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Suboptimal control in the stochastic nonlinear dynamic systems

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Objective. Development of a method for solving task of suboptimal control in stochastic nonlinear dynamical systems.

Subject of study. Stochastic non-linear control systems.

Object of study. Application of splines in the problem of suboptimal control in stochastic nonlinear dynamical systems.

Linear stochastic control systems

The optimization problem for linear stochastic control systems is reduced to two successive steps:

1. *Optimal filtering with using the Kalman-Bucy filter*;

2. *Deterministic control where the state of the system is its evaluation.*

The basic principle underlying the application of such sequence of actions is the **principle of separation** (the separation theorem).

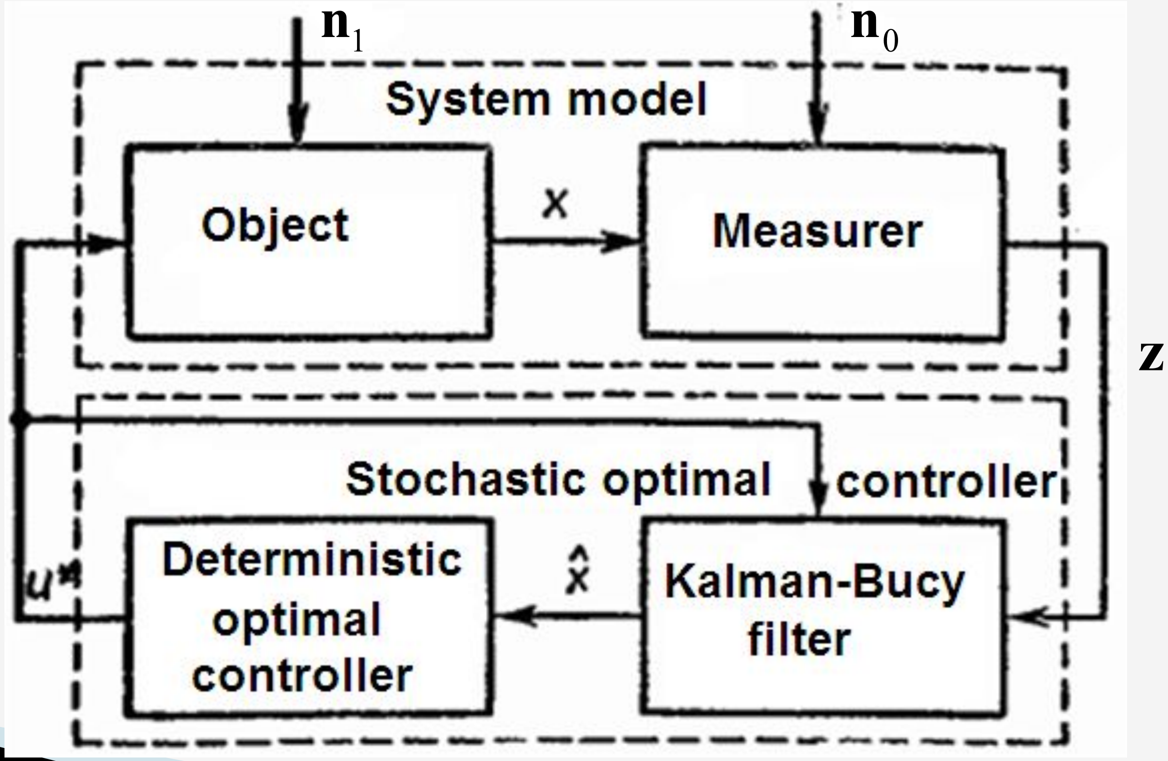
The principle of separation (the principle of stochastic equivalence)

In accordance with the principle of separation, the problem of synthesizing a stochastic linear optimal control system with incomplete information about the state is divided into two:

- *The problem of synthesis of a linear optimal observer;*
- *Deterministic task of synthesis of an optimal system.*

The principle of separation (the principle of stochastic equivalence)

A stochastic linear optimal controller consists of a *linear optimal observer* and a *deterministic optimal controller.*



Nonlinear stochastic control systems

For nonlinear stochastic control systems, the separation principle is not always possible, since it has not been proved for them. On the other hand, the presence of non-linearity leads to the fact that even if the formation noise and observations are Gaussian, then the state is non-Gaussian. The latter negatively affects the work of the Kalman-Bucy filter it becomes not optimal and even inoperative.

Therefore, approaches that allow us to find sub-optimal estimates are relevant. In this connection, the use of splines in the theory of control of stochastic systems is promising and important both from the theoretical and practical point of view.

Spline

Spline is a function that is an algebraic polynomial on every partial interval of interpolation, and on the whole given interval is continuous along with several of its derivatives.

A spline curve is any composite curve formed by polynomial sections that satisfy given conditions of continuity at the boundaries of sections.

There are different types of splines used at the moment and differing in the type of polynomials and certain specific boundary conditions. The spline curve is given through a set of coordinates of points, called control (reference), which indicate the general shape of the curve. Then a piecewise-continuous parametric polynomial function is selected from these points (Fig. 1).

Linear spline

The linear spline is described by the following equation:

$$S(x) = a_i(x - x_i) + b_i$$

where the coefficients can be found by the formulas:

$$a_i = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}, \quad b_i = \frac{f(x_i)x_i - f(x_{i-1})x_{i-1}}{x_i - x_{i-1}}.$$

Linear spline has a number of distinctive advantages: it is a significant reduction in computational costs and universality.

Nonlinear stochastic control systems

Spline approximation of a nonlinear function $f(\mathbf{x})$

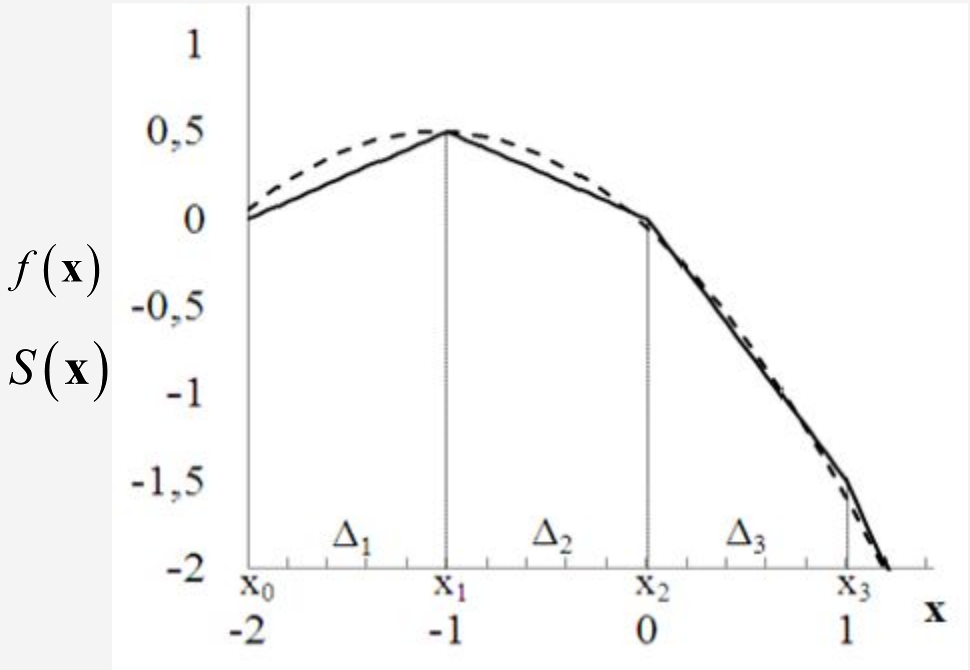


Figure 1. Representation of a function $f(\mathbf{x})$ by linear spline $S(\mathbf{x})$.

Nonlinear stochastic control systems

Let *the control of the object and measurement models* be described by stochastic differential equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) + \mathbf{V}(t)\mathbf{u} + \mathbf{g}_1(t)\mathbf{n}_1(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \qquad (1)$$

$$\mathbf{z} = \mathbf{C}(t)\mathbf{x} + \mathbf{g}_2(t)\mathbf{n}_2(t), \quad \mathbf{z}(t_0) = 0,$$

where $\mathbf{f}(\mathbf{x})$ is a nonlinear smooth function (optional condition).

We represent $\mathbf{f}(\mathbf{x})$ in the form of a spline of the first order: $\mathbf{f}(\mathbf{x}) \approx \mathrm{S}(\mathbf{x}) = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)$.

$$\mathbf{A}(t) = \left\{\sum_{i=1}^{n} h_{ij} \cdot a_{ij}\right\}_{[m\times m]}, \quad \mathbf{B}(t) = \left\{\sum_{i=1}^{n} h_{ij} \cdot b_{ij}\right\}_{[m\times 1]}.$$

Nonlinear stochastic control systems

With the approximation by splines of the expression (1) takes the form:

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t) + \mathbf{V}(t)\mathbf{u} + g_1(t)\mathbf{n}_1(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \\ \mathbf{z} = \mathbf{C}(t)\mathbf{x} + g_2(t)\mathbf{n}_2(t), \quad \mathbf{z}(t_0) = 0. \end{cases} \tag{2}$$

$$\mathbf{A}(t) = \left\{ \sum_{i=1}^{n} h_{ij} \cdot a_{ij} \right\}_{[m\times m]}, \quad \mathbf{B}(t) = \left\{ \sum_{i=1}^{n} h_{ij} \cdot b_{ij} \right\}_{[m\times 1]}, \quad h_i(x_i, x_{i+1}) = \begin{cases} 1 \ if \ \ x \in [x_i, x_{i+1}) \\ 0 \ if \ \ x \notin [x_i, x_{i+1}) \end{cases}.$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ – *the state vector,* $\mathbf{x} \in \mathbf{R}^n$; $\mathbf{z} = (x_1, x_2, \ldots, x_m)$ – *vector of measurements*

$\mathbf{z} \in \mathbf{R}^n$; $t \in T = [t_0, t_1]$

$\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$, $\mathbf{V}(t)$, $g_1(t)$, $g_2(t)$ have dimensions $(n\times n)$, $(n\times q)$, $(m\times n)$, $(n\times q)$, $(n\times k)$, $(m\times l)$;

$\mathbf{u} \in \mathbf{R}^q$; $\mathbf{n}_1(t)$, $\mathbf{n}_2(t)$ – independent white Gaussian noise:

$\langle \mathbf{n}_1(t) \rangle = 0$, $\langle \mathbf{n}_2(t) \rangle = 0$, $\langle \mathbf{n}_1(t)\mathbf{n}_1(t-\tau) \rangle = \mathbf{N}_1\delta(\tau)$, $\langle \mathbf{n}_2(t)\mathbf{n}_2(t-\tau) \rangle = \mathbf{N}_2\delta(\tau)$, $\langle \mathbf{n}_1(t)\mathbf{n}_2(t-\tau) \rangle = 0$.

Nonlinear stochastic control systems

We denote by

$$\mathbf{z}_{t_0}^{t} = \left\{ \mathbf{z}(\tau),\ t_0 \le \tau \le t \right\} \quad \mathbf{R}_1(t) = \mathbf{g}_1(t)\mathbf{g}_1^T(t), \quad \mathbf{R}_2(t) = \mathbf{g}_2(t)\mathbf{g}_2^T(t)$$

Suppose that at the control at time t information about all observations on the time interval $[t_o, t]$ is used.

The set of admissible controls form functions

$$\mathbf{u}(t) = \mathbf{u}\left(t,\ \mathbf{z}_{t_0}^{t}\right) \quad \forall t \in T, \quad \mathbf{u}(t) \in \mathbf{R}^q,$$

which depend on previous observations, for which the system (1) has a unique solution.

Optimal control

Functional of quality of control

$$\mathbf{J} = < \frac{1}{2}\int_{t_0}^{t_1}\left[\mathbf{x}^T(t)\mathbf{S}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{Q}(t)\mathbf{u}(t)\right]dt + \frac{1}{2}\mathbf{x}^T(t_1)\quad \mathrm{x}(t_1) > \qquad (3)$$

where $\mathbf{S}(t)$, $\mathbf{\Lambda}$ – non-negative definite symmetric matrices of dimension $(n \times m)$,

$\mathbf{Q}(t)$ – positive definite symmetric matrix of dimension $(q \times q)$

It is required to find a control $\mathbf{u}^*\left(t,\, \mathbf{z}_{t_0}^{t}\right)$ from the set of admissible ones that ensures the minimum of the functional (3).

Sub-optimal control

Statement. Suboptimal control $\mathbf{u}^*\left(t,\ \mathbf{z}_{t_0}^t\right)$ in the task (1) with the criterion of quality (3) has the form:

$$\mathbf{u}^*(t)=\mathbf{u}^*\left(t,\ \mathbf{z}_{t_0}^t\right)=\mathbf{Q}^{-1}\mathbf{B}^T\mathbf{K}_2(t)\hat{\mathbf{x}}(t).$$

$$\frac{d\hat{\mathbf{x}}}{dt}=\left[\mathbf{A}(t)\hat{\mathbf{x}}+\mathbf{B}-\mathbf{V}(t)\mathbf{u}^*\right]+\mathbf{K}(t)\left[\mathbf{z}(t)-\mathbf{C}(t)\hat{\mathbf{x}}\right],\quad \hat{\mathbf{x}}(t_0)=\mathbf{m}_0$$

$$\frac{d\mathbf{K}_2(t)}{dt}=-\mathbf{A}^T(t)\mathbf{K}_2(t)-\mathbf{K}_2(t)\mathbf{A}(t)-\mathbf{K}_2(t)\mathbf{B}(t)\mathbf{Q}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}_2(t)+\mathbf{S}(t),$$

$$\mathbf{K}(t)\mathbf{C}= \quad (t\mathbf{R}\ ^T(t)\quad {}_2^{-1}(t)\quad \mathbf{K}_2(t)=-\quad .$$

$$\frac{d\Gamma(t)}{dt}=\mathbf{K}\quad \mathbf{FA}\ ^T\ \mathbf{FC}\ \mathbf{R}^T\ \mathbf{C}_2^{-1}\Gamma\quad \mathbf{R}\ {}_1 \qquad \Gamma(t_0)=\mathbf{D}_0^x$$

Sub-optimal control

$\mathbf{K}_2(t)$ - symmetric matrix of the gain factors of the optimal controller,

$\mathbf{K}(t)$ - matrix of the gain factors of filter coefficients dimension (n × m),

$\mathbf{\Gamma}(t) = \langle [\mathbf{x}(t) - \hat{\mathbf{x}}(t)][\mathbf{x}(t) - \hat{\mathbf{x}}(t)]^T \rangle$ - covariance matrix of estimation error,

$\hat{\mathbf{x}}(t) = \langle \mathbf{x}(t) | \mathbf{z}_{t_0}^t \rangle$ - estimation of the state vector of the control object model from the results of observations.

Sub-optimal control

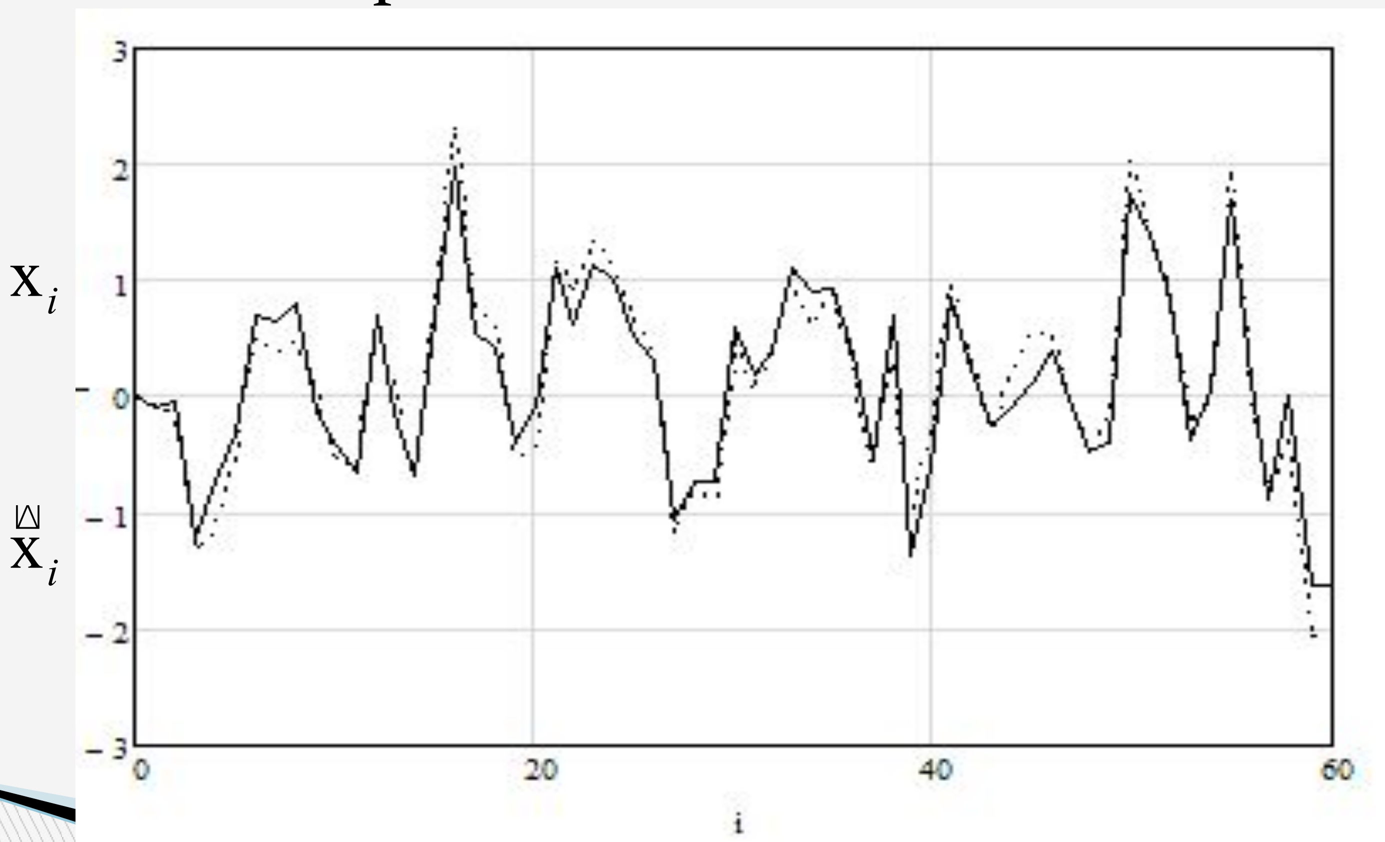
The system of equations on slide 14 is the Kalman-Bucy equation in the representation of a nonlinear function in the form of linear splines and describes the procedure for sub-optimal filtering and control for a nonlinear stochastic system described by equations (1). In general, the use of linear splines not only allowed us to solve the problem, but also circumvented the restriction of the separation theorem, which in principle was proved only for linear systems (for nonlinear systems, the question remains open). Splines allowed for each of the intervals to apply linear filtration and the principle of separation.

As the results of the simulation show, the estimation of the state of the system with the spline approximation very closely coincides with the true value. Moreover, the number of the interval does not influence the quality of the estimation.

Improve the quality by increasing the number of intervals. The increase in the variance of observation noise is also adversely affected in the case of optimal filtering and control, and in the case of spline approximation.

Results of simulation of the filtration task

The state of the system x_i and its estimate $\hat{x}_i$ obtained with the linear spline:



Conclusions

1. The use of splines allows one to solve problems in nonlinear stochastic control systems (NSCS) that are defined not only by a scalar but also by a matrix-vector equation.
2. The spline approximation has the global meaning, not local, as in the case of Taylor series expansion.
3. Extrapolation of the obtained results to the case of parabolic splines does not present difficulties, and the advantages of the proposed approach in comparison with the known sub-optimal methods of nonlinear filtration become even more significant.
4. The application of splines in HSCS of order higher than 2 does not give special advantages in accuracy, but considerably complicates the Kalman-Bucy filter and does not even allow it to be realized.

Bibliography

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