Constructive problems on divisibility, prime and composite numbers

1. Prove that every positive integer greater than 5 can be represented as a sum of simple and composite.

2. The number 100 ... 001 (in the middle of 2000 zeros) is composite. To prove this, specify at least one of its divisors.

3. The product of four consecutive numbers is 303600. Find these numbers.

4. Find the smallest positive integer N such that N! divided by 990.

5. Specify two two-digit numbers, multiples of 5, in which the sum of the numbers does not change when multiplying by all integers from 1 to 9.

6. Find two three-digit numbers, if you know that their sum is a multiple of 498, and the quotient is a multiple of 5.

7. It is known that the ratio of two integers is 0.4, and their sum is a two-digit number and an exact square. Find these numbers.

8. Present the number 186 as the sum of three different terms, the sum of every two of which is divided by the third.

9. The merchant recorded daily profits (income minus losses). As a result, it turned out that every two days, running in a row, he suffered losses, but for the whole week made a profit. Show by example what this could happen.

10. The merchant recorded monthly earnings (income minus losses). As a result, it turned out that every five months, running in a row, he suffered losses, but for the whole year he made a profit. Show by example what this could happen.

11. Is there a set of four weights with a total weight of 40 kg, such that any whole number of kilograms from 1 to 40 kg can be weighed on the pan weights?

12. Find 10 natural numbers, the sum of which is equal to the product.

13. The product of two two-digit numbers consists of only fours. Find these numbers.

14. Indicate two consecutive positive integers, the sum of the digits of each of which is a multiple of 7.

15. Find a four-digit number that, when you rearrange the numbers in the reverse order, increases 4 times.

16. Turn a 3x3 square by 45 degrees and place the numbers from 1 to 9 (each 1 time) in it so that horizontally it turns out 5 exact squares.

17. In a 3x3 square, arrange the numbers from 1 to 9 (each 1 time), so that horizontally you get 3 exact squares.

18. Give an example of such a number that if you add 10 to it, you get an exact cube, and if you subtract 10, you get an exact square.

19. Is there a set of positive numbers, the sum of all numbers is at least 10, and the sum of squares is not more than 0.01?

20. In a four digit number, the first digit is equal to the second, and the third one is equal to the fourth. In addition, the number is an exact square. Find this number.

Simplify the expression: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{99\cdot 100}$.

Simplify the expression: $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{100^2}\right)$.

Strassen algorithm is an algorithm for matrix multiplication

$$\mathbf{C} = \mathbf{A}\mathbf{B} \qquad \mathbf{A}, \mathbf{B}, \mathbf{C} \in R^{2^n \times 2^n}$$

If the matrices *A*, *B* are not of type $2^n \times 2^n$ we fill the missing rows and columns with zeros.

We partition *A*, *B* and *C* into equally sized block matrices

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{C}_{1,1} & \mathbf{C}_{1,2} \\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} \end{bmatrix}$$

with

$$\mathbf{A}_{i,j}, \mathbf{B}_{i,j}, \mathbf{C}_{i,j} \in R^{2^{n-1} \times 2^{n-1}}.$$

The naive algorithm would be:

$$\mathbf{C}_{1,1} = \mathbf{A}_{1,1}\mathbf{B}_{1,1} + \mathbf{A}_{1,2}\mathbf{B}_{2,1}$$
$$\mathbf{C}_{1,2} = \mathbf{A}_{1,1}\mathbf{B}_{1,2} + \mathbf{A}_{1,2}\mathbf{B}_{2,2}$$
$$\mathbf{C}_{2,1} = \mathbf{A}_{2,1}\mathbf{B}_{1,1} + \mathbf{A}_{2,2}\mathbf{B}_{2,1}$$
$$\mathbf{C}_{2,2} = \mathbf{A}_{2,1}\mathbf{B}_{1,2} + \mathbf{A}_{2,2}\mathbf{B}_{2,2}$$

The Strassen algorithm defines instead new matrices:

$$\mathbf{M}_1 := (\mathbf{A}_{1,1} + \mathbf{A}_{2,2})(\mathbf{B}_{1,1} + \mathbf{B}_{2,2})$$
$$\mathbf{M}_2 := (\mathbf{A}_{2,1} + \mathbf{A}_{2,2})\mathbf{B}_{1,1}$$
$$\mathbf{M}_3 := \mathbf{A}_{1,1}(\mathbf{B}_{1,2} - \mathbf{B}_{2,2})$$
$$\mathbf{M}_4 := \mathbf{A}_{2,2}(\mathbf{B}_{2,1} - \mathbf{B}_{1,1})$$
$$\mathbf{M}_5 := (\mathbf{A}_{1,1} + \mathbf{A}_{1,2})\mathbf{B}_{2,2}$$
$$\mathbf{M}_6 := (\mathbf{A}_{2,1} - \mathbf{A}_{1,1})(\mathbf{B}_{1,1} + \mathbf{B}_{1,2})$$
$$\mathbf{M}_7 := (\mathbf{A}_{1,2} - \mathbf{A}_{2,2})(\mathbf{B}_{2,1} + \mathbf{B}_{2,2})$$

only using 7 multiplications (one for each M_k) instead of 8. We may now express the $C_{i,j}$ in terms of M_k:

$$\mathbf{C}_{1,1} = \mathbf{M}_1 + \mathbf{M}_4 - \mathbf{M}_5 + \mathbf{M}_7$$
$$\mathbf{C}_{1,2} = \mathbf{M}_3 + \mathbf{M}_5$$
$$\mathbf{C}_{2,1} = \mathbf{M}_2 + \mathbf{M}_4$$
$$\mathbf{C}_{2,2} = \mathbf{M}_1 - \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_6$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 & 5 & 8 \\ 2 & 3 & 6 & 7 \\ 3 & 2 & 7 & 6 \\ 4 & 1 & 8 & 5 \end{pmatrix}$$

1. $\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}^3$.

2. $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^n$, $a \in \mathbf{R}$.

3. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n$, $\lambda \in \mathbf{R}$.

4. $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^n$.