N.I. Pirogov Russian National Research Medical University

Medicobiologic Faculty
International Faculty

Mathematics

Vector Algebra Essentials

Dr. Oleg B. Shiryaev

General Physics Institute, Russian Academy of Science

https://sites.google.com/site/drolegbshiryaev/

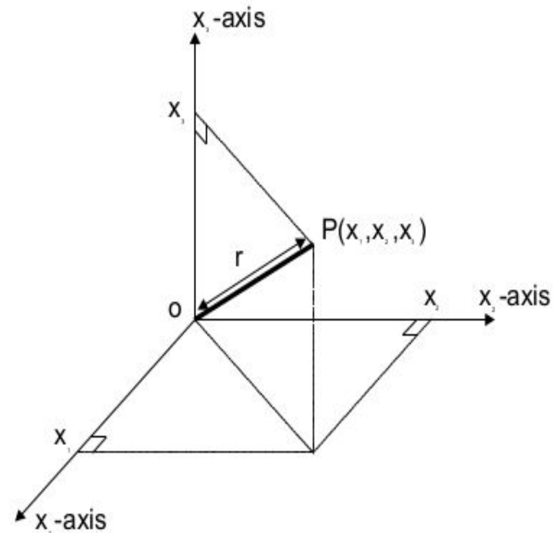
8-916-205-55-82 oleg.b.shiryaev@mail.ru

Math Prelude 1: Vector Algebra Essentials

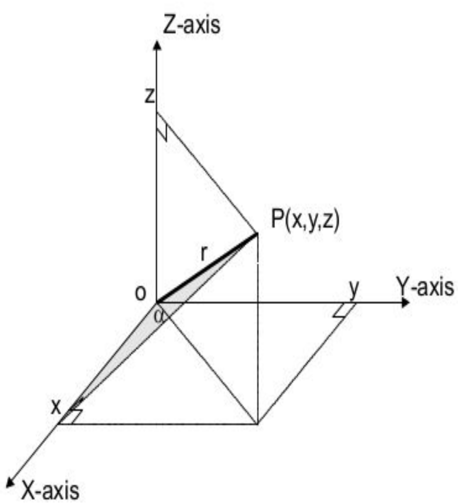
Cartesian coordinate frame

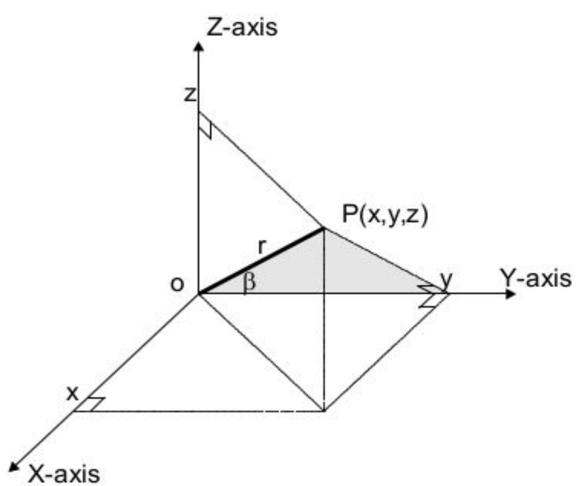
Distance between point and coordinate frame origin:

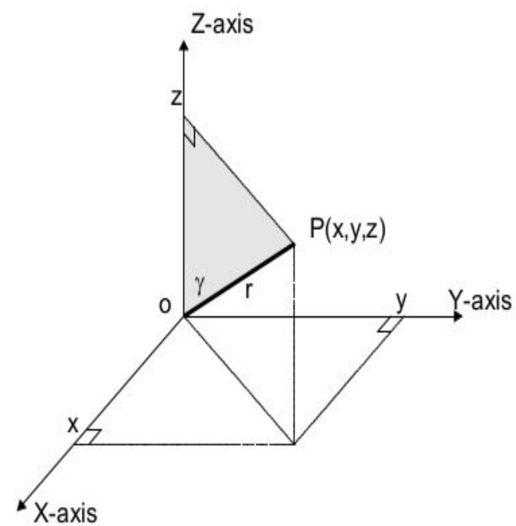
Direction cosines:



x, y, z, axis system

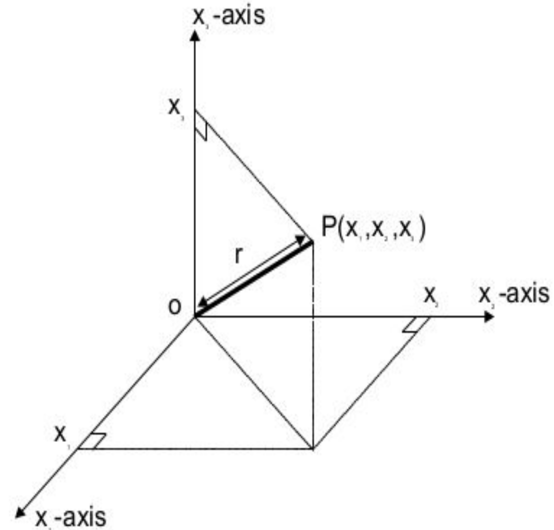






Math Prelude 1: Vector Algebra Essentials

Cartesian coordinate frame

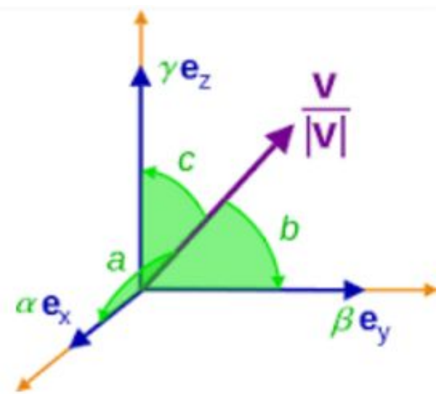


x, y, z, axis system

Distance between point and coordinate frame origin:

Direction cosines:

In analytic geometry, the **direction cosines** (or **directional cosines**) of a vector are the **cosines** of the angles between the vector and the three coordinate axes. Equivalently, they are the contributions of each component of the basis to a unit vector in that **direction**.



Vector Magnitudes and Collinearity

=

Vector length:

Unit vector in same direction:

Equal vectors

Collinear vectors

Unit Vectors in Coordinate Directions

,

Operations on Vectors in Coordinate Representation

Dot Product

Dot Product in Terms of Vector Components

$$\bar{a} \cdot \bar{b} = a_x b_x + a_y b_y + a_z b_z .$$

$$\bar{a} \cdot \bar{b} = (a_x \bar{i} + a_y \bar{j} + a_z \bar{k}) \cdot (b_x \bar{i} + b_y \bar{j} + b_z \bar{k}) =$$

$$\begin{aligned} = \quad & a_x b_x \bar{i}\bar{i} + a_x b_y \bar{i}\bar{j} + a_x b_z \bar{i}\bar{k} + \\ + \quad & a_y b_x \bar{j}\bar{i} + a_y b_y \bar{j}\bar{j} + a_y b_z \bar{j}\bar{k} + \\ + \quad & a_z b_x \bar{k}\bar{i} + a_z b_y \bar{k}\bar{j} + a_z b_z \bar{k}\bar{k} = \end{aligned}$$

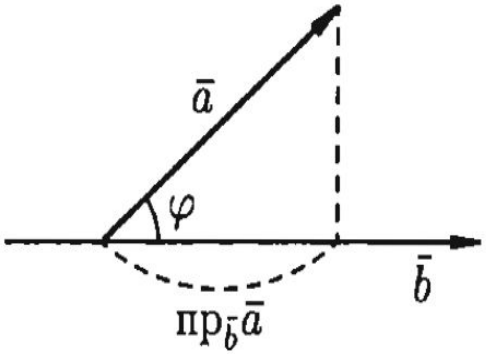
$$= a_x b_x + 0 + 0 + 0 + a_y b_y + 0 + 0 + 0 + a_z b_z;$$

Vector Magnitude

Vectors Orgthogonality Condition

Projection of a Vector on Another Vector's Direction

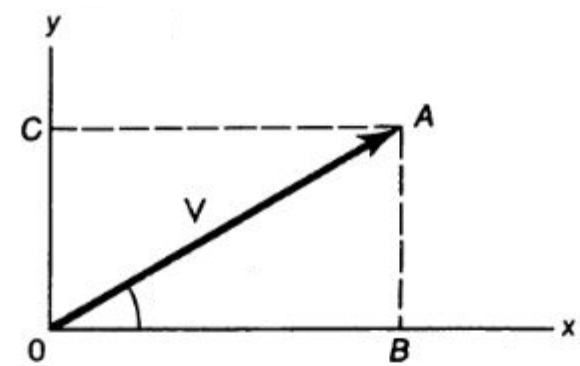
$$= \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{b_x^2 + b_y^2 + b_z^2}}$$

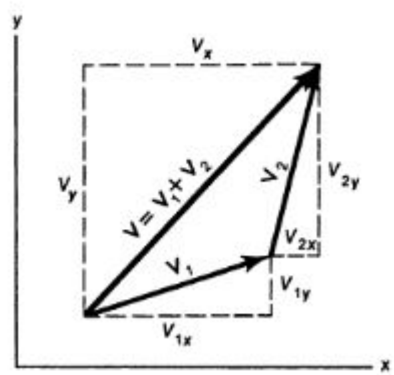


Angle Between Vectors

$$= \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \cdot \sqrt{b_x^2 + b_y^2 + b_z^2}}$$

Vectors on a Plane





Examples

Example: prove that the diagonals are perpendicular for

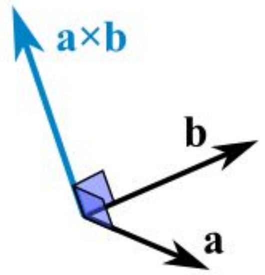
$A(-4;-4;4), \quad B(-3;2;2), \quad C(2;5;1),$

Example: find the length of for $|\bar{a}| = 2, |\bar{b}| = 3, (\widehat{\bar{a},\bar{b}}) = \frac{\pi}{3}$

Example: calculate work performed by the force to move a body from to . Calculate the angle between the force and the displacement

Cross Product

The Cross Product **a × b** of two vectors is another vector that is at right angles to both:

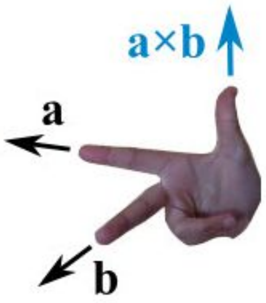


And it all happens in 3 dimensions!

Which Way?

The cross product could point in the completely opposite direction and still be at right angles to the two other vectors, so we have the:

"Right Hand Rule"

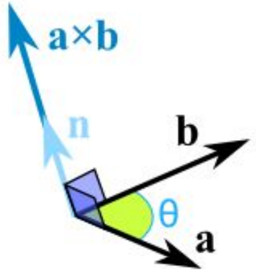


With your right-hand, point your index finger along vector **a**, and point your middle finger along vector **b**: the cross product goes in the direction of your thumb.

We can calculate the Cross Product this way:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$

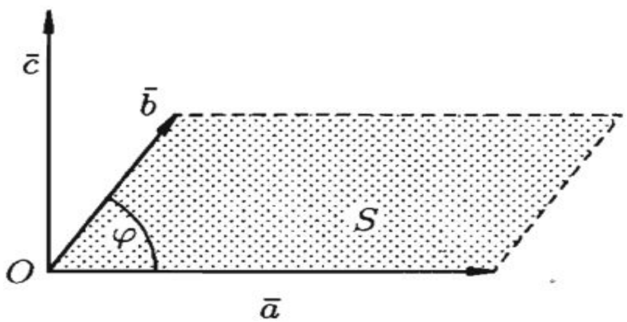
- $|\mathbf{a}|$ is the magnitude (length) of vector **a**
- $|\mathbf{b}|$ is the magnitude (length) of vector **b**
- θ is the angle between **a** and **b**
- **n** is the unit vector at right angles to both **a** and **b**

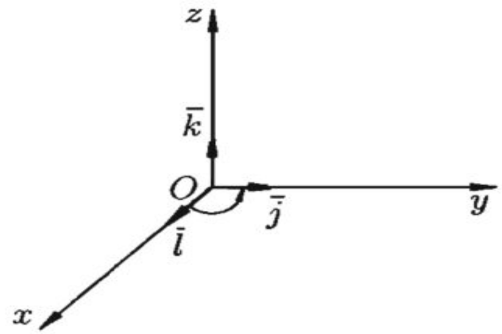


So the **length** is: the length of **a** times the length of **b** times the sine of the angle between **a** and **b**,

Then we multiply by the vector **n** to make sure it heads in the right **direction** (at right angles to both **a** and **b**).

Cross Product





$$|\bar{c}| = |\bar{a}| \cdot |\bar{b}| \sin\varphi,$$

$$\varphi = (\widehat{\bar{a}, \bar{b}})$$

$$\bar{i} \times \bar{j} = \bar{k}, \quad \bar{j} \times \bar{k} = \bar{i}, \quad \bar{k} \times \bar{i} = \bar{j}$$

$$\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$$

$$(\lambda\bar{a}) \times \bar{b} = \bar{a} \times (\lambda\bar{b})$$

$$\bar{a} \parallel \bar{b} \iff \bar{a} \times \bar{b} = \bar{0}$$

$$(\bar{a} + \bar{b}) \times \bar{c} = \bar{a} \times \bar{c} + \bar{b} \times \bar{c}$$

Area of triangle with sides ***a*** and ***b***:

$$S_{\triangle} = \frac{1}{2}|\bar{a} \times \bar{b}|$$

Cross Product in Coordinate Terms

	$\bar{i}$	$\bar{j}$	$\bar{k}$
$\bar{i}$	$\bar{0}$	$\bar{k}$	$-\bar{j}$
$\bar{j}$	$-\bar{k}$	$\bar{0}$	$\bar{i}$
$\bar{k}$	$\bar{j}$	$-\bar{i}$	$\bar{0}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\bar{a} \times \bar{b} = (a_x\bar{i} + a_y\bar{j} + a_z\bar{k}) \times (b_x\bar{i} + b_y\bar{j} + b_z\bar{k}) =$$

$$= a_xb_x(\bar{i} \times \bar{i}) + a_xb_y(\bar{i} \times \bar{j}) + a_xb_z(\bar{i} \times \bar{k}) + a_yb_x(\bar{j} \times \bar{i}) + a_yb_y(\bar{j} \times \bar{j}) +$$

$$+ a_yb_z(\bar{j} \times \bar{k}) + a_zb_x(\bar{k} \times \bar{i}) + a_zb_y(\bar{k} \times \bar{j}) + a_zb_z(\bar{k} \times \bar{k}) =$$

$$= \bar{0} + a_xb_y\bar{k} - a_xb_z\bar{j} - a_yb_x\bar{k} + \bar{0} + a_yb_z\bar{i} + a_zb_x\bar{j} - a_zb_y\bar{i} + \bar{0} =$$

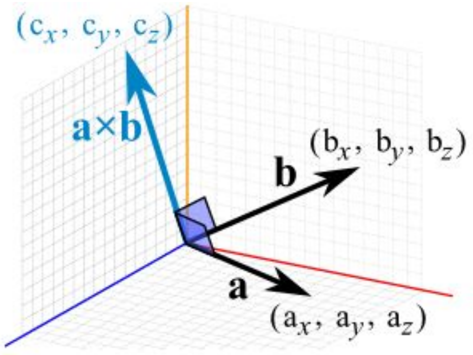
$$= (a_yb_z - a_zb_y)\bar{i} - (a_xb_z - a_zb_x)\bar{j} + (a_xb_y - a_yb_x)\bar{k} =$$

$$= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \bar{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \bar{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \bar{k},$$

Cross Product in Coordinate Terms

When **a** and **b** start at the origin point (0,0,0), the Cross Product will end at:

- $c_x = a_y b_z - a_z b_y$
- $c_y = a_z b_x - a_x b_z$
- $c_z = a_x b_y - a_y b_x$



Example: The cross product of **a** = (2,3,4) and **b** = (5,6,7)

- $c_x = a_y b_z - a_z b_y = 3\times7 - 4\times6 = -3$
- $c_y = a_z b_x - a_x b_z = 4\times5 - 2\times7 = 6$
- $c_z = a_x b_y - a_y b_x = 2\times6 - 3\times5 = -3$

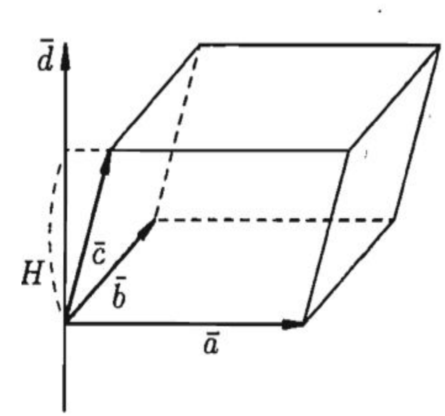
Answer: **a × b** = (−3,6,−3)

Box Product

$$(\bar{a} \times \bar{b}) \cdot \bar{c} = \bar{d} \cdot \bar{c} = |\bar{d}| \cdot \text{пр}_{\bar{d}}\, \bar{c}$$

$$|\bar{d}| = |\bar{a} \times \bar{b}| = S$$

$$(\bar{a} \times \bar{b}) \cdot \bar{c} = S \cdot (\pm H)$$



$$(\bar{a} \times \bar{b}) \cdot \bar{c} =$$

$$(\bar{a} \times \bar{b}) \cdot \bar{c} = (\bar{b} \times \bar{c}) \cdot \bar{a} = (\bar{c} \times \bar{a}) \cdot \bar{b}$$

$$(\bar{a} \times \bar{b}) \cdot \bar{c} = \bar{a} \cdot (\bar{b} \times \bar{c})$$

$$\bar{a}\bar{b}\bar{c} = -\bar{a}\bar{c}\bar{b},\ \bar{a}\bar{b}\bar{c} = -\bar{b}\bar{a}\bar{c},\ \bar{a}\bar{b}\bar{c} =$$

for vectors lying in the same plane

Triple Product in Coordinate Terms

$$(\bar{a}\times\bar{b})\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \cdot (c_x\bar{i} + c_y\bar{j} + c_z\bar{k}) =$$

$$= \left(\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \bar{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \bar{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \bar{k} \right) \cdot (c_x\bar{i} + c_y\bar{j} + c_z\bar{k}) =$$

$$= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \cdot c_x - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \cdot c_y + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \cdot c_z$$

$$\bar{a}\bar{b}\bar{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Example: find prism volume for

$$\bar{a} = \overline{AB} = (-1; -3; -2), \quad \bar{b} = \overline{AC} = (1; 3; -1), \quad \bar{c} = \overline{AD} = (2; -2; -5)$$