
Calculus++ Light
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Sudoku no more!
Playtime's Over

Irrational Numbers

Question 1. The Dirichlet function is defined as follows $f(x)=\begin{cases}1, \text{if } x \text{ is rational}\\0, \text{if } x \text{ is irrational}\end{cases}$

Is this function even or odd or neither?

Is this function periodic? If yes, find a period of this function.

Solution. If x is a rational number, so is $-x$.

$$-\frac{k}{n}=\frac{-k}{n}.$$

If x is a irrational number, so is $-x$.

Hence $f(-x)=f(x)$, and therefore the Dirichlet function is even.

The sum of two rational numbers is a rational number: $\frac{k_1}{n_1}+\frac{k_2}{n_2}=\frac{k_1 n_2 + k_2 n_1}{n_1 n_2}$.

The sum of a rational and an irrational number is an irrational number.

Let x be a rational number, let y be an irrational number, and let us assume that $z = x + y$ is a rational number.

Then $y = z + (-x)$ is also a rational number.

Contradiction!

Hence, the sum of a rational and an irrational number is an irrational number.

Therefore $f(x+y)=f(x)$ for any rational number y.

Thus, the Dirichlet function is periodic.

Any rational number is a period of this function.

However, unlike trigonometric functions $\sin(x)$ or $\cos(x)$, the Dirichlet function does not have minimal (or principal) period T.

Question 2. The numbers $\sqrt{2}$ and $\sqrt{3}$ are irrational. Show that the number $\sqrt{2}+\sqrt{3}$ is irrational too.

Solution. We have $\sqrt{2}+\sqrt{3}=$

$$=\frac{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}{\sqrt{3}-\sqrt{2}}=\frac{3-2}{\sqrt{3}-\sqrt{2}}=\frac{1}{\sqrt{3}-\sqrt{2}}.$$

If $\sqrt{2}+\sqrt{3}$ is a rational number, then

$$\sqrt{2}+\sqrt{3}=\frac{n}{m}\quad\Rightarrow\sqrt{3}-\sqrt{2}=\frac{1}{\sqrt{2}+\sqrt{3}}=\frac{m}{n}$$

$$\Rightarrow\sqrt{2}=\frac{1}{2}\left(\left(\sqrt{3}+\sqrt{2}\right)-\left(\sqrt{3}-\sqrt{2}\right)\right)=\frac{1}{2}\left(\frac{n}{m}-\frac{m}{n}\right)$$

$$\Rightarrow \sqrt{2} = \frac{n^2 - m^2}{2m \cdot n}$$ is a rational number.

Contradiction!!!

Therefore our assumption was incorrect and $\sqrt{2} + \sqrt{3}$ is an irrational number.

Question 3. Let

$$f(x) = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5,$$

and denote $g(x) = f^{-1}(x)$.

Find a general formula for the second derivative of inverse function, $g''(x)$, and calculate $g''(0)$.

Solution. We know that $g'(x) = \dfrac{1}{f'(g(x))}$.

The chain rule yields

$$g''(x) = -\frac{1}{(f'(g(x)))^2} f''(g(x))g'(x).$$

$$\Rightarrow \quad g''(x) = -\frac{f''(g(x))}{(f'(g(x)))^3}.$$

$$f(x) = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5$$

$$g''(x) = -\frac{f''(g(x))}{(f'(g(x)))^3}.$$

Since $f(0) = 0$, we have $g(0) = 0$.

$$f'(x) = 1 + x + x^2 + x^3 + x^4 \Rightarrow f'(g(0)) = 1$$

$$f''(x) = 1 + 2x + 3x^2 + 4x^3 \Rightarrow f''(g(0)) = 1$$

$$\Rightarrow g''(0) = -1.$$

Question. Which of the following conditions imply that a real number x is rational?

I. $\sqrt{x}$ is rational

II. x^2 and x^5 are rational

III. x^2 and x^4 are rational

a) I only b) II only c) I and II only

d) I and III only e) II and III only

Solution: If $\sqrt{x}$ is rational, then $\sqrt{x} = \frac{m}{n}$.

Therefore $x = \frac{m^2}{n^2}$ is also rational.

Counterexample to III: $\sqrt{2}$ is irrational, but $\left(\sqrt{2}\right)^2 = 2$ and $\left(\sqrt{2}\right)^4 = 2^2 = 4$ are rational.

Let now x^2 and x^5 be rational:

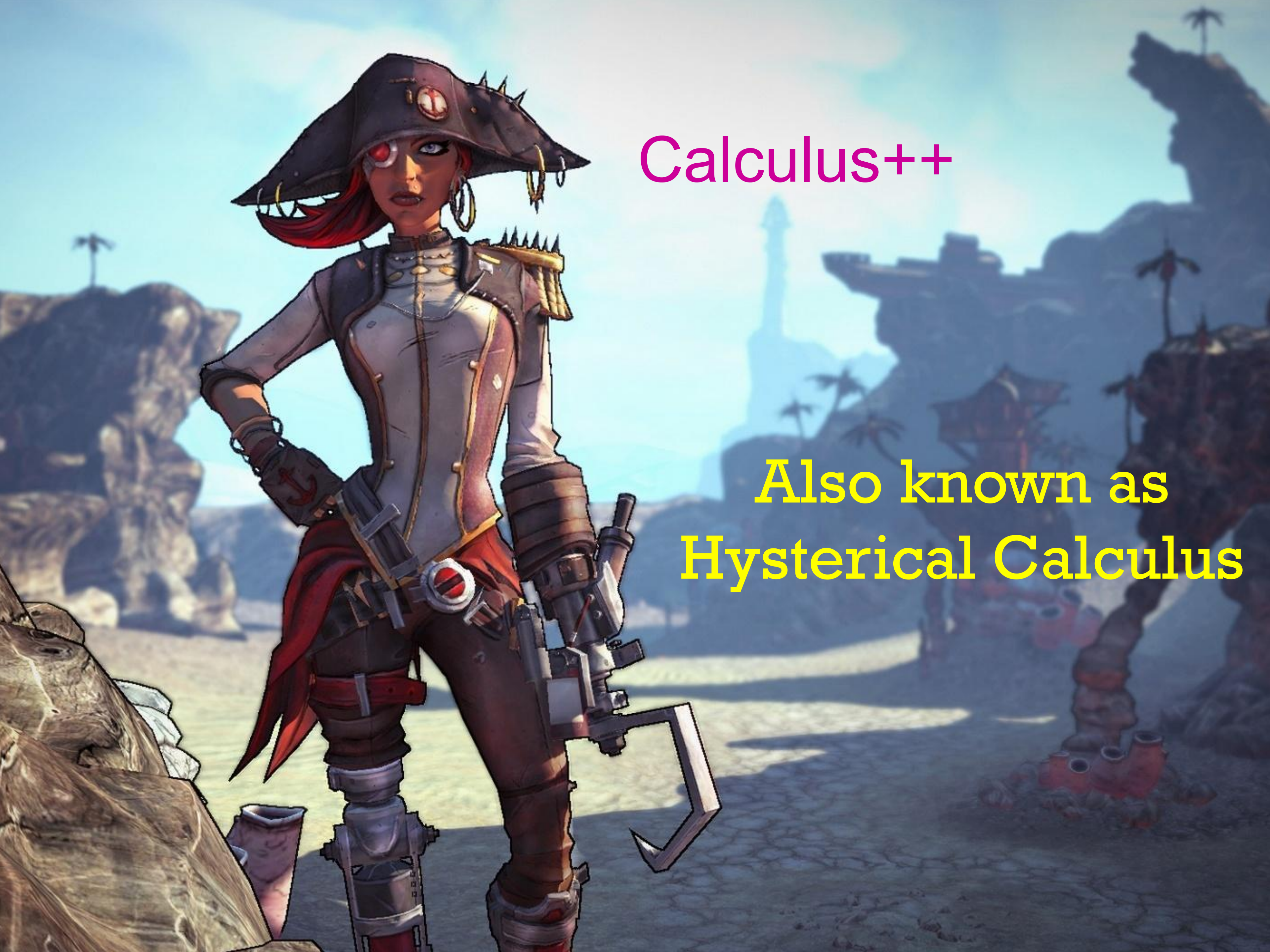
$$x^2 = \frac{m}{n} \quad \text{and} \quad x^5 = \frac{k}{l}.$$

If $m = 0$, then $x^2 = 0$, $x^5 = 0$, and $x = 0$ is a rational number.

In all other cases $x = \frac{x^5}{(x^2)^2} = \frac{k \cdot n^2}{l \cdot m^2}.$

Therefore x is a rational number.

a) I only b) II only c) I and II only
d) I and III only e) II and III only



Calculus++

Also known as
Hysterical Calculus

Question 1. Show that $\sqrt{2}$ is irrational.

Solution. Any integer number n is either even, $n=2k$, or odd, $n=2k+1$, where k is another integer number.

The square of an odd number is odd

$$(2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Hence n^2 can be even only if n is even.

Analogously, the square of an even number is even: $(2k)^2 = 4k^2 = 2(2k^2)$.

Hence n^2 can be odd only if n is odd.

That is n^2 is even (odd), if and only if n is even (odd).

Let us now assume that $\sqrt{2}$ is a rational number, that is $\sqrt{2} = \frac{k}{n}$, where, k and n, do not have common factors.

In particular, either both k and n are odd, or only one of them is even.

Then $2 = \frac{k^2}{n^2} \Rightarrow k^2 = 2n^2$.

That is, k^2 is even, and hence k is also even:

$k = 2m$, where m is another integer number.

But then $2n^2 = 4m^2 \Rightarrow n^2 = 2m^2$.

That is, n^2 is even, and hence n is also even.

Contradiction!

Thus, our assumption that $\sqrt{2}$ is a rational number leads to a contradiction, and hence this number is irrational.

Remark. Using a similar argument one can show that $\sqrt[3]{2}$ is an irrational number.

To show that $\sqrt{3}$ is an irrational number, note that any integer number n is either divisible by 3: $n = 3k$,

or $n = 3k + 1$,

or $n = 3k + 2$.

Higher derivatives

Notations for n-th order derivatives:

$$\frac{d^n}{dx^n} f(x) \quad \text{or} \quad f^{(n)}(x).$$

The following properties are often useful for calculating high-order derivatives:

$$\frac{d^n}{dx^n}(f(x)+g(x)) = \frac{d^n f(x)}{dx^n} + \frac{d^n g(x)}{dx^n}$$

$$\frac{d^n x^k}{dx^n} = 0, \quad \text{if } k < n, \qquad \frac{d^n x^n}{dx^n} = n!,$$

and $$\frac{d^n x^k}{dx^k} = \frac{k!}{(k-n)!} x^{k-n}, \quad \text{if } k > n.$$

Question 5. Find the n-th derivative of the function

$$f(x)=\frac{x^n}{1-x}.$$

Solution. Recall the formula for the sum of a geometrical series

$$1+x+x^2+\cdots+x^{n-1}=\frac{1-x^n}{1-x}.$$

Hence $f(x)=\dfrac{x^n-1+1}{1-x}=\dfrac{1}{1-x}-\dfrac{1-x^n}{1-x}$

$$=\frac{1}{1-x}-(1+x+x^2+\cdots+x^{n-1}).$$

Therefore

$$\frac{d^n}{dx^n} f(x) = \frac{d^n}{dx^n}\frac{1}{1-x} - \frac{d^n}{dx^n}(1+x+x^2+\cdots+x^{n-1})$$

$$= 0$$

$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\frac{d^2}{dx^2}\frac{1}{1-x} = \frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}$$

$$\frac{d^3}{dx^3}\frac{1}{1-x} = \frac{d}{dx}\frac{2}{(1-x)^3} = \frac{2\cdot 3}{(1-x)^4}$$

$$\frac{d^4}{dx^4}\frac{1}{1-x} = \frac{d}{dx}\frac{2\cdot 3}{(1-x)^4} = \frac{2\cdot 3\cdot 4}{(1-x)^5}$$

Thus $$\frac{d^n}{dx^n} f(x) = \frac{n!}{(1-x)^{n+1}}.$$