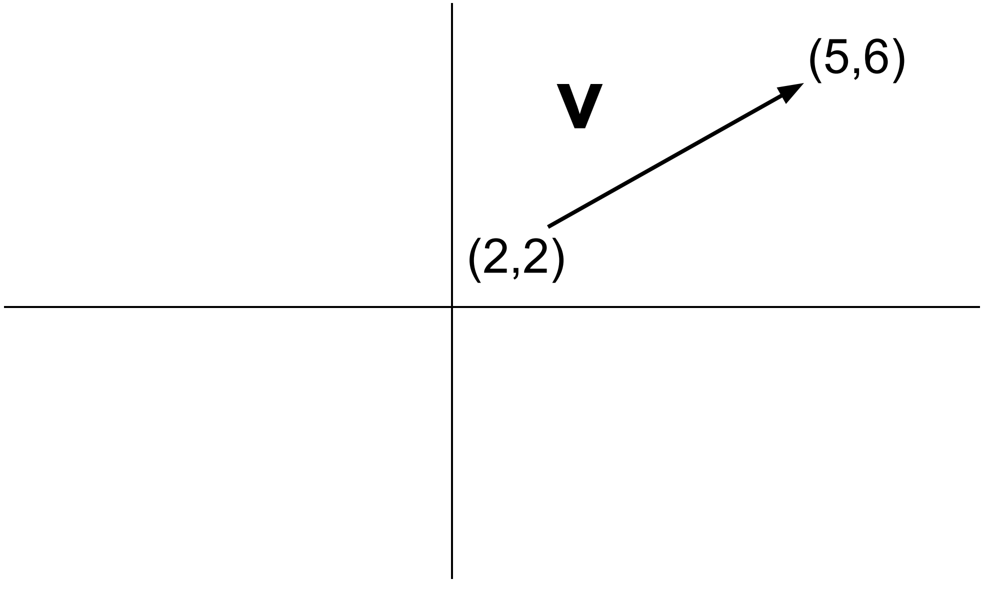


Components

The first component of **v** is 5 -2 = 3.

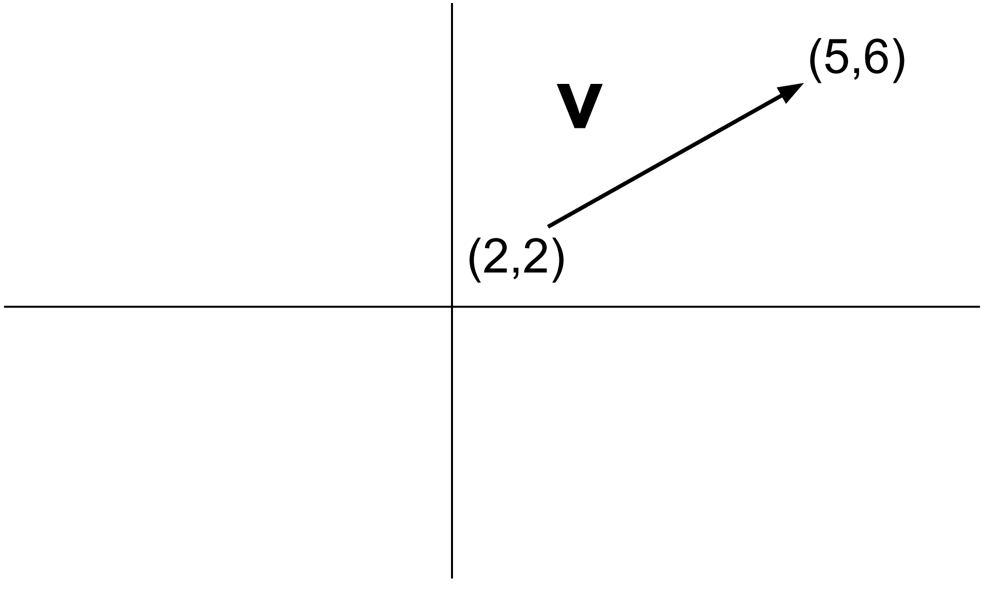
The second is 6 -2 = 4.

We write **v** = <3,4>



Magnitude

The magnitude of the vector is the length of the segment, it is written ||**v**||.



Scalar Multiplication

Once we have a vector in component form, the arithmetic operations are easy.

To multiply a vector by a real number, simply multiply each component by that number.

Example: If **v** = <3,4>, -2**v** = <-6,-8>

Addition

To add vectors, simply add their components.

For example, if **v** = <3,4> and **w** = <-2,5>, then **v** + **w** = <1,9>.

Other combinations are possible.
For example: 4**v** – 2**w** = <16,6>.

Unit Vectors

A **unit vector** is a vector with magnitude 1.

Given a vector **v**, we can form a unit vector by multiplying the vector by 1/||**v**||.

For example, find the unit vector in the direction <3,4>:

Special Unit Vectors

A vector such as <3,4> can be written as 3<1,0> + 4<0,1>.

For this reason, these vectors are given special names: **i** = <1,0> and **j** = <0,1>.

A vector in component form **v** = <a,b> can be written a**i** + b**j**.

Three dimensional space:

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a}\cdot\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1\mathbf{i}\cdot\mathbf{i} + a_1b_2\mathbf{i}\cdot\mathbf{j} + a_1b_3\mathbf{i}\cdot\mathbf{k} + a_2b_1\mathbf{j}\cdot\mathbf{i} + a_2b_2\mathbf{j}\cdot\mathbf{j} + a_2b_3\mathbf{j}\cdot\mathbf{k} + a_3b_1\mathbf{k}\cdot\mathbf{i} + a_3b_2\mathbf{k}\cdot\mathbf{j} + a_3b_3\mathbf{k}\cdot\mathbf{k}$$

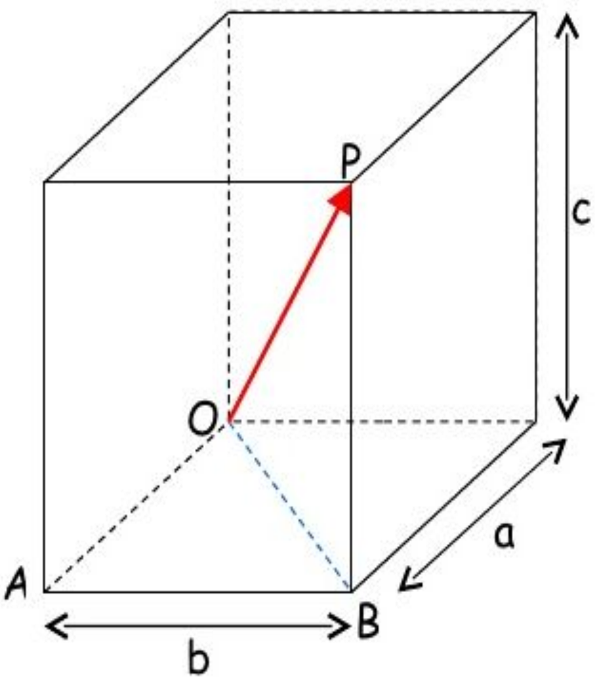
The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ have length 1 and are at 90° to each other and so any unit vector when scalar product combined with itself will give:

$$\mathbf{i}\cdot\mathbf{i} = 1 \times 1 \times \cos 0° = 1$$

Whilst any unit vector when scalar product combined with a different one will give: $\mathbf{i}\cdot\mathbf{j} = 1 \times 1 \times \cos 90° = 0$

Therefore $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Three dimensional space:



From the diagram the length of OP in terms of the side lengths can be determined as follows:

$OP^2 = OB^2 + BP^2$ and $OB^2 = OA^2 + AB^2$

Thus $OP^2 = OA^2 + AB^2 + BP^2$

$OP^2 = a^2 + b^2 + c^2$ $\quad OP = \sqrt{a^2 + b^2 + c^2}$

For our two vectors: $a = \sqrt{a_1^2 + a_2^2 + a_3^2}$ and $b = \sqrt{b_1^2 + b_2^2 + b_3^2}$

Using, $\cos\theta = \dfrac{a \cdot b}{ab} = \dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \times \sqrt{b_1^2 + b_2^2 + b_3^2}}$

Spanning Sets and Linear Independence

- Linear combination :

A vector $\mathbf{u}$ in a vector space V is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in V if $\mathbf{u}$ can be written in the form

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k,$$

where $c_1, c_2, \ldots, c_k$ are real-number scalars

- Ex : Finding a linear combination

$$\mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (0,1,2) \quad \mathbf{v}_3 = (-1,0,1)$$

Prove (a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Sol:

(a) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$(1,1,1) = c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1)$$
$$= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3)$$

$$\Rightarrow \begin{aligned} c_1 \qquad\quad - c_3 &= 1 \\ 2c_1 + \quad c_2 \qquad\quad &= 1 \\ 3c_1 + 2c_2 + c_3 &= 1 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1 + t\,,\ c_2 = -1 - 2t\,,\ c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

$$\overset{t=2}{\Rightarrow} \mathbf{w} = 3\mathbf{v}_1 - 5\mathbf{v}_2 + 2\mathbf{v}_3$$

⊠

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$\Rightarrow$ This system has no solution since the third row means

$$0 \cdot c_1 + 0 \cdot c_2 + 0 \cdot c_3 = 7$$

$\Rightarrow$ $\mathbf{w}$ can not be expressed as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

- The span of a set: span(S)

If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the span of S is the set of all linear combinations of the vectors in S,

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid \forall c_i \in R\}$$

(the set of all linear combinations of vectors in S)

- Definition of a spanning set of a vector space:

If every vector in a given vector space V can be written as a linear combination of vectors in a set S, then S is called a **spanning set** of the vector space V

- Note: The above statement can be expressed as follows

$$\text{span}(S) = V$$
$$\Leftrightarrow \ S \text{ spans (generates) } V$$
$$\Leftrightarrow \ V \text{ is spanned (generated) by } S$$
$$\Leftrightarrow \ S \text{ is a spanning set of } V$$

- Ex 4:

(a) The set $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\mathbf{u} = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$$

(b) The set $S = \{1, x, x^2\}$ spans P_2 because any polynomial function $p(x) = a + bx + cx^2$ in P_2 can be written as

$$p(x) = a(1) + b(x) + c(x^2)$$

- Ex 5: A spanning set for R^3

Show that the set $S=\{(1,2,3),(0,1,2),(-2,0,1)\}$ spans R^3

Sol:

We must determine whether an arbitrary vector $\mathbf{u}=(u_1,u_2,u_3)$ in R^3 can be expressed as a linear combination of $\mathbf{v}_1=(1,2,3)$, $\mathbf{v}_2=(0,1,2)$, and $\mathbf{v}_3=(-2,0,1)$

$$\text{If } \mathbf{u}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 \Rightarrow \begin{aligned} c_1 \qquad\quad - 2c_3 &= u_1 \\ 2c_1 + c_2 \qquad\quad &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3 \end{aligned}$$

The above problem thus reduces to determining whether this system is consistent for all values of u_1, u_2, and u_3

$$\because \quad |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

$\therefore A\mathbf{x} = \mathbf{u}$ has exactly one solution for every $\mathbf{u}$

$\Rightarrow \text{span}(S) = R^3$

Definitions of Linear Independence (L.I.) and Linear Dependence (L.D.) :

$S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_k\}$: a set of vectors in a vector space V

For $c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k=\mathbf{0}$

(1) If the equation has only the trivial solution $(c_1=c_2=\cdots=c_k=0)$ then S (or $\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_k$) is called **linearly independent**

(2) If the equation has a nontrivial solution (i.e., not all zeros), then S (or $\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_k$) is called **linearly dependent** (The name of linear dependence is from the fact that in this case, there exist a $\mathbf{v}_i$ which can be represented by the linear combination of $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{i-1},\mathbf{v}_{i+1},\ldots\ \mathbf{v}_k\}$ in which the coefficients are not all zero.

- Ex : Testing for linear independence

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}=\{(1,\ 2,\ 3),(0,\ 1,\ 2),(-2,\ 0,\ 1)\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}\ \Rightarrow\ \begin{aligned} c_1 \qquad\quad -2c_3 &= 0\\ 2c_1+\ c_2+\qquad\ &= 0\\ 3c_1+2c_2+\ c_3 &= 0\end{aligned}$$

$$\Rightarrow\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0\\ 2 & 1 & 0 & 0\\ 3 & 2 & 1 & 0\end{array}\right]\xrightarrow{\text{G.-J. E.}}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

$$\Rightarrow c_1=c_2=c_3=0\ \ (\text{only the trivial solution})$$

$$(\text{or } \det(A)=-1\neq 0,\ \text{so there is only the trivial solution})$$

$\Rightarrow S$ is (or $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ are) linearly independent

- EX: Testing for linear independence

Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}=\{1+x-2x^2, 2+5x-x^2, x+x^2\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$$

i.e., $c_1(1+x-2x^2)+c_2(2+5x-x^2)+c_3(x+x^2)=0+0x+0x^2$

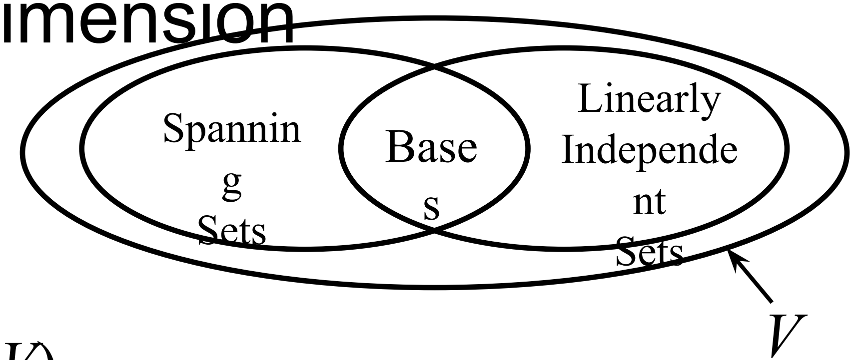
$$\begin{aligned} c_1+2c_2 \quad\quad &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1-c_2+c_3 &= 0 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G. E.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system has infinitely many solutions

(i.e., this system has nontrivial solutions, e.g., $c_1=2$, $c_2=-1$, $c_3=3$)

S is (or $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are) linearly dependent

Basis and Dimension



- Basis :

  V: a vector space

  $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ □ V

  (1) S spans V (i.e., span(S) = V)

  (For any $\mathbf{u} \in V$, $\sum c_i \mathbf{v}_i = A\mathbf{x} = \mathbf{u}$ has a solution ($\det(A) \neq 0$),

  see Ex 5 on Slide 4.44)

  (2) S is linearly independent

  (For $\sum c_i \mathbf{v}_i = A\mathbf{x} = \mathbf{0}$, there is only the trivial solution ($\det(A) \neq 0$),

  □ S is called a basis for V

- Notes:

  A basis S must have enough vectors to span V, but not so many vectors that one of them could be written as a linear combination of the other vectors in S

- Notes:

(1) the **standard basis** for R^3:

$\{i, j, k\}$ $i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1)$

(2) the **standard basis** for R^n :

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ $\mathbf{e}_1=(1,0,\ldots,0), \mathbf{e}_2=(0,1,\ldots,0),\ldots, \mathbf{e}_n=(0,0,\ldots,1)$

Ex: For R^4, $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

(3) the **standard basis** matrix space:

Ex: 2×2 matrix space:

$$\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$$

(4) the **standard basis** for $P_n(x)$:

$\{1, x, x^2, \ldots, x^n\}$

Ex: $P_3(x)$ $\{1, x, x^2, x^3\}$

- Ex 2: The nonstandard basis for R^2

Show that $S=\{\mathbf{v}_1,\mathbf{v}_2\}=\{(1,1),\ (1,-1)\}$ is a basis for R^2

$$\text{(1) For any } \mathbf{u}=(\mathrm{u}_1,\mathrm{u}_2)\in R^2,\ c_1\mathbf{v}_1+c_2\mathbf{v}_2=\mathbf{u}\ \Rightarrow \begin{cases} c_1+c_2=\mathrm{u}_1 \\ c_1-c_2=\mathrm{u}_2 \end{cases}$$

Because the coefficient matrix of this system has a **nonzero determinant**, the system has a unique solution for each **u**. Thus you can conclude that S spans R^2

$$\text{(2) For } c_1\mathbf{v}_1+c_2\mathbf{v}_2=\mathbf{0}\ \Rightarrow \begin{cases} c_1+c_2=0 \\ c_1-c_2=0 \end{cases}$$

Because the coefficient matrix of this system has a **nonzero determinant**, you know that the system has only the trivial solution. Thus you can conclude that S is linearly independent

According to the above two arguments, we can conclude that S is a (nonstandard) basis for R^2