NUFYP Mathematics

5.1 Differentiation 1

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Lecture Outline

Rates of change

- Tangent lines and slopes
- Average rate of changes
- Instantaneous rate of changes

Defining derivatives using limits

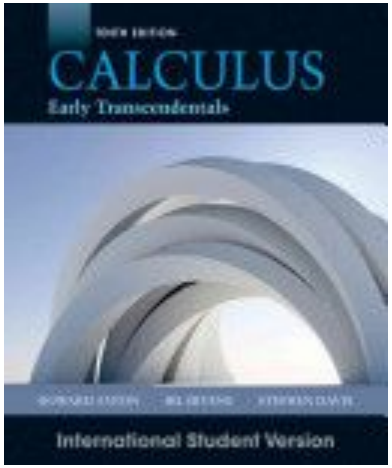
- Definition of a derivative of a function
- Differentiation using limits
- Differentiability

Differentiation rules

- Constant
- Power function
- Constant-times function
- Sum & difference
- Product
- Quotient

Reference

Anton, H., Bivens I., Davis, S.
Calculus Early Transcendentals, 10th edition



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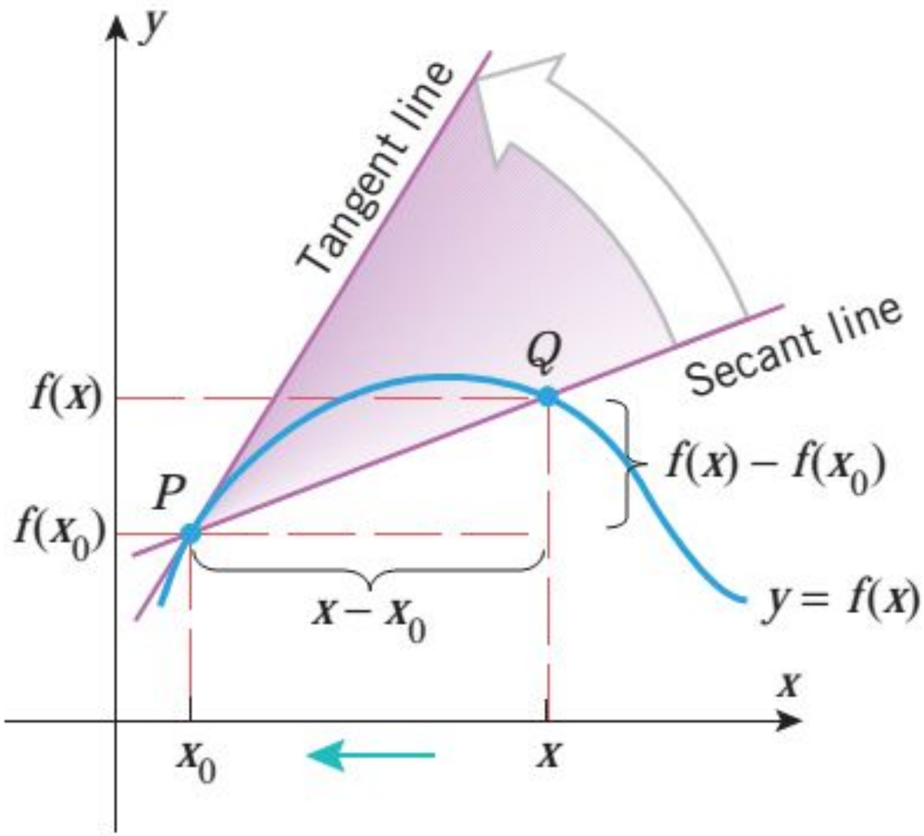
Introduction

Many real-world phenomena involve changing quantities:

- the speed of a rocket,
- the inflation of currency,
- the number of bacteria in a culture,
- the shock intensity of an earthquake,
- the voltage of an electrical signal, and so forth.

In this lecture we will develop the concept of a “derivative”, which is the mathematical tool for studying the rate at which one quantity changes relative to another.

Tangent lines



Consider a point $Q(x, f(x))$ on the curve that is distinct from P

Slope m_{PQ} of the secant line through P and Q:

$$m_{PQ} = \frac{f(x) - f(x_0)}{x - x_0}$$

If the slope m_{PQ} of the secant line through P and Q approaches a limit as $x \to x_0$, then we regard that limit to be the slope m_{tan} of the tangent line at P.

Tangent lines

Definition

Suppose that x_0 is in the domain of the function f.
The ***tangent line*** to the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the line with equation

$$y - f(x_0) = m_{\tan}(x - x_0)$$

where

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided the limit exists. For simplicity, we will also call this the tangent line to $y = f(x)$ at x_0.

Tangent lines

Example 1. Use the definition to find an equation for the tangent line to the parabola $y = x^2$ at the point $P(1,1)$.

Solution

Applying the formula with $f(x) = x^2$ and $x_0 = 1$, we have

$$m_{\tan} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1} (x + 1) = 2$$

Thus, the tangent line to $y = x^2$ at $(1,1)$ has equation

$$y - 1 = 2(x - 1) \quad \text{or equivalently} \quad y = 2x - 1$$

Tangent lines

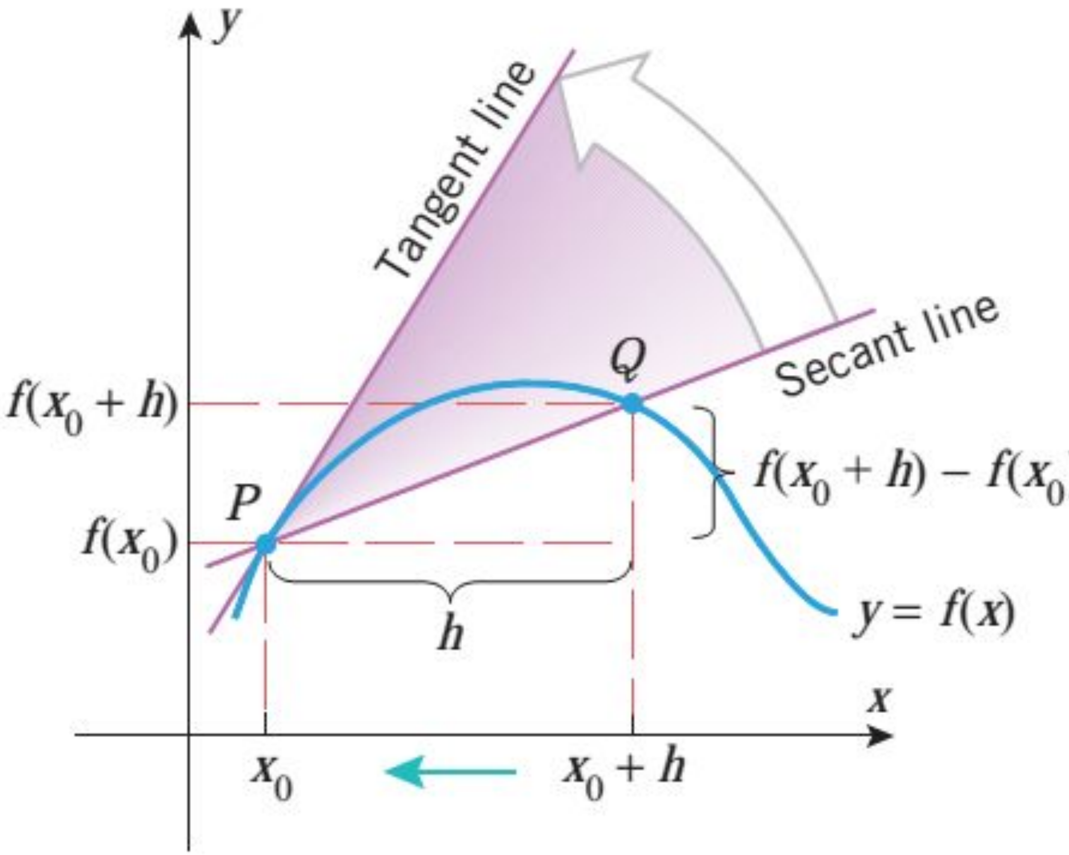
$(x_0 + h)$-notation.

If we let $h = x - x_0$ then the statement $x \to x_0$ is equivalent to the statement $h \to 0$, so we can rewrite

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

in terms of x_0 and h as

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



Tangent lines

$(x_0 + h)$-notation.

Example 2. Find an equation for tangent line to the curve $y = 2/x$ at the point $(2,1)$ on this curve.

Solution

$$m_{\tan} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{2+h} - 1}{h} = \lim_{h \to 0} \frac{\left(\frac{2-(2+h)}{2+h}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(2+h)} = -\left(\lim_{h \to 0} \frac{1}{2+h}\right) = -\frac{1}{2}$$

Tangent lines

$(x_0 + h)$-notation.

Example 2. Find an equation for tangent line to the curve $y = 2/x$ at the point $(2,1)$ on this curve.

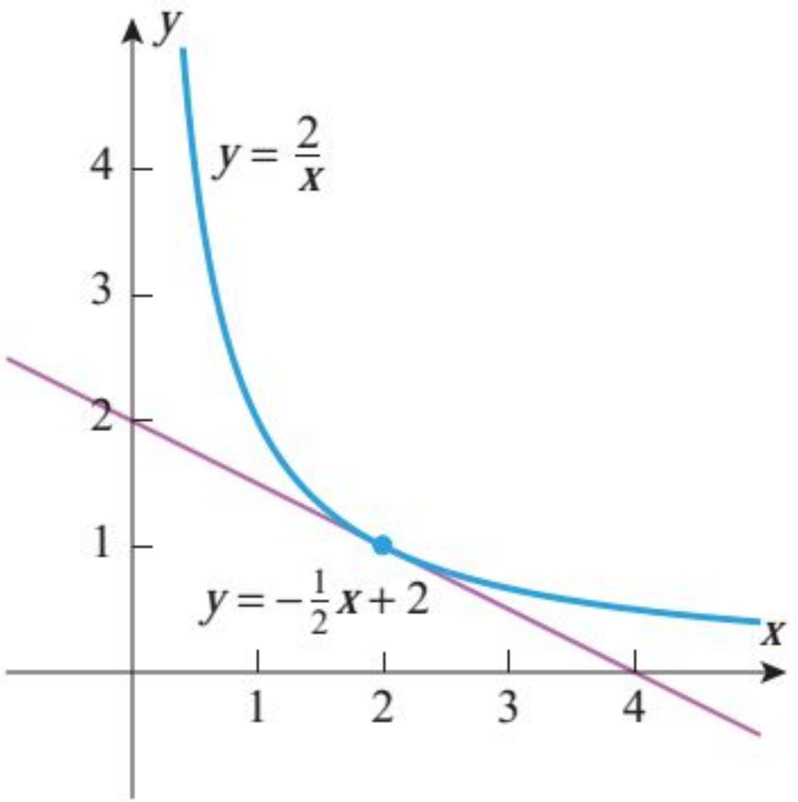
Solution

Thus, an equation of the tangent line at $(2,1)$ is

$$y - 1 = -\frac{1}{2}(x - 2)$$

or equivalently

$$y = -\frac{1}{2}x + 2$$

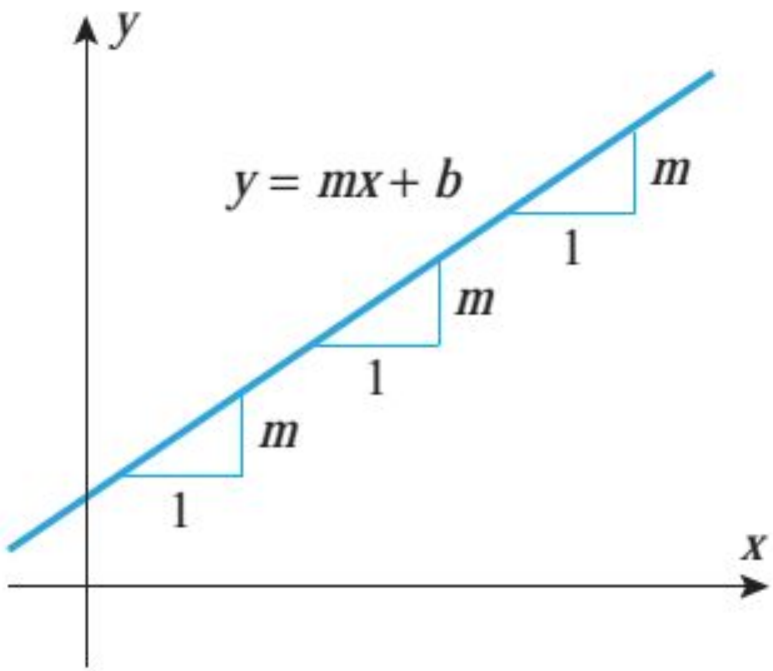


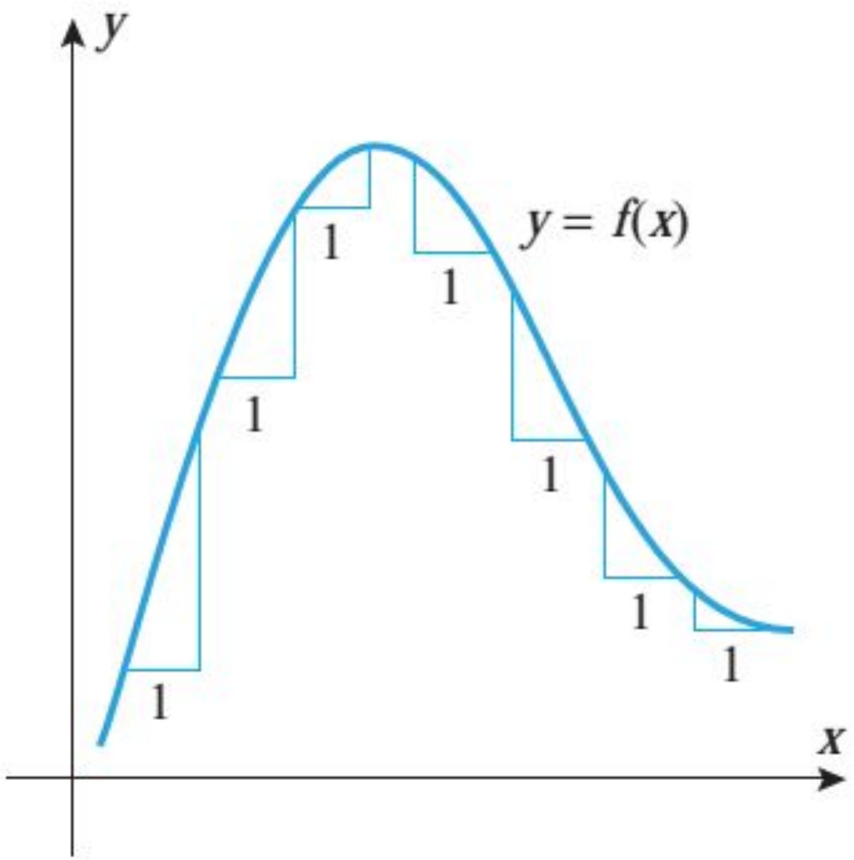
Rates of change

Rates of change occur in many applications, for example:

- A microbiologist might be interested in the rate at which the number of bacteria in a colony changes with time.
- An engineer might be interested in the rate at which the length of a metal rod changes with temperature.
- An economist might be interested in the rate at which production cost changes with the quantity of a product that is manufactured.
- A medical researcher might be interested in the rate at which the radius of an artery changes with the concentration of alcohol in the bloodstream.

Rates of change





For linear case, each 1-unit increase in x anywhere along the line produces *constantly* an m-unit change in y.

For general (nonlinear) case this change is not constant.

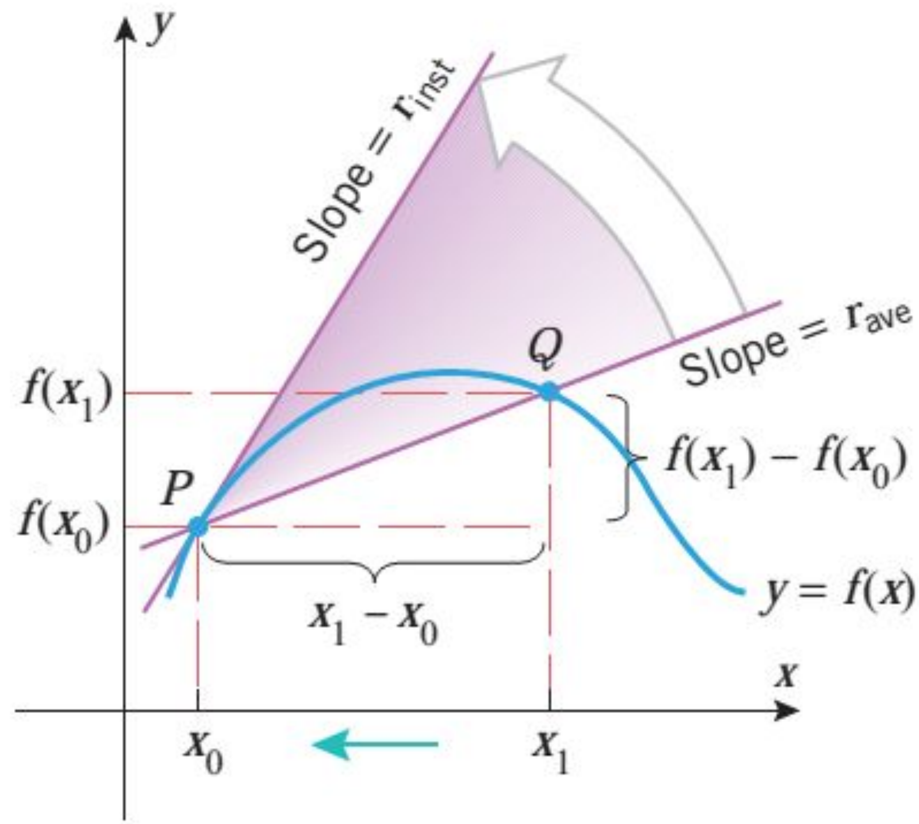
Rates of change

Geometrically,

the **average rate of change of y with respect to x over the interval** $[x_0, x_1]$ is the slope of the secant line through the points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$,

and

the **instantaneous rate of change of y with respect to x at x_0** is the slope of the tangent line at the point $P(x_0, f(x_0))$.



Rates of change

If $y = f(x)$, then

the ***average rate of change of y with respect to x over the interval*** $[x_0, x_1]$ is

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

the ***instantaneous rate of change of y with respect to x at x_0*** is

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

or if we let $h = x_1 - x_0$

$$r_{\text{ave}} = \frac{f(x_0 + h) - f(x_0)}{h}$$

$$r_{\text{inst}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Rates of change

Example 3. Let $y = x^2 + 1$

(a) Find the average rate of change of y with respect to x over the interval $[3,5]$.

(b) Find the instantaneous rate of change of y with respect to x when $x = -4$.

Solution (a) $f(x) = x^2 + 1, x_0 = 3, x_1 = 5.$

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(5) - f(3)}{5 - 3} = \frac{26 - 10}{2} = 8$$

Thus, y increases an average of 8 units per unit increase in x over the interval $[3, 5]$.

Rates of change

Example 3. Let $y = x^2 + 1$

(a) Find the average rate of change of y with respect to x over the interval $[3,5]$.

(b) Find the instantaneous rate of change of y with respect to x when $x = -4$.

Solution (b) $f(x) = x^2 + 1, x_0 = -4.$

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to -4} \frac{f(x_1) - f(-4)}{x_1 - (-4)} = \lim_{x_1 \to -4} \frac{(x_1^2 + 1) - 17}{x_1 + 4}$$

$$= \lim_{x_1 \to -4} \frac{x_1^2 - 16}{x_1 + 4} = \lim_{x_1 \to -4} \frac{(x_1 + 4)(x_1 - 4)}{x_1 + 4} = \lim_{x_1 \to -4} (x_1 - 4) = -8$$

Thus, a small increase in x from $x = -4$ will produce approximately an 8-fold decrease in y.

Derivative function

Definition

The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called the ***derivative of f with respect to x***. The domain of f' consists of all x in the domain of f for which the limit exists.

The term "derivative" is used because the function f' is *derived* from the function f by a limiting process.

Derivative function

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative = Slope of a tangent line

Derivative = Instantaneous rate of change

Slope of a tangent line = Instantaneous rate of change

Derivative function

Example 4. Find the derivative with respect to x of $f(x) = x^2$, and use it to find the equation of the tangent line to $y = x^2$ at $x = 2$.

Solution

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} (2x + h) = 2x
\end{aligned}
$$

Thus, the slope of the tangent line to $y = x^2$ at $x = 2$ is $f'(2) = 4$. Since $y = 4$ if $x = 2$, the point-slope form of the tangent line is

$$y - 4 = 4(x - 2)$$

which we can rewrite in slope-intercept form as $y = 4x - 4$.

Derivative function

Example 4. Find the derivative with respect to x of $f(x) = x^2$, and use it to find the equation of the tangent line to $y = x^2$ at $x = 2$.

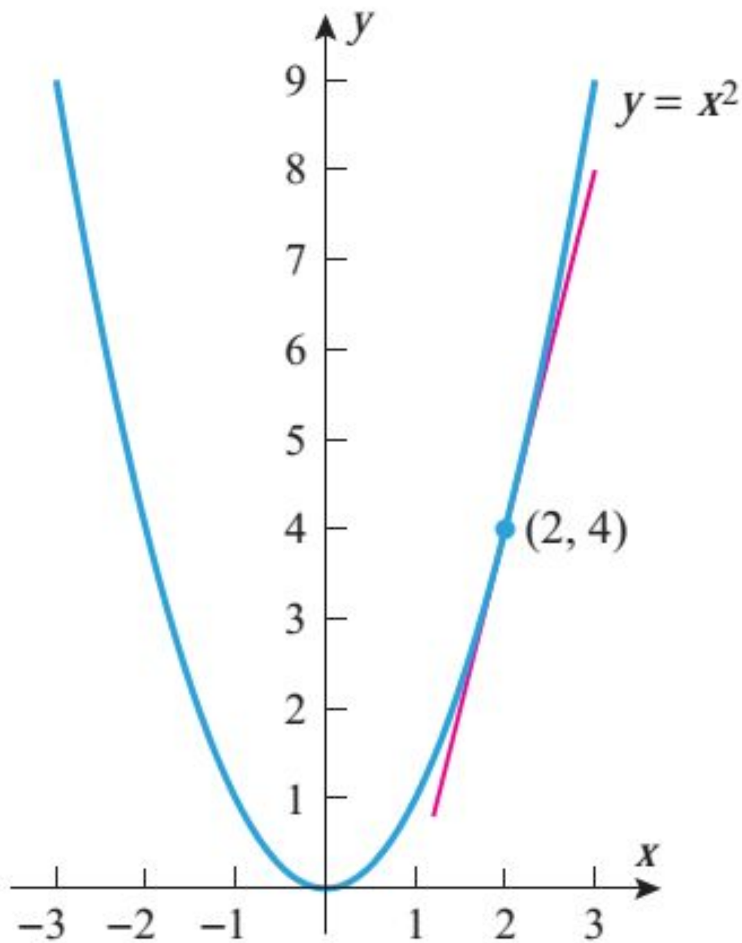
Solution

$$f(x) = x^2$$

$$f'(x) = 2x$$

Tangent line (point-slope form)

$$y - 4 = 4(x - 2)$$



Derivative function

In general, if $f'(x)$ is defined at $x = x_0$, then the point-slope form of the equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$ may be found using the following steps.

Finding an Equation for the Tangent Line to $y = f(x)$ at $x = x_0$.

Step 1. Evaluate $f(x_0)$; the point of tangency is $(x_0, f(x_0))$.

Step 2. Find $f'(x)$ and evaluate $f'(x_0)$, which is the slope m of the line.

Step 3. Substitute the value of the slope m and the point $(x_0, f(x_0))$ into the point-slope form of the line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

or, equivalently,

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Derivative function

Example 5. (a) Find the derivative with respect to x of $f(x) = x^3 - x$.
(b) Graph f and f' together, and discuss their relationship.

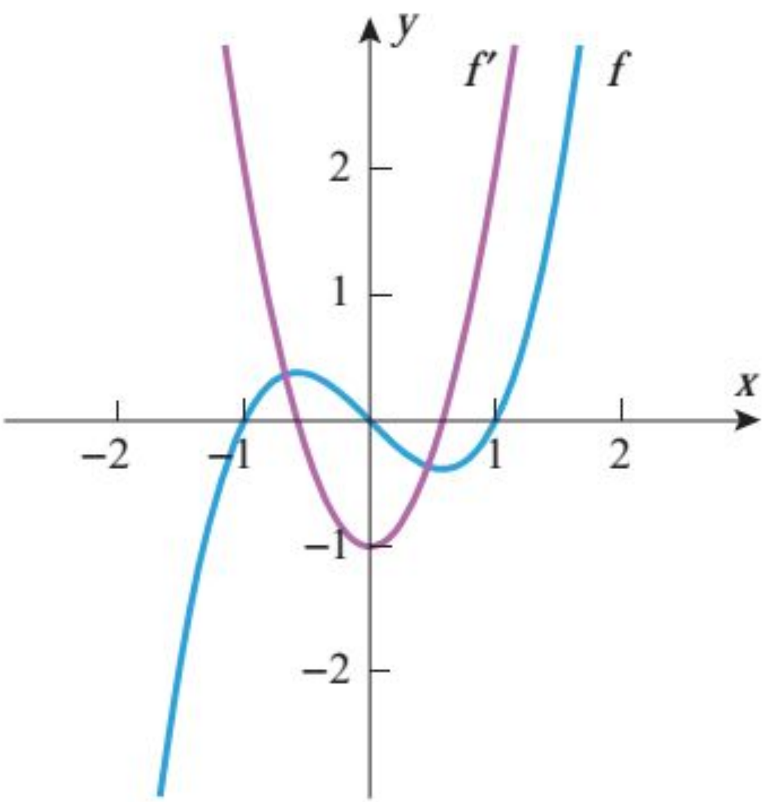
Solution (a)

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\ &= \lim_{h \to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - x - h] - [x^3 - x]}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\ &= \lim_{h \to 0} [3x^2 + 3xh + h^2 - 1] = 3x^2 - 1 \end{aligned}$$

Derivative function

Example 5. (a) Find the derivative with respect to x of $f(x) = x^3 - x$.
(b) Graph f and f' together, and discuss their relationship.

Solution (b)



Since $f'(x)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at x, it follows that $f'(x)$ is positive where the tangent line has positive slope, is negative where the tangent line has negative slope, and is zero where the tangent line is horizontal.

Differentiability

Is a function always differentiable at its certain point x_0?

A function f is said to be ***differentiable at x_0*** if the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

exists.

If f is differentiable at each point of the open interval (a, b), then we say that it is differentiable on (a, b), and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the last case we say that f is differentiable everywhere.

Differentiability

Recall from the previous section that $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists if

$$\lim_{h \to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$$

Left-hand derivative
$f'_-(x_0)$

Right-hand derivative
$f'_+(x_0)$

So, a function f is ***differentiable at x_0*** or has derivative $f'(x_0)$ if

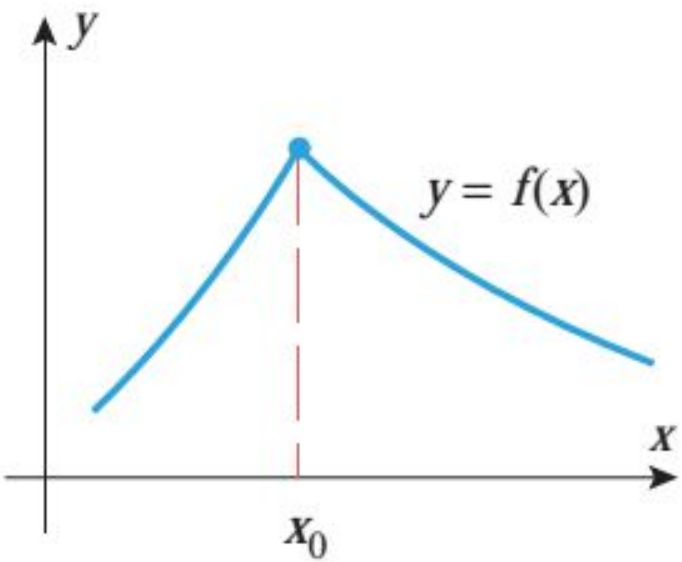
$$f'_-(x_0) = f'_+(x_0) = f'(x_0) \qquad \text{or}$$

$$\lim_{h \to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$$

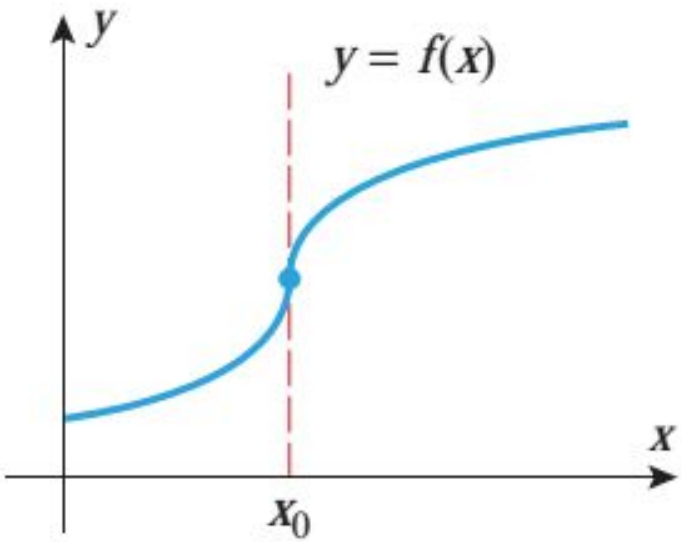
Differentiability

Geometrically a function is not differentiable at:

- corner points
- points of vertical tangency



Corner point



Point of vertical tangency

Differentiability

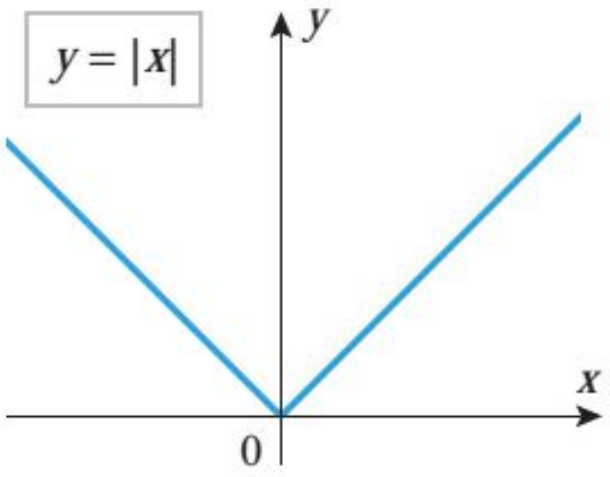
Example 6. (a) Prove that $f(x) = |x|$ is not differentiable at $x = 0$.
(b) Determine a formula for $f'(x)$.

Solution (a)

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} =$$

$$= \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = -1 \qquad \lim_{h \to 0^+} \frac{|h|}{h} = 1$$



$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$ does not exists.

$\therefore\ f(x) = |x|$ is not differentiable at $x = 0$.

Differentiability

Example 6. (a) Prove that $f(x) = |x|$ is not differentiable at $x = 0$.
(b) Determine a formula for $f'(x)$.

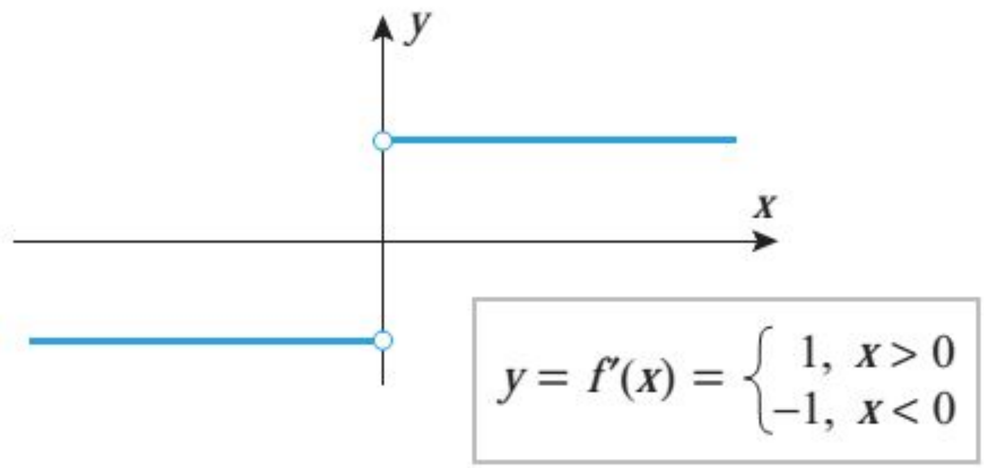
Solution (b)

Separate the cases $x > 0$ and $x < 0$.

If $x > 0$, then $f(x) = x$ and $f'(x) = 1$.

If $x < 0$, then $f(x) = -x$ and $f'(x) = -1$.

Thus, $f'(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

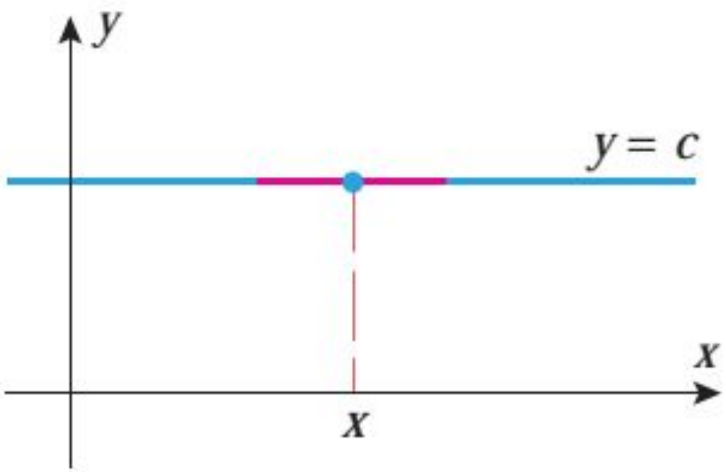


Differentiation rules

Derivative of a constant

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$



The tangent line to the graph of $f(x) = c$ has slope 0 for all x.

The derivative of a constant function is 0; that is, if c is any real number, then

$$\frac{d}{dx}[c] = 0$$

Differentiation rules

Derivative of power functions (Power rule)

If n is a positive integer, then

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

in particular

$$\frac{d}{dx}[x] = 1$$

Proof

Let $f(x) = x^n$. Thus, from the definition of a derivative and the binomial formula for expanding the expression $(x+h)^n$, we obtain

$$\frac{d}{dx}[x^n] = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$

Differentiation rules

Derivative of power functions (Power rule)

If n is a positive integer, then $\frac{d}{dx}[x^n] = nx^{n-1}$ in particular $\frac{d}{dx}[x] = 1$

Proof

Let $f(x) = x^n$. Thus, from the definition of a derivative and the binomial formula for expanding the expression $(x+h)^n$, we obtain

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

$$= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1} \right]$$

$$= nx^{n-1} + 0 + \cdots + 0 + 0 = nx^{n-1}$$

Differentiation rules

Derivative of power functions (Power rule)

Extended power rule

If r is any real number, then

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

In the next lecture we provide a proof using derivatives of a logarithmic function

Differentiation rules

Derivative of a constant times a function

(Constant Multiple Rule) *If f is differentiable at x and c is any real number, then cf is also differentiable at x and*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

$$\frac{d}{dx}[cf(x)] = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c\frac{d}{dx}[f(x)] \quad \blacksquare$$

Differentiation rules

Derivative of sums and differences

If f and g are differentiable at x, then so are $f+g$ and $f-g$ and

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

$$\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]
\end{aligned}$$

Differentiation rules

Derivative of a product

If f and g are differentiable at x, then so is the product $f \cdot g$

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left[f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + g(x) \cdot \frac{f(x+h) - f(x)}{h} \right]$$

Differentiation rules

Derivative of a product

If f and g are differentiable at x, then so is the product $f \cdot g$

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

$$= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \left[\lim_{h\to 0} f(x+h)\right]\frac{d}{dx}[g(x)] + \left[\lim_{h\to 0} g(x)\right]\frac{d}{dx}[f(x)]$$

$$= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Differentiation rules

Derivative of a quotient

If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} =$$

$$= \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$

Differentiation rules

Derivative of a quotient

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

$$= \lim_{h \to 0} \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x) - f(x)\cdot g(x+h) + f(x)\cdot g(x)}{h\cdot g(x)\cdot g(x+h)}$$

$$= \lim_{h \to 0} \frac{\left[g(x)\cdot \frac{f(x+h)-f(x)}{h}\right] - \left[f(x)\cdot \frac{g(x+h)-g(x)}{h}\right]}{g(x)\cdot g(x+h)}$$

$$= \frac{\lim_{h \to 0} g(x)\cdot \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} - \lim_{h \to 0} f(x)\cdot \lim_{h \to 0} \frac{g(x+h)-g(x)}{h}}{\lim_{h \to 0} g(x)\cdot \lim_{h \to 0} g(x+h)}$$

Differentiation rules

Derivative of a quotient

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

$$= \frac{\left[\lim_{h\to 0} g(x)\right]\cdot \frac{d}{dx}[f(x)] - \left[\lim_{h\to 0} f(x)\right]\cdot \frac{d}{dx}[g(x)]}{\lim_{h\to 0} g(x)\cdot \lim_{h\to 0} g(x+h)}$$

$$= \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Differentiation rules

Example 7

(a) $\frac{d}{dx}[1] = ?$

(c) $\frac{d}{dx}[x^4] = ?$

(b) $\frac{d}{dx}[\pi] = ?$

(d) $\frac{d}{dx}[x^5] = ?$

(e) $\frac{d}{dx}[x^\pi]_{|x=1} = ?$

(f) $\frac{d}{dt}[t^{12}]_{|t=-1} = ?$

Differentiation rules

Example 7

(a) $\frac{d}{dx}[1] = 0$

(c) $\frac{d}{dx}[x^4] = 4x^3$

(b) $\frac{d}{dx}[\pi] = 0$

(d) $\frac{d}{dx}[x^5] = 5x^4$

(e) $\frac{d}{dx}[x^\pi]_{|x=1} = \pi x^{\pi-1}{}_{|x=1} = \pi$

(f) $\frac{d}{dx}[t^{12}]_{|t=-1} = 12t^{11}{}_{|t=-1} = -12$

Differentiation rules

Example 8

(a) $\frac{d}{dx}\left[\frac{1}{x}\right] =?$

(b) $\frac{d}{d\omega}\left[\frac{1}{\omega^{100}}\right] =?$

(c) $\frac{d}{dx}\left[x^{4/5}\right] =?$

(d) $\frac{d}{dx}\left[\frac{1}{\sqrt[3]{x}}\right] =?$

Differentiation rules

Example 8

(a) $\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}\left[x^{-1}\right] = -x^{-2} = -\frac{1}{x^2}$

(b) $\frac{d}{d\omega}\left[\frac{1}{\omega^{100}}\right] = \frac{d}{dx}\left[\omega^{-100}\right] = -\omega^{-101} = -\frac{100}{\omega^{101}}$

(c) $\frac{d}{dx}\left[x^{4/5}\right] = \frac{4}{5}x^{\frac{4}{5}-1} = \frac{4}{5}x^{-\frac{1}{5}}$

(d) $\frac{d}{dx}\left[\frac{1}{\sqrt[3]{x}}\right] = \frac{d}{dx}\left[x^{-1/3}\right] = -\frac{1}{3}x^{-\frac{1}{3}-1} = -\frac{1}{3}x^{-\frac{4}{3}}$

Differentiation rules

Example 9

(a) $\frac{d}{dx}[(4x^2-1)(7x^3+x)] =?$

(b) $\frac{d}{dt}\left[(1+t)\sqrt{t}\right] =?$

Differentiation rules

Example 9

(a) $\frac{d}{dx}[(4x^2-1)(7x^3+x)] =$

$$= (4x^2-1)\frac{d}{dx}[7x^3+x] + (7x^3+x)\frac{d}{dx}[4x^2-1] =$$

$$= (4x^2-1)(21x^2+1) + (7x^3+x)(8x) = 140x^4 - 9x^2 - 1$$

(b) $\frac{d}{dt}\left[(1+t)\sqrt{t}\right] =$

$$= (1+t)\frac{d}{dt}\left[\sqrt{t}\right] + \sqrt{t}\frac{d}{dt}[1+t] = \frac{1+t}{2\sqrt{t}} + \sqrt{t} = \frac{1+3t}{2\sqrt{t}}$$

Differentiation rules

Example 10

(a) $\frac{d}{dx}\left[\frac{x^3+2x^2-1}{x+5}\right]=?$

(b) $\frac{d}{dx}\left[\frac{x^2-1}{x^4+1}\right]=?$

Differentiation rules

Example 10

(a) $$\frac{d}{dx}\left[\frac{x^3+2x^2-1}{x+5}\right]=\frac{(x+5)\frac{d}{dx}[x^3+2x^2-1]-(x^3+2x^2-1)\frac{d}{dx}[x+5]}{(x+5)^2}=$$

$$=\frac{(x+5)(3x^2+4x)-(x^3+2x^2-1)(1)}{(x+5)^2}=\frac{2x^3+17x^2+20x+1}{(x+5)^2}$$

(b) $$\frac{d}{dx}\left[\frac{x^2-1}{x^4+1}\right]=\frac{(x^4+1)\frac{d}{dx}[x^2-1]-(x^2-1)\frac{d}{dx}[x^4+1]}{(x^4+1)^2}=$$

$$=\frac{(x^4+1)(2x)-(x^2-1)(4x^3)}{(x^4+1)^2}=-\frac{2x(x^4-2x^2-1)}{(x^4+1)^2}$$

Learning outcomes

5.1.1. Find an equation of a tangent line to a function.

5.1.2. Find a derivative of a function using limits.

5.1.3. Determine whether a function differentiable at some point x_0.

5.1.4. Find a derivative of a function using differentiation rules (Derivative of a constant, Power rule, Constant-times function rule, Derivatives of a sum, difference, product, and quotient).

Formulae

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$r_{\text{ave}} = \frac{f(x_0 + h) - f(x_0)}{h}$$

$$r_{\text{inst}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$y - f(x_0) = f'(x_0)(x - x_0)$$ or

$$y = f(x_0) + f'(x_0)(x - x_0)$$

RULES FOR DIFFERENTIATION

$\frac{d}{dx}[c] = 0$	$(f+g)' = f' + g'$	$(f \cdot g)' = f \cdot g' + g \cdot f'$	$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$
$(cf)' = cf'$	$(f-g)' = f' - g'$	$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$	$\frac{d}{dx}[x^r] = rx^{r-1}$

Preview activity: Differentiation 2

Using definition of a derivative show that

(a) $\frac{d}{dx}[\sin x] = \cos x$

(b) $\frac{d}{dx}[\ln x] = \frac{1}{x}$