Calculation of the determinants + **verifiation**

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{12}a_{32}a_{13} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23}$$

$$\det A = \sum_{k=1}^{n} a_{ik} \cdot (-1)^{i+k} \det A^{(i,k)}$$

Dodgson condensation
Charles Lutwidge Dodgson
(better known as Lewis Carroll)

This algorithm can be described in the following four steps:

1. Let A be the given *n* × *n* matrix. Arrange A so that no zeros occur in its interior. An explicit definition of interior would be all $a_{i,j}$ with $i, j \neq 1, n$. One can do this using any operation that one could normally perform without changing the value of the determinant, such as adding a multiple of one row to another.
2. Create an (*n* − 1) × (*n* − 1) matrix B, consisting of the determinants of every 2 × 2 submatrix of A. Explicitly, we write $b_{i,j} = \begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix}$.
3. Using this (*n* − 1) × (*n* − 1) matrix, perform step 2 to obtain an (*n* − 2) × (*n* − 2) matrix C. Divide each term in C by the corresponding term in the interior of A so $c_{i,j} = \begin{vmatrix} b_{i,j} & b_{i,j+1} \\ b_{i+1,j} & b_{i+1,j+1} \end{vmatrix} / a_{i+1,j+1}$.
4. Let A = B, and B = C. Repeat step 3 as necessary until the 1 × 1 matrix is found; its only entry is the determinant.

One wishes to find

$$\begin{vmatrix} -2 & -1 & -1 & -4 \\ -1 & -2 & -1 & -6 \\ -1 & -1 & 2 & 4 \\ 2 & 1 & -3 & -8 \end{vmatrix}.$$

We make a matrix of its 2 × 2 submatrices.

$$\begin{bmatrix} \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} -1 & -1 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} -1 & -4 \\ -1 & -6 \end{vmatrix} \\ \\ \begin{vmatrix} -1 & -2 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} -1 & -6 \\ 2 & 4 \end{vmatrix} \\ \\ \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ -3 & -8 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ -1 & -5 & 8 \\ 1 & 1 & -4 \end{bmatrix}.$$

We then find another matrix of determinants:

$$\begin{bmatrix} \begin{vmatrix} 3 & -1 \\ -1 & -5 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ -5 & 8 \end{vmatrix} \\ \\ \begin{vmatrix} -1 & -5 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -5 & 8 \\ 1 & -4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -16 & 2 \\ 4 & 12 \end{bmatrix}.$$

We must then divide each element by the corresponding element of our original matrix. The interior of the original matrix is $\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}$, so after dividing we get $\begin{bmatrix} 8 & -2 \\ -4 & 6 \end{bmatrix}$. The process must be repeated to arrive at a 1 × 1 matrix. $\begin{bmatrix} \begin{vmatrix} 8 & -2 \\ -4 & 6 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 40 \end{bmatrix}$. Dividing by the interior of the 3 × 3 matrix, which is just −5, gives $\begin{bmatrix} -8 \end{bmatrix}$ and −8 is indeed the determinant of the original matrix.

Find the determinant of the matrix M:

a) $M = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$

b) $M = \begin{pmatrix} 15 & 10 \\ 3 & 2 \end{pmatrix}$

c) $M = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ 3 & 2 & -1 \end{pmatrix}$

d) $M = \begin{pmatrix} 1 & -6 & 5 \\ 2 & 2 & 5 \\ -1 & -4 & 1 \end{pmatrix}$

e) $M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{pmatrix}$

f) $M = \begin{pmatrix} 2 & 1 & 0 & 4 \\ 1 & 2 & 1 & 4 \\ 0 & 3 & 2 & 2 \\ 2 & 1 & 3 & 3 \end{pmatrix}$

g) $M = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}$

h) $M = \begin{pmatrix} 17 & -11 \\ 6 & -3 \end{pmatrix}$

i) $M = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 4 & -5 \end{pmatrix}$

j) $M = \begin{pmatrix} 15 & 4 & 8 \\ -12 & -7 & 5 \\ 0 & -5 & 15 \end{pmatrix}$

k) $M = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 1 & 1 & 1 \\ 3 & 6 & 5 & 4 \\ 3 & 3 & 2 & 2 \end{pmatrix}$

l) $M = \begin{pmatrix} 3 & 3 & 3 & 1 \\ 2 & 4 & 5 & 2 \\ 3 & 4 & 5 & 1 \\ 2 & 2 & 3 & 4 \end{pmatrix}$

m) $M = \begin{pmatrix} 0 & -2 \\ 6 & 30 \end{pmatrix}$

n) $M = \begin{pmatrix} -27 & 54 \\ 24 & -13 \end{pmatrix}$

o) $M = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 7 & 3 \\ 3 & 1 & -5 \end{pmatrix}$

p) $M = \begin{pmatrix} -2 & 5 & 8 \\ -7 & 1 & 12 \\ -6 & 5 & 17 \end{pmatrix}$

r) $M = \begin{pmatrix} 0 & 0 & 3 & 3 \\ 3 & 0 & 1 & 2 \\ 1 & 0 & 2 & 4 \\ 2 & 1 & 3 & 2 \end{pmatrix}$

s) $M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$

Solve the equation given by the determinant:

a) $\begin{vmatrix} 6 & 2 \\ 3 & x \end{vmatrix} = 0$

b) $\begin{vmatrix} x-1 & x & x+2 \\ 1 & 2 & 1 \\ 1 & x & 2 \end{vmatrix} = 0$

c) $\begin{vmatrix} 8 & 6 & 4 \\ x & 5 & 5 \\ 7 & x & x-2 \end{vmatrix} = 0$

d) $\begin{vmatrix} 2 & x & 1 \\ x & 2 & 1 \\ 3 & 4 & x \end{vmatrix} = 0$

e) $\begin{vmatrix} 3 & x+7 & 5 \\ 1 & 2 & x \\ -x & 5 & 6 \end{vmatrix} = 0$

f) $\begin{vmatrix} x-1 & 4 \\ 1 & x+2 \end{vmatrix} = 0$

g) $\begin{vmatrix} 3 & 4 & 5 \\ 7 & 7 & 7 \\ x & x+1 & 9 \end{vmatrix} = 0$

h) $\begin{vmatrix} 7 & 7 & x+2 \\ 4 & x & x-3 \\ 5 & 5 & 7 \end{vmatrix} = 0$

i) $\begin{vmatrix} 5 & x & 3 \\ x & 3 & 4 \\ -2 & -2 & x \end{vmatrix} = 58$

j) $\begin{vmatrix} 6-\lambda & 5 \\ 6 & 5-\lambda \end{vmatrix} = 0$

k) $\begin{vmatrix} x^2+1 & x \\ x(x-3) & x-2 \end{vmatrix} = 0$

l) $\begin{vmatrix} x & 1 & x+1 \\ 2 & x & 3 \\ x+1 & 4 & x \end{vmatrix} = -2x^2+11$

m) $\begin{vmatrix} x & 4 & 5 \\ 3 & -1 & x \\ 3 & x & -1 \end{vmatrix} = 0$

n) $\begin{vmatrix} 4 & x & 6 \\ 0 & 1 & x-5 \\ x & 2 & 5 \end{vmatrix} = 0$

o) $\begin{vmatrix} 4-\lambda & 1 & 0 \\ 2 & 6-\lambda & 1 \\ 0 & 1 & 4-\lambda \end{vmatrix} = 0$

Constructive problems on divisibility

1. How many zeros ends the number 100!
2. How many six-digit numbers exist in which all digits are odd?
3. How many six-digit numbers exist that have at least one even digit in the record?
4. Can the sum of 19 consecutive positive integers be divisible by 87?
5. Can the sum of 12 consecutive positive integers be divisible by 4?